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# Fuzzy Spheres in *pp* Wave Matrix String Theory

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## Abstract

The behaviour of matrix string theory in the background of a type IIA *pp* wave at small string coupling, $g_s \ll 1$, is determined by the combination $Mg_s$ where $M$ is a dimensionless parameter proportional to the strength of the Ramond-Ramond background. For $Mg_s \ll 1$, the matrix string theory is conventional; only the degrees of freedom in the Cartan subalgebra contribute, and the theory reduces to copies of the perturbative string. For $Mg_s \gg 1$, the theory admits degenerate vacua representing fundamental strings blown up into fuzzy spheres with nonzero lightcone momenta. We determine the spectrum of small fluctuations around these vacua. Around such a vacuum all $N^2$ degrees of freedom are excited with comparable energies. The spectrum of masses has a spacing which is independent of the radius of the fuzzy sphere, in agreement with expected behaviour of continuum giant gravitons. Furthermore, for fuzzy spheres characterized by reducible representations of SU(2) and vanishing Wilson lines, the boundary conditions on the field are characterized by a set of continuous angles which shows that generically the blown up strings do not "close".

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## I. INTRODUCTION

The *pp* wave is one of the few nontrivial backgrounds in which perturbative string theory is exactly solvable [1]. Its realization as a Penrose limit of $AdS \times S$ spacetimes [2] provides a candidate for a nonperturbative definition of string theory, and has recently led to progress in understanding the correspondence between gauge theories and IIB string theories in such backgrounds [3]. In a slightly different direction, BMN [3] also proposed a definition of a BFSS type Matrix theory [4] in *pp* wave backgrounds, thus giving a nonperturbative definition of M-theory in a nontrivial background. The *pp* wave Matrix theory has been studied quite extensively [5, 6, 7, 8] from various points of view.

In this paper we begin an investigation of several nonperturbative aspects of IIA strings in *pp* wave backgrounds using matrix string theory. The BMN matrix model represents DLCQ M-theory in the *pp* wave background. To derive a matrix string theory we first need to compactify an additional spacelike compact direction. Circle compactifications of *pp* wave backgrounds were found in [9], by identifying an appropriate isometry direction. Following a standard procedure [10, 11, 12], we may then obtain a matrix string theory on a IIA *pp* wave background. (This matrix string theory was constructed in [13, 14] and the corresponding perturbative string has been studied in [14, 15, 16]. Some properties of the matrix string theory for the maximally supersymmetric type IIB *pp* wave [17] have been discussed in [18].) Applying standard arguments involving a "9-11" flip, the corresponding IIA string theory has no zerobrane charge, but has nonzero light cone momentum.

The IIA *pp* wave matrix string theory is a deformed 1+1 dimensional SU($N$) Yang-Mills theory which should describe IIA strings with a compact lightcone direction of radius $R$ and a momentum $p_- = N/R$ along it. The dynamics is characterized by two dimensionless parameters, the string coupling $g_s$ and the quantity

$$M = \frac{\mu \ell_s^2}{R}, \tag{1.1}$$

where $\mu$ is the strength of the Ramond-Ramond (RR) field strength, $\ell_s$ is the string length and $R$ is the radius of a null direction. By using a chain of dualities the corresponding matrix string theory is essentially two-dimensional Yang-Mills theory with additional terms that represent the nontrivial background. The dimensional coupling constant of the gauge theory $g_{YM}$ is related to $g_s$ by

$$g_{YM} = \frac{R}{g_s \ell_s^2}, \tag{1.2}$$

in units for which the circumference of the spatial circle on which the Yang-Mills theory lives is

$$L = \frac{2\pi \ell_s^2}{R}. \tag{1.3}$$

The various regimes of the theory are characterized by the combination $Mg_s$. Observe that this quantity is proportional to the *effective string coupling* in this background $g_{\text{eff}} \sim g_s \mu p_- l_s^2$ [18, 19]. When $g_s \to 0$ with $M$ finite, so that $Mg_s \ll 1$, the vacua of matrix string theory are essentially the same as in flat space, *i.e.* the matrices $X^i$ can be all chosen to be diagonal. Well known arguments then lead to light cone strings in the IIA *pp* wave with various lengths whose sum is the total lightcone momentum.

However, matrix string theory is in principle a nonperturbative definition of string theory. The *pp* wave is in fact the only known nontrivial background in which a matrix string theory can be formulated unambiguously. We therefore have a unique opportunity to probe nonperturbative aspects of string theory in a controlled setting. Indeed, when $g_s \to 0$, but $M \to \infty$ keeping $Mg_s$ finite, there are degenerate vacua corresponding to fuzzy spheres along the directions of the RR field. In the IIB language these are actually "fuzzy cylinders" since there is an additional compact direction along which the ground state solution is constant. By the chain of dualities reviewed below this means that IIA fundamental strings with momenta along a compact null direction have blown up into fuzzy spheres. In the following we will, therefore, refer to these simply as fuzzy spheres. In the ground state the matrices which correspond to the other transverse directions are constrained to vanish, *except* for one of the directions, for which the matrix can have a set of "$\theta$-parameters", the number of which is determined by the SU(2) representation content of the fuzzy sphere. Such fuzzy sphere configurations retain all eight linearly realized supersymmetries of the theory. The mechanism is in fact the Myers' effect [20, 21] with two crucial differences. First the various fuzzy sphere configurations have zero lightcone energy and are, therefore, degenerate with the trivial vacuum for which all the matrices vanish. This is what happens in the giant graviton effect [22]. Indeed we have an *exact* microscopic description of giant gravitons in a nontrivial background. In earlier works giant gravitons have been described in certain D-brane backgrounds in an approximate fashion [23].[1] Secondly, the gauge field, though non-dynamical, plays an important role, particularly in the emergence of the $\theta$-parameters.

We then investigate the spectra of small fluctuations around these degenerate fuzzy vacua. This is done in the limit of large $Mg_s$ where a perturbative analysis is valid. The matrix theory fuzzy sphere fluctuation spectrum has been investigated earlier in [5]. For matrix strings, there are several important modifications since the gauge field couples nontrivially and renders the analysis rather involved. Nevertheless the spectrum is rather simple. For the fuzzy sphere in the irreducible representation, the fluctuations are described by a set of $1+1$ dimensional fields labeled by angular momentum quantum numbers $(\ell, m)$ with[2] $0 \le \ell \le N-1$ and $-\ell \le m \le \ell$, with masses of the form $Mf(\ell)$, where $f(\ell)$ is a linear function of $\ell$ that depends on the specific field. Significantly, the mass does not depend directly on $N$. This feature is in fact quite characteristic of fluctuations of giant gravitons.

When $N \gg 1$ the fuzzy spheres may be approximated by continuous spheres. In this limit we show that the radius of the giant graviton agrees with that of the fuzzy sphere, which is

$$r_0 = \frac{\pi \mu g_s \ell_s^3 N}{3R} = (\frac{Mg_s}{3})\pi N \ell_s. \tag{1.4}$$

A simple argument then shows that the fluctuation spectrum of these giant gravitons is governed by a level spacing which is given by $\mu$ and *independent of the radius* $r_0$. This kind of spectrum has been noticed for giant gravitons in $AdS \times S$ backgrounds earlier [28]. Their description in terms of the holographic Yang-Mills theory [29, 30] when applied to the *pp* wave background also reveals a similar spectrum [31, 32].

For matrix strings in flat space, the length of the string is determined by the boundary conditions of the fields in a description in which there is no background Wilson line. As is

[1] An exact description, based in part on the extension [24, 25] of matrix theory in weak backgrounds [21] to matrix string theory, and on the nonabelian Born-Infeld action defined using the symmetric trace prescription, was proposed in [26]. However, as the symmetric trace prescription is known to break down by order $F^6$ [27], this proposal should be viewed cautiously.

[2] This bound on $\ell$ holds only for irreducible vacua; the more general statement is given in Sec. VI D.

well known these boundary conditions correspond to conjugation by U($N$) group elements which belong to the subgroup $S_N$. They represent a collection strings whose lengths add up to the total light cone momentum. In the *pp* wave background there is a similar story for $Mg_s \ll 1$. However for $Mg_s \gg 1$, the fuzzy spheres generically form $N\times N$ dimensional reducible representations of SU(2). Sticking to a gauge in which the background Wilson lines vanish, we will find that for $Mg_s$ large, the fluctuations in each of the irreducible blocks are necessarily periodic as we go around the compact circle. However the boundary conditions on the fluctuations in the off-diagonal blocks are characterized by a set of continuous angles. This implies that the momenta along the circle may be written in the form $\frac{1}{R}(n+\frac{1}{\chi})$, with $\chi$ generically irrational. In other words the blown up string never closes. Alternatively, in a gauge in which the fields are strictly periodic, allowed Wilson lines are characterized by a set of U(1) angles.

The presence of fuzzy sphere vacua suggests that the density of states at intermediate energies has a rather different behaviour than the Hagedorn density of states appropriate for strings. In fact, as discussed in Sec. VII, this appears to be an overly naïve expectation, and the Hagedorn transition persists. Certainly, though, there are nontrivial consequences for the thermodynamics of strings in this regime.

This paper is organized as follows. In Sec. II we describe the basic setup which relates Type IIA strings in *pp* wave backgrounds to a matrix string theory. In Sec. III we show the existence of stable giant gravitons in these *pp* wave backgrounds and discuss their fluctuation spectra. Sec. IV contains our derivation of the *pp* wave matrix string action. In Sec. V we discuss supersymmetry properties of the action and its vacua in various regimes, with particular attention to fuzzy sphere vacua for large $Mg_s$. Sec. VI contains the bulk of our results for fluctuations around fuzzy sphere vacua and allowed boundary conditions. In Sec. VII we make brief comments about implications to thermodynamics and Sec. VIII contains conclusions and comments. The appendix presents a detailed and self-contained discussion of fuzzy spherical harmonics.

## II. THE SETUP

The *pp* wave background in $M$-theory is given by the metric

$$\begin{aligned} ds^2 &= 2dx^+dx^- - \Big[(\frac{\mu}{3})^2(x^a)^2 + (\frac{\mu}{6})^2(x^{a'})^2\Big](dx^+)^2 + (dx^a)^2 + (dx^{a'})^2, \\ F_{+123} &= \mu, \end{aligned} \tag{2.1}$$

where $a = 1\cdots 3$ and $a' = 4\cdots 9$. To make a spacelike isometry explicit it is necessary to perform a coordinate transformation

$$\begin{aligned} x^+ = \hat{x}^+, \qquad x^a = \hat{x}^a, \qquad & x^{a''} = \hat{x}^{a''}\,(a'' = 4\cdots 7) \\ x^- &= \hat{x}^- - \frac{\mu}{6}\hat{x}^8\hat{x}^9, \\ x^8 &= \hat{x}^8\cos(\frac{\mu}{6}\hat{x}^+) + \hat{x}^9\sin(\frac{\mu}{6}\hat{x}^+), \\ x^9 &= -\hat{x}^8\sin(\frac{\mu}{6}\hat{x}^+) + \hat{x}^9\cos(\frac{\mu}{6}\hat{x}^+), \end{aligned} \tag{2.2}$$

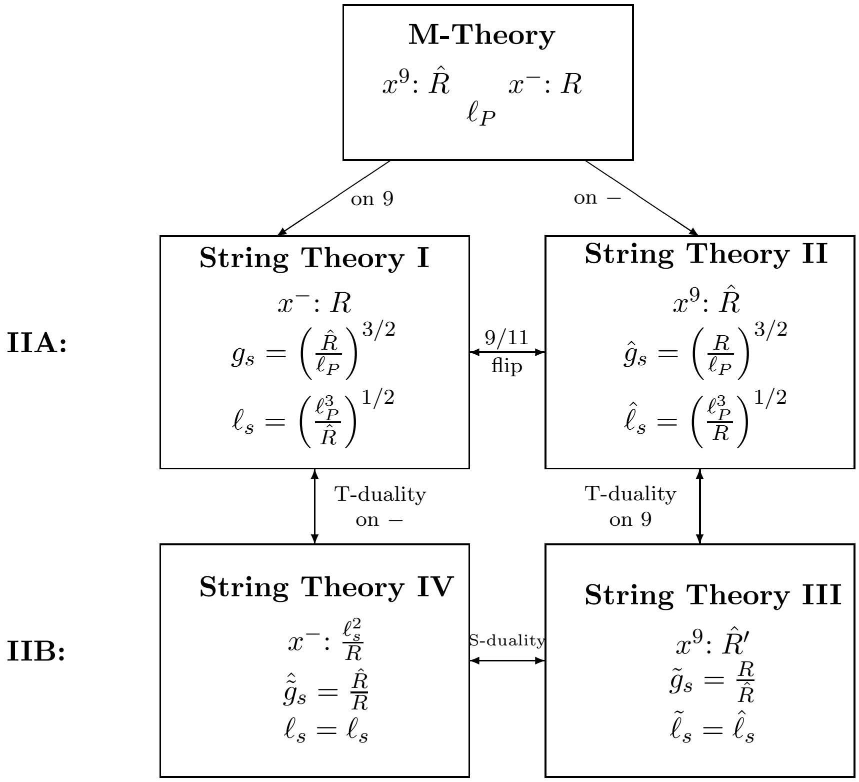


FIG. 1: The chain of dualities that leads from matrix theory to the matrix string. For each theory, the length and direction of the compact dimensions are given. String Theory IV has been included for completeness, but would be more accurately titled "String Theory not appearing in this paper" (with apologies to Monty Python).

so that the background becomes

$$\begin{aligned}
&ds^2 = 2d\hat{x}^+ d\hat{x}^- - 2\frac{\mu}{3}\hat{x}^8 d\hat{x}^9 d\hat{x}^+ - \Big[(\frac{\mu}{3})^2(\hat{x}^a)^2 + (\frac{\mu}{6})^2(\hat{x}^{a''})^2\Big](d\hat{x}^+)^2 + (d\hat{x}^a)^2 + (d\hat{x}^{a'})^2, \\
&\hat{F}_{+123} = \mu.
\end{aligned} \tag{2.3}$$

From now on we will use these new coordinates and remove the hats from the $x$'s. Since translations along the $\hat{x}^9$-direction are isometries one can now compactify along $x^9$ with a radius $\hat{R}$ which will lead to a IIA string theory with the string frame metric and RR fields given by

$$\begin{aligned}
ds^2 = 2dx^+dx^- - \Big[(\frac{\mu}{3})^2\big((x^a)^2 + (x^8)^2\big) + (\frac{\mu}{6})^2(x^{a''})^2\Big](dx^+)^2 + (dx^a)^2 + (dx^{a''})^2 + (dx^8)^2, \\
A_+ = -\frac{\mu}{3}x^8, \quad C_{+ab} = \mu\epsilon_{abc}x^c.
\end{aligned} \tag{2.4}$$

This IIA theory will be called String Theory No. I in the following (see Fig. 1). It has a string coupling

$$g_s = \left(\frac{\hat{R}}{\ell_P}\right)^{3/2}, \tag{2.5}$$

and a string length $\ell_s$ given by

$$\ell_s = \left(\frac{\ell_P^3}{\hat{R}}\right)^{1/2}. \tag{2.6}$$

We will take $x^-$ to be a compact direction with radius $R$ and consider a state in the M-theory with a momentum

$$p_- = \frac{N}{R} \tag{2.7}$$

along this direction. Then String Theory No. I has a null compact direction $x^-$ with a momentum $p_-$ given above.

We can construct another IIA string theory, which we will call String Theory No. II where the null direction $x^-$ is considered as the extra M-theory direction. In this string theory the state under consideration has a zerobrane charge $N$ while the string coupling $\hat{g}_s$ and the string length $\hat{\ell}_s$ are given by

$$\hat{g}_s = \left(\frac{R}{\ell_P}\right)^{3/2}, \qquad \hat{\ell}_s = \left(\frac{\ell_P^3}{R}\right)^{1/2}. \tag{2.8}$$

String Theory No. II has a compact spacelike direction $x^9$ with radius $\hat{R}$.

T-dualizing String Theory No. II along $x^9$ gives rise to a IIB string theory (String Theory No. III) with D1-brane charge $N$ defined on a radius $\hat{R}'$

$$\hat{R}' = \frac{\hat{\ell}_s^2}{\hat{R}} = \frac{\ell_P^3}{R\hat{R}}, \tag{2.9}$$

and string coupling $\tilde{g}_s$ and string length $\tilde{\ell}_s$ given by

$$\tilde{g}_s = \hat{g}_s \frac{\hat{\ell}_s}{\hat{R}} = \frac{R}{\hat{R}}, \qquad \tilde{\ell}_s = \hat{\ell}_s. \tag{2.10}$$

This IIB theory thus has a set of D1-branes wrapped on a circle with a total D1-brane charge $N$ and is therefore described by a 1+1 dimensional Yang-Mills theory with gauge group SU($N$). The dimensional coupling constant of this Yang-Mills theory is given by

$$g_{YM}^2 = \frac{R^2}{\hat{R}\ell_P^3}. \tag{2.11}$$

The dimensionless combination of the Yang-Mills coupling and the size of the circle is easily seen to be

$$g_{YM}\hat{R}' = \frac{1}{g_s}, \tag{2.12}$$

so that String Theory No. I at weak coupling corresponds to strongly coupled Yang-Mills theory. This is the matrix string theory which we discuss in this paper.

As we will see, the matrix string theory is characterized by the combination $Mg_s$, where $M = \frac{\mu\ell_s^2}{R}$ is also dimensionless. In view of the results of [19] the effective string coupling of String Theory No. I is given by the quantity $g_{\text{eff}} = g_s\mu p_- l_s^2$. Since $p_- = N/R$, this implies $g_{\text{eff}} \sim Mg_sN$. Thus one would expect that when $Mg_s \ll 1$ one has the usual perturbative IIA string, while for $Mg_s \sim 1$ one starts probing nonperturbative behaviour. We will find that $1/(Mg_s)$ is essentially the coupling constant of the deformed Yang-Mills theory. Thus for $Mg_s \gg 1$ the Yang-Mills theory becomes weakly coupled and one can discuss nonperturbative string theory in a controlled fashion.

## III. GIANT GRAVITONS AND THEIR FLUCTUATIONS

In this section we will consider a single spherical M2-brane in the background of the M-theory *pp* wave (2.1) and show that there is a stable solution in the presence of a nonzero momentum $p_-$ in the $x^-$ direction. We will then examine the nature of the fluctuation spectrum of these branes.

### A. The Classical Solution

The reparametrization invariant action for this brane is given by

$$S = T_2 \int d^3\xi \left[ -\sqrt{-\det g} + \frac{1}{6}\epsilon^{abc}\partial_a X^\mu \partial_b X^\nu \partial_c X^\lambda C_{\mu\nu\lambda}(X) \right], \tag{3.1}$$

where $\xi^a$ denotes the worldvolume coordinates, $X^\mu(\xi)$ are the target space coordinates, and $g_{ab}$ is the induced metric on the worldvolume

$$g_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \tag{3.2}$$

with $G_{\mu\nu}$ the *pp* wave metric given in eq. (2.1). $C_{\mu\nu\lambda}$ is a 3-form RR potential which gives rise to the 4-form gauge field strength appearing in eq. (2.1).

We will use polar coordinates $(r, \theta, \phi)$ in the plane defined by the Cartesian coordinates $x^a, a = 1 \cdots 3$. In these coordinates the field strength is

$$F_{+r\theta\phi} = \mu r^2 \sin\theta, \tag{3.3}$$

and we can choose a gauge such that the potential is

$$C_{+\theta\phi} = \frac{1}{3}\mu r^3 \sin\theta, \tag{3.4}$$

with all other components zero.

Let us first fix a gauge in which the spatial coordinates on the worldvolume are identified with the angles $\theta$ and $\phi$. We want to restrict our dynamics to the sector where the remaining coordinates are independent of $\theta, \phi$. This is a consistent truncation which respects the equations of motion. Then all worldvolume fields depend only on the worldvolume time $\tau$ and the action (3.1) becomes

$$S = 4\pi T_2 \int d\tau \left[ -r^2(\tau)\sqrt{-G_{AB}\partial_\tau X^A \partial_\tau X^B} + \frac{\mu}{3}r^3(\tau)\partial_\tau X^+ \right], \tag{3.5}$$

where $A, B$ stand for $(r, X^{a'}, X^\pm)$

Let us define the quantity

$$D \equiv \sqrt{-G_{AB}\partial_\tau X^A \partial_\tau X^B}. \tag{3.6}$$

Then the canonical momenta are given by

$$P_A = 4\pi T_2 \left[ \frac{1}{D} r^2 G_{AB}\partial_\tau X^B + \frac{\mu r^3}{3}\delta_A^+ \right]. \tag{3.7}$$

The identity

$$G^{AB}(P_A - \frac{4\pi T_2}{3}\mu r^3 \delta_A^+)(P_B - \frac{4\pi T_2}{3}\mu r^3 \delta_B^+) = -(4\pi T_2 r^2)^2 \tag{3.8}$$

yields

$$P_+ = \frac{1}{2P_-}[G_{++}P_-^2 - P_{a'}^2 - P_r^2 - (4\pi T_2 r^2)^2] + \frac{4\pi T_2}{3}\mu r^3. \tag{3.9}$$

The above gauge choice still allows arbitrary reparametrization of the worldvolume time coordinate $\tau$. We fix this by choosing a gauge

$$\tau = X^+. \tag{3.10}$$

Then the canonical Hamiltonian in this gauge is given by

$$H = -P_+. \tag{3.11}$$

Using the specific form of $G_{++}$ given in (2.4), $H$ may be written as a sum of squares

$$H = \frac{1}{2P_-}\left[P_{a'}^2 + P_r^2 + \left(\frac{\mu}{6}\right)^2 P_-^2 (X^{a'})^2 + r^2(4\pi T_2 r - \frac{\mu}{3}P_-)^2\right]. \tag{3.12}$$

Clearly the ground state solutions are static with $P_r = P_{a'} = 0$, $X^{a'} = 0$, and either $r = 0$ or

$$r = r_0 = \frac{\mu}{12\pi T_2}P_-. \tag{3.13}$$

This second solution is the giant graviton. It has lightcone energy zero, and is therefore degenerate with the lightcone vacuum.

An interesting feature of this solution is that while the momentum $P_-$ in the direction $X^-$ is nonzero, there is no *velocity* along $x^-$. This may be easily seen from the expressions for $P_\pm$

$$\begin{aligned} P_+ &= \frac{(4\pi T_2 r^2)}{D}(\partial_\tau X^- + G_{++}) + \frac{4\pi T_2}{3}\mu r^3, \\ P_- &= \frac{(4\pi T_2 r^2)}{D}, \end{aligned} \tag{3.14}$$

by plugging in the classical solution for a giant graviton given in eq. (3.13).

The M2-brane giant graviton will appear as a stable spherical D2-brane in String Theory No. I with a nonzero light cone momentum $p_- = N/R$. Using

$$T_2 = \frac{1}{4\pi^2 \ell_P^3} = \frac{1}{4\pi^2 g_s \ell_s^3}, \tag{3.15}$$

one sees that the radius of the D2-brane is

$$r_0 = \frac{\mu g_s \ell_s^3}{R}\frac{\pi N}{3} = \ell_s \frac{M g_s}{3}\pi N. \tag{3.16}$$

As explained above, although there is a nonzero $p_-$, the brane is actually static.

In String Theory No. II the giant graviton also appears as a static D2-brane, but with a D0-brane charge $N$ and no other momentum. This D2-brane has a compact transverse direction. T-dualizing along the compact direction yields a D3-brane in String Theory No. III having a shape $S^2 \times S^1$. This is the continuum analog of the "fuzzy cylinder" configuration of matrix string theory discussed in this paper.

### B. Fluctuations

The fluctuation spectrum around the giant graviton may be obtained by an analysis similar to [28]. For our purposes, however, it is sufficient to examine some general features. The effective action for some mode of transverse fluctuation $\Phi$ of the giant graviton would be given, in the linearized approximation, by

$$S = \int d^3\xi \sqrt{-g} g^{ab} \partial_a \Phi \partial_b \Phi, \tag{3.17}$$

where $g_{ab}$ denotes the induced metric on the brane worldvolume, whose components are given by

$$g_{ab} = G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu, \tag{3.18}$$

The target space metric $G$ has to be evaluated *on the classical solution*. In the light cone gauge used above this solution is described by

$$\begin{aligned} \theta &= \sigma, \qquad \phi = \rho, \\ X^+ &= \tau, \\ r &= r_0 = \frac{\mu}{12\pi T_2} P_-, \\ X^{a'} &= 0, \\ X^- &= \text{constant}, \end{aligned} \tag{3.19}$$

where $\sigma$ and $\rho$ denote the two angular spatial coordinates on the worldvolume. Thus on the giant graviton solution one has

$$\begin{aligned} g_{\tau\tau} &= -(\frac{\mu}{3})^2 r_0^2, \\ g_{\sigma\sigma} &= r_0^2, \\ g_{\rho\rho} &= r_0^2 \sin^2\theta, \end{aligned} \tag{3.20}$$

so that the action for fluctuations (3.17) becomes

$$S \sim \frac{\mu}{3} r_0 \int dt d\theta d\phi \sin\theta \left[ -(\frac{3}{\mu})^2 (\partial_\tau \Phi)^2 + (\partial_\theta \Phi)^2 + \frac{1}{\sin^2\theta} (\partial_\phi \Phi)^2 \right]. \tag{3.21}$$

It is now clear that the frequencies of oscillation $\omega$ are independent of $r_0$ and the scale is set entirely by $\mu$. In fact from eq. (3.21) one would get

$$\omega = (\frac{\mu}{3}) \ell(\ell + 1), \tag{3.22}$$

where $\ell$ is the angular momentum quantum number. It would be interesting to verify the above argument by a detailed calculation of the fluctuation spectrum. It is clear that while the details of the spectrum would not agree precisely with (3.22), the fact that it is independent of $r_0$ and hence $p_-$ would persist.

In the following sections we set up the matrix string theory for this problem and study its vacua and fluctuations.

## IV. THE MATRIX STRING ACTION

We start with the matrix theory action of BMN [3], modulo some conventions; see also [5, 7]. Specifically, we normalize the worldline coordinate $\tau$ as in BMN and our $R$ differs from their $R_{11}$ by a factor of 2.[3] We also follow the usual convention of setting the 11-dimensional Planck length, $\ell_P = 1$, (for now). The conventions for indices are $i = 1 \cdots 9$, $a = 1, 2, 3$, $a' = 4 \cdots 9$ and (later) $a'' = 4 \cdots 7$.

$$S = R \int d\tau \, \mathrm{Tr} \left\{ \frac{1}{2R^2} (D_\tau X^i)^2 + \frac{i}{R} \Psi^\mathsf{T} D_\tau \Psi + \Psi^\mathsf{T} \Gamma^i \left[ X^i, \Psi \right] + \frac{1}{4} \left[ X^i, X^j \right]^2 \right.$$
$$\left. - \frac{1}{2} \big(\frac{\mu}{3R}\big)^2 (X^a)^2 - \frac{1}{2} \big(\frac{\mu}{6R}\big)^2 (X^{a'})^2 - i \frac{\mu}{4R} \Psi^\mathsf{T} \Gamma^{123} \Psi - i \frac{\mu}{3R} \epsilon_{abc} X^a X^b X^c \right\} . \quad (4.1)$$

The first part of this section will resemble ref. [14] (see also [15]) and the latter is similar to [12].

The first step toward making this a matrix string action is to make the *field redefinition* (*cf.* eq. (2.2))

$$X^8 = \hat{X}^8 \cos \frac{\mu}{6} \tau + \hat{X}^9 \sin \frac{\mu}{6} \tau, \qquad X^9 = -\hat{X}^8 \sin \frac{\mu}{6} \tau + \hat{X}^9 \cos \frac{\mu}{6} \tau, \quad (4.2a)$$
$$\Psi = e^{\frac{\mu}{12} \Gamma^{89} \tau} \hat{\Psi}. \quad (4.2b)$$

The field redefinition (4.2a) constitutes part of the coordinate transformation (the full coordinate transformation involves $X^-$, which has already been eliminated via use of the infinite momentum frame) to make $\frac{\partial}{\partial \hat{X}^9}$ manifestly Killing [9, 14]. This, of course, is just a $\tau$-dependent rotation of the $(X^8, X^9)$ plane, and so motivates the additional redefinition (4.2b). Then the interaction term $\Psi^\mathsf{T} \Gamma^i [X^i, \Psi]$ is invariant. Upon dropping the hats, the action is

$$S = R \int d\tau \, \mathrm{Tr} \left\{ \frac{1}{2R^2} (D_\tau X^i)^2 + \frac{i}{R} \Psi^\mathsf{T} D_\tau \Psi + \Psi^\mathsf{T} \Gamma^i \left[ X^i, \Psi \right] + \frac{1}{4} \left[ X^i, X^j \right]^2 \right.$$
$$- \frac{1}{2} \big(\frac{\mu}{3R}\big)^2 (X^a)^2 - \frac{1}{2} \big(\frac{\mu}{6R}\big)^2 (X^{a''})^2 - \frac{\mu}{3R^2} X^8 D_\tau X^9 - i \frac{\mu}{3R} \epsilon_{abc} X^a X^b X^c$$
$$\left. - i \frac{\mu}{4R} \Psi^\mathsf{T} (\Gamma^{123} - \frac{1}{3} \Gamma^{89}) \Psi \right\} , \quad (4.3)$$

where we have dropped a total derivative term $\frac{\mu R}{3} \int d\tau \, \mathrm{Tr} \left[ D_\tau (X^8 X^9) \right]$. We see that $X^9$ only appears differentiated in the action, and so we can consider compactifying the $X^9$ direction to a circle of radius $\hat{R}$.[4]

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[3] Also, we use the convention that complex conjugation interchanges the order of Grassmann variables. The gauge theory conventions are $D_\mu = \partial_\mu + i [A_\mu, \cdot]$; $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} + i [A_\mu, A_\nu]$. Under gauge transformations, $A_\mu \to -iU \partial_\mu U^{-1} + U A_\mu U^{-1}$; $X^i \to U X^i U^{-1}$; $\Psi \to U \Psi U^{-1}$. Unless stated otherwise, $\Psi^\mathsf{T}$ denotes the transpose of the spinor $\Psi$, without affecting the U($N$) matrices. So under gauge transformations, $\Psi^\mathsf{T} \to U \Psi^\mathsf{T} U^{-1}$.

[4] The very observant reader will note that choosing to subtract instead of add the total derivative would

We compactify and T-dualize simultaneously, in the usual way [33]. This means that we insert an $\int d\sigma$, with $0 \le \sigma < 2\pi$ and replace

$$X^9 = -i\hat{R}D_\sigma, \tag{4.4}$$

where $D_\sigma$ is the gauge covariant derivative. This implies that

$$D_\tau X^9 = \hat{R}F_{\tau\sigma}. \tag{4.5}$$

So now the action reads

$$\begin{aligned} S = R\int d\tau d\sigma \,\mathrm{Tr}\Big\{ & \frac{\hat{R}^2}{2R^2}F_{\tau\sigma}^2 + \frac{1}{2R^2}(D_\tau X^i)^2 - \frac{\hat{R}^2}{2}(D_\sigma X^i)^2 + \frac{i}{R}\Psi^\mathsf{T} D_\tau\Psi - i\hat{R}\Psi^\mathsf{T}\Gamma^9 D_\sigma\Psi \\ & +\Psi^\mathsf{T}\Gamma^i\left[X^i,\Psi\right] + \frac{1}{4}\left[X^i,X^j\right]^2 - \frac{1}{2}\big(\frac{\mu}{3R}\big)^2(X^a)^2 - \frac{1}{2}\big(\frac{\mu}{6R}\big)^2(X^{a''})^2 \\ & -\frac{\mu\hat{R}}{3R^2}X^8F_{\tau\sigma} - i\frac{\mu}{4R}\Psi^\mathsf{T}(\Gamma^{123} - \frac{1}{3}\Gamma^{89})\Psi - i\frac{\mu}{3R}\epsilon_{abc}X^aX^bX^c\Big\}. \end{aligned} \tag{4.6}$$

The plan is now to put this in a canonical form.

To do this, rescale

$$\tau \to \frac{1}{\hat{R}R}\tau \quad X^i \to \frac{1}{\sqrt{\hat{R}}}X^i. \tag{4.7}$$

After taking the transformation properties of $D_\tau$ and $F_{\tau\sigma}$ into account and reinserting appropriate powers of $\ell_P$ (so as to ensure that the new $X^i$ is dimensionless, as are $\tau,\sigma$), we obtain

$$\begin{aligned} S = \int d\tau d\sigma \,\mathrm{Tr}\Big\{ & \frac{1}{2}\big(\frac{\hat{R}}{\ell_P}\big)^3F_{\tau\sigma}^2 + \frac{1}{2}(D_\tau X^i)^2 - \frac{1}{2}(D_\sigma X^i)^2 + i\Psi^\mathsf{T} D_\tau\Psi - i\Psi^\mathsf{T}\Gamma^9 D_\sigma\Psi \\ & +\big(\frac{\ell_P}{\hat{R}}\big)^{3/2}\Psi^\mathsf{T}\Gamma^i\left[X^i,\Psi\right] + \frac{1}{4}\big(\frac{\ell_P}{\hat{R}}\big)^3\left[X^i,X^j\right]^2 - \frac{1}{2}\big(\frac{\mu\ell_P^2}{3R}\big)^2\big(\frac{\ell_P}{\hat{R}}\big)^2(X^a)^2 - \frac{1}{2}\big(\frac{\mu\ell_P^2}{6R}\big)^2\big(\frac{\ell_P}{\hat{R}}\big)^2(X^{a''})^2 \\ & -\frac{\mu\ell_P^2}{3R}\sqrt{\frac{\hat{R}}{\ell_P}}X^8F_{\tau\sigma} - i\frac{\mu\ell_P^2}{4R}\frac{\ell_P}{\hat{R}}\Psi^\mathsf{T}(\Gamma^{123} - \frac{1}{3}\Gamma^{89})\Psi - i\frac{\mu\ell_P^2}{3R}\big(\frac{\ell_P}{\hat{R}}\big)^{5/2}\epsilon_{abc}X^aX^bX^c\Big\}. \end{aligned} \tag{4.8}$$

Observe that, with the 9/11 flip, all coefficients may be expressed in terms of the quantities

$$\frac{\mu\ell_P^2}{R} = \frac{\mu\alpha'}{R}g_s^{2/3} \equiv Mg_s^{2/3}, \quad \text{and} \quad \frac{\hat{R}}{\ell_P} = g_s^{2/3}. \tag{4.9}$$

---

lead to $\hat{X}^8$ rather than $\hat{X}^9$ as the direction of the circle. Equivalently, we could have chosen the rotation in the field redefinition to be in the opposite direction, and still have obtained an isometry along $\hat{X}^9$. At first sight, this is discomforting. The latter statement is, in fact, true, though in the full coordinate transformation it would require $X^- = \hat{X}^- + \frac{\mu}{6}\hat{X}^8\hat{X}^9$ rather than the $X^- = \hat{X}^- - \frac{\mu}{6}\hat{X}^8\hat{X}^9$ in eq. (2.2). In other words, M-theory is parity invariant ($\hat{X}^9 \to -\hat{X}^9$).

(Note also that, as defined, $M$ is *dimensionless*.) Our final expression is then

$$S = \int d\tau d\sigma \operatorname{Tr}\left\{\frac{1}{2}g_s^2 F_{\tau\sigma}^2 + \frac{1}{2}(D_\tau X^i)^2 - \frac{1}{2}(D_\sigma X^i)^2 + i\Psi^\mathsf{T} D_\tau \Psi - i\Psi^\mathsf{T}\Gamma^9 D_\sigma \Psi \right.$$
$$+\frac{1}{g_s}\Psi^\mathsf{T}\Gamma^i\left[X^i,\Psi\right] + \frac{1}{4g_s^2}\left[X^i,X^j\right]^2 - \frac{1}{2}(\frac{M}{3})^2(X^a)^2 - \frac{1}{2}(\frac{M}{6})^2(X^{a''})^2$$
$$\left. -\frac{M}{3}g_s X^8 F_{\tau\sigma} - i\frac{M}{4}\Psi^\mathsf{T}(\Gamma^{123} - \frac{1}{3}\Gamma^{89})\Psi - i\frac{M}{3g_s}\epsilon_{abc}X^a X^b X^c\right\}. \quad (4.10)$$

Note that all powers of $g_s$ are integer, and that the mass terms have no powers of $g_s$. Finally, each power of $F_{\tau\sigma}$ carries a power of $g_s$.

## V. PROPERTIES OF THE MATRIX STRING ACTION

### A. Supersymmetry and Vacua

The Matrix quantum mechanics (4.1) admits the classical solutions[5]

$$X^a = \frac{\mu}{3R}J^a, \qquad X^8 = X_0^8\cos\frac{\mu}{6}\tau, \qquad X^9 = -X_0^8\sin\frac{\mu}{6}\tau, \quad (5.1a)$$
$$\left[J^a,J^b\right] = i\epsilon_{abc}J^c, \qquad \left[X_0^8,J^a\right] = 0, \quad (5.1b)$$

with all other fields vanishing. Here $J^a$ is an $N$-dimensional representation of the $\mathfrak{su}(2)$ algebra, and $X_0^8$ is a constant matrix. When $J^a$ is nontrivial, these describe a fuzzy sphere, rotating in a circle in the $X^8$-$X^9$ plane. When $X_0^8 = 0$ these are fully supersymmetric vacua with vanishing vacuum energy. For $X_0^8 \neq 0$, these are half-supersymmetric, saturating a BPS bound which relates the energy to the angular momentum in the $X^8$-$X^9$ plane.

Under the reduction to the matrix string (4.10), the solution (5.1a) becomes

$$X^a = \frac{Mg_s}{3}J^a, \qquad X^8 = X_0^8, \qquad \left[J^a,X_0^8\right] = 0, \quad (5.2)$$

with all other fields (such as $A_\sigma$) vanishing. The "rotation" that is included in the field-redefinition (4.2a) modifies the Hamiltonian by an angular momentum term [9]. Also, the supersymmetries broken by the angular momentum are precisely those that are broken by the compactification. Thus, in the matrix string theory, all fuzzy spheres are fully

[5] These are related to solutions which appeared in [3]. While they seemed to only consider either fuzzy spheres ($X_0^8 = 0$) or rotating solutions ($J^a = 0$), the BPS conditions allow them to be superposed under certain conditions (namely, $\left[X^8,J^a\right] = 0$). Bak [34], considered rotating fuzzy spheres, but in that work some of the supersymmetries were broken by adding time dependence to, and deforming, the $\mathfrak{su}(2)$ generators, so as to obtain rotating ellipsoids. There are much simpler states with the same supersymmetry as Bak's, for which one adds an extra term to $X^a$ that commutes with the $\mathfrak{su}(2)$ generators, and which adds a rotation in a plane. These are the oscillations that we believe were misidentified in [5] as being associated with Bak's states. The additional rotations—that we just described as being in a plane that intersects the fuzzy spheres—can also be done in other directions, such as the $X^8, X^9$-directions that we use here.

supersymmetric and have vanishing energy for all values of $X_0^8$. The fact that $X_0^8$ must commute with the $\mathfrak{su}(2)$ generators implies that the fuzzy spheres have definite position in the 8-direction. We will eventually see, and explain, that despite this translation invariance, there is no associated massless mode in the matrix string spectrum.

Explicitly, it can be checked that the action (4.10) is invariant under the (4,4) supersymmetries, (*cf.* [3, 5, 15] and *e.g.* [35] for related *pp* wave and matrix string, supersymmetry transformations)

$$\delta X^i = i\Psi^\intercal \Gamma^i \epsilon, \qquad \delta A_\tau = \frac{i}{g_s}\Psi^\intercal \epsilon, \qquad \delta A_\sigma = \frac{i}{g_s}\Psi^\intercal \Gamma^9 \epsilon, \tag{5.3a}$$

$$\begin{aligned}\delta\Psi = \frac{1}{2}g_s F_{\tau\sigma}\Gamma^9\epsilon + \frac{1}{2}D_\tau X^i\Gamma^i\epsilon - \frac{1}{2}D_\sigma X^i\Gamma^i\Gamma^9\epsilon + \frac{M}{6}X^a\Gamma^a\Gamma^{123}\epsilon + \frac{i}{4g_s}\left[X^i, X^j\right]\Gamma^{ij}\epsilon \\ - \frac{M}{12}X^{a''}\Gamma^{a''123}\epsilon,\end{aligned} \tag{5.3b}$$

where $\epsilon$ obeys the constraint

$$\epsilon = \Gamma^{12389}\epsilon \Leftrightarrow \epsilon = \Gamma^{4567}\epsilon. \tag{5.4}$$

The action is also invariant under the 16 nonlinearly realized supersymmetries

$$\delta\Psi = e^{\frac{M}{4}(\Gamma^{123} - \frac{1}{3}\Gamma^{89})\tau}\chi, \tag{5.5}$$

where $\chi$ is an unconstrained Majorana spinor. We will not discuss the nonlinearly realized supersymmetries further.

It is straightforward to check that the vacua (5.2) preserve all eight (linearly realized) supersymmetries. Conversely, by linear independence of antisymmetrized products of $\Gamma$-matrices (though in principle it should have been necessary to take (5.4) into account) a fully supersymmetric background requires

$$\begin{gathered}F_{\tau\sigma} = D_\tau X^i = D_\sigma X^i = X^{a''} = \left[X^a, X^{a''}\right] = \left[X^a, X^8\right] = \left[X^{a''}, X^8\right] = \left[X^{a''}, X^{b''}\right] = 0, \\ \left[X^a, X^b\right] = i\frac{Mg_s}{3}\epsilon_{abc}X^c.\end{gathered} \tag{5.6}$$

Note that, this does not imply $X^8 = 0$, but allows for a constant value of $X^8$ that commutes with $J^a$. This is precisely eq. (5.2). Moreover the $X^a$ generically higgses the $\mathrm{U}(N)$ gauge group to $\mathrm{U}(1)^r$, where $r$ is the number of irreducible representations in $J^a$. Then $X^8$ has $r$ components that can be turned on, and this (roughly) induces $r$ "$\theta$-angles" for the corresponding $\mathrm{U}(1)$s.

After a tedious calculation, one finds the on-shell algebra,

$$\begin{aligned}\left[\delta_1, \delta_2\right] X^i = i(\epsilon_1^\intercal\epsilon_2)D_\tau X^i - i(\epsilon_1^\intercal\Gamma^9\epsilon_2)D_\sigma X^i + i(\epsilon_1^\intercal\Gamma^a\epsilon_2)\left\{-\frac{M}{3}\delta_b^i\epsilon_{abc}X^c - \frac{i}{g_s}\left[X^a, X^i\right]\right\} \\ - i\frac{i}{g_s}(\epsilon_1^\intercal\Gamma^8\epsilon_2)\left[X^8, X^i\right] + i\frac{M}{6}(\epsilon_1^\intercal\Gamma^{a''b''89}\epsilon_2)\delta_{a''}^i X^{b''},\end{aligned} \tag{5.7a}$$

$$\left[\delta_1, \delta_2\right] A_\tau = i(\epsilon_1^\intercal\Gamma^9\epsilon_2)F_{\tau\sigma} + \frac{i}{g_s}(\epsilon_1^\intercal\Gamma^a\epsilon_2)D_\tau X^a + \frac{i}{g_s}(\epsilon_1^\intercal\Gamma^8\epsilon_2)D_\tau X^8, \tag{5.7b}$$

$$[\delta_1, \delta_2]\, A_\sigma = i(\epsilon_1^\mathsf{T}\epsilon_2)F_{\tau\sigma} + \frac{i}{g_s}(\epsilon_1^\mathsf{T}\Gamma^a\epsilon_2)D_\sigma X^a + \frac{i}{g_s}(\epsilon_1^\mathsf{T}\Gamma^8\epsilon_2)D_\sigma X^8, \tag{5.7c}$$

$$\begin{aligned}[\delta_1, \delta_2]\, \Psi = i(\epsilon_1^\mathsf{T}\epsilon_2)D_\tau\Psi - i(\epsilon_1^\mathsf{T}\Gamma^9\epsilon_2)D_\sigma\Psi + i(\epsilon_1^\mathsf{T}\Gamma^a\epsilon_2)\left\{-\frac{M}{12}\epsilon_{abc}\Gamma^{bc}\Psi - \frac{i}{g_s}\left[X^a, \Psi\right]\right\} \\ + \frac{1}{g_s}(\epsilon_1^\mathsf{T}\Gamma^8\epsilon_2)\left[X^8, \Psi\right] + i\frac{M}{24}(\epsilon_1^\mathsf{T}\Gamma^{a''b''89}\epsilon_2)\Gamma^{a''b''}\Psi.\end{aligned} \tag{5.7d}$$

We can summarize these equations as

$$\begin{aligned}\{Q_\alpha, Q_\beta\} = \mathbb{P}_{\alpha\beta}\big[H - G(A_\tau)\big] + (\Gamma^9\mathbb{P})_{\alpha\beta}\big[P + G(A_\sigma)\big] - (\Gamma^a\mathbb{P})_{\alpha\beta}\big[\frac{M}{3}\mathcal{J}^a - \frac{1}{g_s}G(X^a)\big] \\ + (\Gamma^8\mathbb{P})_{\alpha\beta}\big[-\frac{1}{g_s}G(X^8)\big] + \frac{M}{12}(\Gamma^{a''b''89}\mathbb{P})_{\alpha\beta}\mathcal{J}_-^{a''b''}; \qquad \mathbb{P} = \frac{1}{2}\left(\mathbb{1} + \Gamma^{12389}\right),\end{aligned} \tag{5.8}$$

where $G(\Lambda)$ is the Hermitian generator of infinitesimal gauge transformations with parameter $\Lambda$,[6] $H$ is the Hamiltonian, $P$ generates translations in $\sigma$, $\mathcal{J}^a$ is the SU(2) rotational generator in the 123-directions and $\mathcal{J}_-^{a''b''}$ is the anti-selfdual SU(2) rotational generator in the 4567-directions. The projection operator, $\mathbb{P}$, enforces the constraint (5.4). This explains why only $\mathcal{J}_-^{a''b''}$ appears, and not the full SO(4), as

$$\Gamma^{a''b''89}\mathbb{P} = \frac{1}{2}\Gamma^{c''d''89}\left(\delta_{a''c''}\delta_{b''d''} - \frac{1}{2}\epsilon_{a''b''c''d''}\right), \tag{5.9}$$

which projects out the self-dual SU(2).[7] Thus we see that—as for the 11-dimensional Matrix Theory [5]—the supersymmetry algebra contains angular momentum.

Of course, it is well known (see *e.g.* [36]) that the supersymmetry algebra cannot contain angular momentum. This follows by first considering the $(P, P, Q)$ Jacobi identity, which (in flat space) implies that $Q$ commutes with all momentum generators. Then the $(Q, Q, P)$ Jacobi identity implies that the supersymmetry algebra cannot contain angular momentum. However, in the *pp* wave background, the spacetime momentum and boost/rotational generators are replaced by Heisenberg generators which do not commute. In particular, the Hamiltonian does not commute with "momentum". So, there is no contradiction; indeed the anti-de Sitter superalgebra is a well-known example in which the "theorem" is "violated." Other explicit examples of *pp* wave supersymmetry algebras which contain angular momentum in the anticommutation relations can be found in [37].

Although so far we have isolated the gauge transformations in the supersymmetry algebra from the usual generators, in the fuzzy sphere background, the gauge transformations are crucial to rotational invariance. Let us define the operator

$$\hat{\mathcal{J}}^a \equiv \mathcal{J}^a - \frac{3}{Mg_s}G(X_0^a), \tag{5.10}$$

where $X_0^a$ is the background value of $X^a$, eq. (5.2). It is clear that, for the fuzzy sphere background (5.2), $\hat{\mathcal{J}}$ also obeys the $\mathfrak{su}(2)$ algebra,

$$\left[\hat{\mathcal{J}}^a, \hat{\mathcal{J}}^b\right] = i\epsilon_{abc}\hat{\mathcal{J}}^c. \tag{5.11}$$

[6] Explicitly, $[G(\Lambda), A_\mu] = i\partial_\mu\Lambda + [\Lambda, A_\mu]$, and *e.g.* $\left[G(\Lambda), X^i\right] = \left[\Lambda, X^i\right]$.

[7] Incidentally, eq. (5.9) is crucial for eliminating extraneous terms that would otherwise have appeared in (5.7d).

Moreover, it is a symmetry of the background:

$$\left[\hat{\mathcal{J}}^a, X_0^b\right] = \left[\mathcal{J}^a, X_0^b\right] - \left[J^a, X_0^b\right] = i\epsilon_{abc}X_0^c - i\epsilon_{abc}X_0^c = 0. \tag{5.12}$$

Thus, we rewrite eq. (5.8) as

$$\begin{aligned}\{Q_\alpha, Q_\beta\} = \mathbb{P}_{\alpha\beta}\big[H - G(A_\tau)\big] + (\Gamma^9\mathbb{P})_{\alpha\beta}\big[P + G(A_\sigma)\big] - \frac{M}{3}(\Gamma^a\mathbb{P})_{\alpha\beta}\big[\hat{\mathcal{J}}^a - \frac{3}{Mg_s}G(X^a - X_0^a)\big] \\ + (\Gamma^8\mathbb{P})_{\alpha\beta}\big[-\frac{1}{g_s}G(X^8)\big] + \frac{M}{12}(\Gamma^{a''b''89}\mathbb{P})_{\alpha\beta}\mathcal{J}_-^{a''b''}.\end{aligned} \tag{5.13}$$

From the algebra (5.13), we see that there are short BPS multiplets of mass $m = \frac{M}{3}n$, with $n$ an integer. More precisely, multiplets with quantum numbers $(\ell_1, \ell_2)$ under $(\hat{\mathcal{J}}^a, \mathcal{J}_-^{a''b''})$ can be short if the mass obeys one of

$$\text{mass}^2 = \begin{cases} \left(\frac{M}{3}\right)^2(\ell_1+\ell_2)^2, \\ \left(\frac{M}{3}\right)^2(\ell_1-\ell_2+1)^2, \\ \left(\frac{M}{3}\right)^2(\ell_1-\ell_2-1)^2, \\ \left(\frac{M}{3}\right)^2(\ell_1+\ell_2+2)^2. \end{cases} \tag{5.14}$$

This follows from eq. (A.45).[8] The result (5.14) agrees with the spectrum we give in Sec. VI. Namely, we have four short multiplets: two with $\ell_1 = \ell$, $\ell_2 = 0$, and two with $\ell_1 = \ell$, $\ell_2 = \frac{1}{2}$. It is interesting—though required by counting of supermultiplet components—that the $X^{a''} \in (1/2, 1/2)$ of SO(4) form two degenerate multiplets, and not a longer multiplet.

It is tempting to speculate that the other gauge transformations in the action are related to other BPS conditions. The $G(A_\mu)$ terms are simply the gauge completions of the Hamiltonian $(i\frac{d}{d\tau})$ and the momentum generator $(P = -i\frac{d}{d\sigma})$, but can play an important role if there are Wilson lines. The presence of $G(X^8)$, and the fact that there are vacua with non-trivial values of $X^8$, is particularly intriguing.

### B. Perturbation Theory and Fuzzy Spheres

In eq. (4.10), we wrote down an action in which the fields were scaled in such a way that the matrix string action had mass parameters where expected, and coupling constant dependence that is familiar from the flat-space matrix string. There is another useful rescaling of the fields and coordinates, which isolates the combination $Mg_s$. Specifically, let

$$\tau = \frac{\hat{\tau}}{M}, \qquad \sigma = \frac{\hat{\sigma}}{M}, \qquad A_\mu = M\hat{A}_\mu, \qquad X^i = Mg_s\hat{X}^i, \qquad \Psi = M^{\frac{3}{2}}g_s\hat{\Psi}, \tag{5.15}$$

[8] For the anti-selfdual rotations, this follows by decomposing the four-dimensional spinors into two-dimensional Weyl spinors. Then the two-dimensional spinors can be decomposed in terms of spinor spherical harmonics, and eq. (A.45) is easily applied.

Then,

$$S = (Mg_s)^2 \int d\hat{\tau} d\hat{\sigma}\, \mathrm{Tr} \left\{ \frac{1}{2} \hat{F}_{\hat{\tau}\hat{\sigma}}^2 + \frac{1}{2} (\hat{D}_{\hat{\tau}} \hat{X}^i)^2 - \frac{1}{2} (\hat{D}_{\hat{\sigma}} \hat{X}^i)^2 + i \hat{\Psi}^{\mathsf{T}} \hat{D}_{\hat{\tau}} \hat{\Psi} - i \hat{\Psi}^{\mathsf{T}} \Gamma^9 \hat{D}_{\hat{\sigma}} \hat{\Psi} \right.$$
$$+ \hat{\Psi}^{\mathsf{T}} \Gamma^i \left[ \hat{X}^i, \hat{\Psi} \right] + \frac{1}{4} \left[ \hat{X}^i, \hat{X}^j \right]^2 - \frac{1}{18} (\hat{X}^a)^2 - \frac{1}{72} (\hat{X}^{a''})^2 - \frac{1}{3} \hat{X}^8 \hat{F}_{\hat{\tau}\hat{\sigma}}$$
$$\left. - i \frac{1}{4} \hat{\Psi}^{\mathsf{T}} (\Gamma^{123} - \frac{1}{3} \Gamma^{89}) \hat{\Psi} - i \frac{1}{3} \epsilon_{abc} \hat{X}^a \hat{X}^b \hat{X}^c \right\}, \quad (5.16)$$

where $\hat{D}_\mu = \hat{\partial}_\mu + i[\hat{A}_\mu, \cdot]$ and $\hat{F} = d\hat{A} + i\hat{A} \wedge \hat{A}$. In this expression $Mg_s$ has been completely factored out of the action, which is otherwise independent of both $M$ and $g_s$. Thus, the dimensionless $Mg_s$ acts as an inverse coupling constant, and perturbation theory in $(Mg_s)^{-1}$ is valid.

Since, for the validity of the matrix string theory, we should take $g_s$ small, we take $M \gg g_s^{-1}$ for the perturbative realm. Alternatively, the regime of $Mg_s \ll 1$ requires a strong coupling expansion.

### 1. *Strong Coupling and Perturbative IIA Strings*

For $Mg_s \ll 1$, the Yang-Mills theory is at strong coupling, and we expect to be able to compare to type IIA perturbation theory. In fact, type IIA perturbation theory is an expansion around $g_s = 0$, and is therefore strictly at $Mg_s = 0$. For this value, there are no fuzzy spheres; the fuzzy sphere radii are strictly zero and the background values are $X^a = \frac{Mg_s}{3} J^a = 0$.

Moreover, in the limit $Mg_s \to 0$, finiteness of the energy (derived from (4.10)) sets the commutator terms in the action to zero [12, 38]. Thus one is left with a quadratic (but massive) action for the elements of the Cartan subalgebra—that is, $N$ copies of the IIA string on this background.

There are actually some subtleties with this discussion. Let us consider the action for a single element of the Cartan subalgebra. It is

$$S = (Mg_s)^2 \int d\hat{\tau} d\hat{\sigma}\, \mathrm{Tr} \left\{ \frac{1}{2} \left( \hat{F}_{\hat{\tau}\hat{\sigma}} - \frac{1}{3} \hat{X}^8 \right)^2 + \frac{1}{2} (\dot{\hat{X}}^i)^2 - \frac{1}{2} (\hat{X}'^i)^2 + i \hat{\Psi}^{\mathsf{T}} \dot{\hat{\Psi}} - i \hat{\Psi}^{\mathsf{T}} \Gamma^9 \hat{\Psi}' \right.$$
$$\left. - \frac{1}{18} (\hat{X}^a)^2 - \frac{1}{18} (\hat{X}^8)^2 - \frac{1}{72} (\hat{X}^{a''})^2 - i \frac{1}{4} \hat{\Psi}^{\mathsf{T}} (\Gamma^{123} - \frac{1}{3} \Gamma^{89}) \hat{\Psi} \right\}. \quad (5.17)$$

If we ignore the first term—this is justified in flat space for standard $D = 2$ gauge-theoretic reasons [11, 12]—then we have written the action for the IIA perturbative string, albeit with nonstandard normalizations. Equivalently, we could treat $\hat{F}_{\hat{\tau}\hat{\sigma}}$ as an auxiliary field and integrate it out. However, neither procedure can be justified.

Instead, we should integrate out the gauge potential. In order to do this, we introduce an auxiliary scalar field $\phi$, and replace the term $\frac{1}{2} \hat{F}_{\hat{\tau}\hat{\sigma}}^2$ inside the brackets in (5.17) by

$$-\frac{1}{2} \phi^2 + \phi \hat{F}_{\hat{\tau}\hat{\sigma}}. \quad (5.18)$$

The resulting action is equivalent to (5.17), but now the gauge potential appears only linearly. Varying $\hat{A}_{\hat{\mu}}$ gives the constraint

$$\partial_{\hat{\mu}}(\phi - \frac{1}{3}\hat{X}^8) = 0, \tag{5.19}$$

which together with the subsequent equation of motion for $\phi$ tells us that $\phi = \frac{1}{3}(\hat{X}^8 - \hat{X}^8_0)$. Substituting this for $\phi$ in (5.18) gives the equivalent action

$$S = (Mg_s)^2 \int d\hat{\tau} d\hat{\sigma}\,\mathrm{Tr}\left\{\frac{1}{2}(\dot{\hat{X}}^i)^2 - \frac{1}{2}(\hat{X}'^i)^2 + i\hat{\Psi}^\intercal\dot{\hat{\Psi}} - i\hat{\Psi}^\intercal\Gamma^9\hat{\Psi}' \right.$$
$$\left. -\frac{1}{18}(\hat{X}^a)^2 - \frac{1}{18}(\hat{X}^8 - \hat{X}^8_0)^2 - \frac{1}{72}(\hat{X}^{a''})^2 - i\frac{1}{4}\hat{\Psi}^\intercal(\Gamma^{123} - \frac{1}{3}\Gamma^{89})\hat{\Psi}\right\}. \tag{5.20}$$

The result—consistent with eq. (5.2)—is that $\hat{X}^8$ can take any constant value $\hat{X}^8_0$. Clearly, the spectrum is independent of $\hat{X}^8_0$. Thus, we do obtain the IIA string; however, there are an infinite number of choices, with identical physics, parametrized in the matrix string theory by $\hat{X}^8_0$, although this value is invisible to the perturbative string theory. Furthermore, fluctuations of $\hat{X}^8$ about this value are massive; the spectrum contains no Nambu-Goldstone boson.

This perhaps surprising result is reminiscent of the bosonized massive Schwinger model [39]. More generally, recall that a massless scalar field $\varphi$ in $D$ dimensions has an infinite number of vacua parametrized by the vacuum expectation value of $\varphi$. This is just spontaneous symmetry breaking, where $\varphi$ is its own Nambu-Goldstone boson, and the vacuum expectation value characterizes a superselection sector (for $D > 2$). Of course, for $D = 2$, the typical story is that fluctuations of the Nambu-Goldstone boson permeate space and destroy the spontaneous symmetry breaking.

However, spontaneous symmetry breaking does occur in the two-dimensional bosonized massive Schwinger model [39]. For identical reasons, spontaneous symmetry breaking via the background value of $\hat{X}^8$ also occurs in the action (5.17). The symmetry which is broken by the vacuum expectation value for $\hat{X}^8$ is (classically) anomalous. More precisely, the conserved current is

$$J_\mu = \partial_\mu\hat{X}^8 + \frac{1}{6}\epsilon_\mu{}^\nu A_\nu, \tag{5.21}$$

which is gauge-variant. So, there is no (physical) Nambu-Goldstone boson in the spectrum to mix up the vacua, and the vacuum expectation value of $\hat{X}^8$ characterizes a superselection sector of otherwise identical vacua. The eventual Type IIA Lagrangian is independent of $X^8_0$, although the IIA string obtained is only one element of an infinite family.

Nevertheless, since $\hat{X}^8$ has a target space interpretation, we should expect that, in a partition function, we should sum over all vacua, which means integrating over the superselection sectors, thereby obtaining a factor of the coordinate length of $\hat{X}^8$. This does not appear to be visible in perturbation theory.

We expect that an argument parallel to those in [40] should yield a thermodynamic behaviour identical to that for the perturbative IIA string on this background, and we further expect that thermodynamics to be qualitatively identical to that already worked out [41] for the IIB string on the maximally supersymmetric *pp* wave. Indeed the recent work [42] explicitly gave the expected thermodynamics of the IIA string. For earlier work on *pp* wave thermodynamics, see [43, 44, 45, 46].

*2. Weak Coupling and Fuzzy Spheres*

If $Mg_s$ is large, then no degrees of freedom are suppressed. Instead, [in the original variables (4.10)] the fuzzy sphere vacua

$$\left[X^a, X^b\right] = i\frac{Mg_s}{3}\epsilon_{abc}X^c, \tag{5.22}$$

solve the equations of motion. This is solved by

$$X^a = \frac{Mg_s}{3}J^a, \tag{5.23}$$

where $J^a$ is an $N$-dimensional representation of $\mathfrak{su}(2)$. One might expect that since no degrees of freedom are suppressed, the weakly-coupled matrix model could display a drastically different thermodynamics compared to $Mg_s \ll 1$. In fact, this appears not to be the case, as is further discussed in Sec. VII.

In Sec. VI, we study the fluctuations around these vacua.

## VI. THE MATRIX STRING SPECTRUM

In this section we compute the perturbative spectrum of the matrix string theory. In Sec. VI B, we give the bosonic spectrum about an irreducible vacuum; the corresponding fermionic spectrum is degenerate, as shown in Sec. VI C. General vacua are discussed in Sec. VI D. Nontrivial boundary conditions are applied and discussed in Sec. VI E, but to facilitate the discussion, we start in Sec. VI A with a discussion of boundary conditions in a general gauge theory.

### A. Boundary Conditions in Gauge Theory

The most general boundary condition one can imagine writing for a U($N$) gauge theory on a circle[9] only requires the fields to return to themselves up to a gauge transformation $U(\tau,\sigma)$,

$$\begin{aligned} A_\mu(\tau,\sigma+2\pi) &= U(\tau,\sigma)A_\mu(\tau,\sigma)U(\tau,\sigma)^{-1} - iU(\tau,\sigma)\partial_\mu U(\tau,\sigma)^{-1}, \\ X^i(\tau,\sigma+2\pi) &= U(\tau,\sigma)X^i(\tau,\sigma)U(\tau,\sigma)^{-1}. \end{aligned} \tag{6.1}$$

[9] This is easily extended to a general topology in an arbitrary dimension following [47]. However, though ref. [47] claims that homogeneity of the space restricts the boundary conditions to be independent of position, we do not assume that here. In any case, we will shortly show that our boundary conditions can be taken to be constant. Ref. [47] also allows for the matter fields to change by a constant phase after going around a nontrivial cycle; however, except for a sign this is incompatible with hermiticity of an adjoint field, and the sign is generically not compatible with the symmetries of the action. Of course, we could, more generally, consider *e.g.* $X^{a''}(\sigma+2\pi) = \mathcal{O}^{a''}{}_{b''}UX^{b''}(\sigma)U^{-1}$ with $\mathcal{O}$ a constant SO(4) matrix. Such boundary conditions appear in twisted sectors of *pp* wave *orbifolds*.

Of course, a (not necessarily periodic) gauge transformation $\Omega$ need not preserve these boundary conditions. The boundary conditions of the fields

$$\begin{aligned} A'_\mu &= \Omega A_\mu \Omega^{-1} - i\Omega\partial_\mu\Omega^{-1}, \\ X'^i &= \Omega X^i \Omega^{-1}, \end{aligned} \tag{6.2}$$

are of the form (6.1), but with $U$ replaced by $U'$ where

$$U'(\tau,\sigma) = \Omega(\tau,\sigma+2\pi)U(\tau,\sigma)\Omega(\tau,\sigma)^{-1}. \tag{6.3}$$

Thus, the *field redefinition* (6.2) preserves the action, and changes the boundary conditions according to eq. (6.3).

Also, eq. (6.3) tells us that the unbroken gauge *symmetries* (*i.e.,* gauge transformations which are periodic), for a given boundary condition $U(\tau,\sigma)$, consist of those $\Omega(\tau,\sigma)$ which commute with $U$. This follows since a gauge symmetry is necessarily single-valued.

Now, following [47], suppose the vacuum is such that $F_{\mu\nu} = 0$. Then, there is a unitary matrix $V(\tau,\sigma)$ (not necessarily periodic) such that

$$\langle A_\mu \rangle = -iV^{-1}\partial_\mu V. \tag{6.4}$$

Here, $\langle A_\mu \rangle$ simply means the background, or vacuum, value for $A_\mu$. For a gauge field of the form (6.4), the boundary condition (6.1) is equivalent to

$$\partial_\mu \left[ V(\tau,\sigma+2\pi)U(\tau,\sigma)V(\tau,\sigma)^{-1} \right] = 0. \tag{6.5}$$

Under the field redefinition (6.2) with $\Omega = V$,

$$\begin{aligned} \langle A'_\mu \rangle &= 0, \\ U' &= V(\tau,\sigma+2\pi)U(\tau,\sigma)V(\tau,\sigma)^{-1}. \end{aligned} \tag{6.6}$$

Observe that, because of eq. (6.5), $U'$ is *constant*. Moreover, both the vanishing gauge field and the constant boundary conditions are clearly preserved by constant (or global) "gauge" transformations.

To summarize, it follows that, without loss of generality, one can take $\langle A_\mu \rangle = 0$ and boundary conditions

$$A_\mu(\tau,\sigma+2\pi) = UA_\mu(\tau,\sigma)U^{-1}, \quad X^i(\tau,\sigma+2\pi) = UX^i(\tau,\sigma)U^{-1}, \tag{6.7}$$

where $U$ is a *constant* matrix in the gauge group. Moreover, for any constant $\Omega$ in the gauge group, $U$ and $\Omega U \Omega^{-1}$ are equivalent. Thus, the set of boundary conditions consist of conjugacy classes of the gauge group. For U($N$), this is the maximal torus $U(1)^N/S_N$, where $S_N$ is the group of permutations on the $N$ eigenvalues of the unitary matrix.

Thus, without loss of generality, we can take $\langle A_\mu \rangle = 0$ and boundary conditions

$$U = \begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & \ddots \end{pmatrix} \tag{6.8}$$

where the $\theta_i$'s are constant and ordered. Note that this explicit form fixes the gauge.

Conversely, suppose we have $\langle A_\mu \rangle = 0$ and a boundary condition given by a constant $U$, possibly—though this is not necessary for the following result—of the form (6.8). Under the field-redefinition (6.2), with unitary $\Omega = e^{\frac{\sigma}{2\pi}\ln U^{-1}}$, the theory is equivalently described by

$$\begin{aligned} \langle A'_\sigma \rangle &= -\frac{i}{2\pi}\ln U^{-1}, \\ U' &= \mathbb{1}. \end{aligned} \tag{6.9}$$

Therefore, a nontrivial boundary condition but vanishing gauge field is equivalent to a nontrivial (constant) Wilson line along the circle, and periodic boundary conditions. In particular, this shows that, in the path integral, fixing the boundary conditions and summing over Wilson lines is equivalent to fixing $\langle A_\mu \rangle = 0$ and summing over boundary conditions.[10] This point has been previously emphasized by [48, 49, 50].

For example, consider strongly coupled $d$=2, $\mathcal{N}$=(8,8) super Yang-Mills theory—that is, flat-space matrix string theory. We start with the fact that for the strongly-coupled two-dimensional super Yang-Mills, the gauge field kinetic term is irrelevant [51] and, at each point, the matter fields live in a Cartan subalgebra, $[X^i, X^j] = 0$ [38, 51]. Because the gauge field kinetic term is an irrelevant operator, the gauge field equation of motion—or equivalently, the constraint equation for having gone to $\langle A_\mu \rangle = 0$ gauge—is

$$\left[X^i, D_\mu X^i\right] = 0. \tag{6.10}$$

This is just the vanishing of the charge current [38].

Since this equation involves a sum over $i$, we cannot immediately conclude that the Cartan subalgebra does not rotate. That is, just because the $X^i$ commute at every point does not (immediately) imply that $X^i$'s at different points commute. However, eq. (6.10) implies that

$$\begin{aligned} 0 &= \mathrm{Tr}\left[X^i, D_\mu X^i\right]\left[X^j, D_\mu X^j\right], \\ &= \mathrm{Tr}\left[D_\mu X^j, X^i\right]\left[D_\mu X^i, X^j\right] + \mathrm{Tr}\left[X^i, X^j\right]\left[D_\mu X^i, D_\mu X^j\right], \\ &= \mathrm{Tr}\left[X^i, D_\mu X^j\right]^2. \end{aligned} \tag{6.11}$$

There need not be a sum over $\mu$ here. In the first step the Jacobi identity with cyclicity of the trace was used; since, at each $(\tau, \sigma)$, the $X^i$ live in a Cartan subalgebra, the second term vanishes. This also implies that $[D_\mu X^i, X^j] = D_\mu [X^i, X^j] - [X^i, D_\mu X^j] = -[X^i, D_\mu X^j]$, which was used in the second step. Since the result is (minus) a sum of squares, each term in the sum must separately vanish, and since U($N$) is compact we can conclude that

$$\left[X^i, D_\mu X^j\right] = 0. \tag{6.12}$$

Let us now take, without loss of generality as shown in the discussion which led to eq. (6.6), $\langle A_\mu \rangle = 0$ and a constant boundary condition $U$ of the form (6.8). Although the $X^i$ can be diagonalized, it is not generically true that the boundary condition $U$ and the fields $X^i$ can be simultaneously diagonalized. What we do know is that there is a gauge in which $\langle A_\mu \rangle = 0$, and for which the boundary conditions are constant. We also know

[10] This also explains the empirical observation of [40] that (flat space) IIA thermodynamics is reproduced by matrix string theory under the prescription that boundary conditions are summed with unit weight.

that the degrees of freedom in the $X^i$ are constrained to a Cartan subalgebra [38, 51], and by eq. (6.12), this Cartan subalgebra does not rotate. If this Cartan subalgebra can be simultaneously diagonalized with the boundary condition $U$, then this is the sector of short strings. In particular, the $U(1)^N$ commutes with the Cartan subalgebra, and so the maximal torus of boundary conditions is invisible. Otherwise, consistency demands that the matrix $U$ permutes the elements of the Cartan subalgebra; this leads to the sectors of long strings. So, for flat-space matrix string theory, we reproduce the result that there are $N!$ (the number of elements of $S_N$) sectors of boundary conditions, each of which contains, in principle, a maximal torus of boundary conditions, but this $U(1)^N$ acts trivially on the restricted fields, and so can be neglected.

It is useful to see all this explicitly for the case $N = 2$. The usual description is to take the fields $X^i$ to be diagonal and a vanishing Wilson line. Let us denote any one of these fields by $\Phi$. Then

$$\Phi = \begin{pmatrix} \phi_1 & \\ & \phi_2 \end{pmatrix} \tag{6.13}$$

For the trivial boundary condition, the fields $\phi_1$ and $\phi_2$ are periodic, while for the twisted boundary condition one has

$$\phi_1(\tau, \sigma + 2\pi) = \phi_2(\tau, \sigma) \qquad \phi_2(\tau, \sigma + 2\pi) = \phi_1(\tau, \sigma) \tag{6.14}$$

This means

$$\Phi(\tau, \sigma + 2\pi) = U\Phi(\tau, \sigma)U^\dagger \tag{6.15}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.16}$$

By a global gauge rotation one may bring the field $\Phi$ to the form

$$\Phi = \frac{1}{2}\begin{pmatrix} \phi_1 + \phi_2 & \phi_2 - \phi_1 \\ \phi_2 - \phi_1 & \phi_1 + \phi_2 \end{pmatrix} \tag{6.17}$$

As argued above, the general form of the matrix which implements boundary conditions is now given by

$$U' = \begin{pmatrix} e^{i\theta_1} & \\ & e^{i\theta_2} \end{pmatrix} \tag{6.18}$$

which means that $\phi_1 + \phi_2$ is periodic while $\phi_2 - \phi_1$ acquires a phase $e^{i(\theta_1 - \theta_2)}$. However since both the fields $\phi_1, \phi_2$ are real the only nontrivial value of $\theta_1 - \theta_2$ which is allowed is $\pi$. This gives the twisted boundary condition above.

It is clear, however, that acting on a general $2\times 2$ hermitian matrix the boundary condition $U'$ in (6.18) puts in arbitrary phases in the off-diagonal elements.

## B. The Bosonic Spectrum in the Irreducible Vacuum

For now we will ignore the fermions. Starting with the matrix string action (4.10), and completing a square gives ($a = 1, 2, 3$, $a'' = 4, 5, 6, 7$)

$$S = \int d\tau d\sigma \,\mathrm{Tr}\left\{\frac{1}{2}\left(g_s F_{\tau\sigma} - \frac{M}{3}X^8\right)^2 + \frac{1}{2}(D_\tau X^i)^2 - \frac{1}{2}(D_\sigma X^i)^2 - \frac{1}{2}(\frac{M}{6})^2(X^{a''})^2\right.$$
$$-\frac{1}{2}(\frac{M}{3})^2(X^8)^2 - \frac{1}{2g_s^2}\left(\frac{Mg_s}{3}X^a + \frac{i}{2}\epsilon_{abc}\left[X^b, X^c\right]\right)^2 + \frac{1}{2g_s^2}\left[X^a, X^{a''}\right]^2 + \frac{1}{2g_s^2}\left[X^a, X^8\right]^2$$
$$\left. + \frac{1}{2g_s^2}\left[X^{a''}, X^8\right]^2 + \frac{1}{4g_s^2}\left[X^{a''}, X^{b''}\right]^2\right\}. \tag{6.19}$$

One might expect to be able to set the first term to zero, by treating $F_{\tau\sigma}$ as an auxiliary field.[14] However, as in Sec. V B 1, we will be more careful. In particular, we need to take account of the gauge field in the covariant derivatives. This would not be an issue (or at least not a serious issue) were $M = 0$, as then the equation of motion would (essentially) set $A_\mu = 0$. However, here as the naïve equation of motion sets $F_{\tau\sigma} = \frac{M}{3}X^8$, and therefore turns on the gauge field, we will immediately have an interaction between the gauge field and $X^i$ in the $X^i$ kinetic term.

It is convenient to decompose the fields into a sum over a basis of $\mathrm{U}(N)$ matrices. For now, we will only treat the irreducible vacuum—that is, the vacuum (5.23) for which $J^a$ generate an irreducible representation of $\mathfrak{su}(2)$. So, let us write

$$X^{a''}(\tau,\sigma) = \sum_{\ell=0}^{N-1}\sum_{m=-\ell}^{\ell} X^{a''}_{\ell,m}(\tau,\sigma)Y_{\ell m}, \quad X^8(\tau,\sigma) = \sum_{\ell=0}^{N-1}\sum_{m=-\ell}^{\ell} X^8_{\ell,m}(\tau,\sigma)Y_{\ell m}, \tag{6.20a}$$

$$A_\mu(\tau,\sigma) = \sum_{\ell,m} A_{\mu\ell,m}(\tau,\sigma)Y_{\ell,m}, \qquad X^a = \frac{Mg_s}{3}J^a + \sum_{\ell=0}^{N-1}\sum_{j=|\ell-1|}^{\ell+1}\sum_{m=-\ell}^{\ell} X_{j\ell m}Y^a_{j\ell m}, \tag{6.20b}$$

$$X^{a''}_{\ell,-m} = (-1)^m X^{a''*}_{\ell,m}, \quad X^8_{\ell,-m} = (-1)^m X^{8*}_{\ell,m}, \quad A_{\ell,-m} = (-1)^m A^*_{\mu\ell m},$$
$$X_{j,\ell,-m} = (-1)^{m+(j-\ell)}X^*_{j\ell m}. \tag{6.20c}$$

Here we have used the scalar and vector spherical harmonics; see Appendix A 2. Plugging eq. (6.20) into eq. (6.19) gives, to quadratic order,

$$S = \int d^2\sigma \frac{N}{2}\sum_{\ell,m}\left\{\left|g_s\dot{A}_{\sigma\ell m} - g_s A'_{\tau\ell m} - \frac{M}{3}X^8_{\ell m}\right|^2 + \left|\dot{X}_{\ell\ell m} - i\frac{Mg_s}{3}\sqrt{\ell(\ell+1)}A_{\tau\ell m}\right|^2\right.$$
$$+\left|\dot{X}_{\ell-1,\ell m}\right|^2 + \left|\dot{X}_{\ell+1,\ell m}\right|^2 + \left|\dot{X}^{a''}_{\ell m}\right|^2 + \left|\dot{X}^8_{\ell m}\right|^2 - \left|X'_{\ell\ell m} - i\frac{Mg_s}{3}\sqrt{\ell(\ell+1)}A_{\sigma\ell m}\right|^2$$
$$-\left|X'_{\ell-1,\ell m}\right|^2 - \left|X'_{\ell+1,\ell m}\right|^2 - \left|X'^{a''}_{\ell m}\right|^2 - \left|X'^8_{\ell m}\right|^2 - (\frac{M}{3})^2(\ell+1)^2\left|X_{\ell+1,\ell,m}\right|^2$$
$$\left.-(\frac{M}{3})^2\ell^2\left|X_{\ell-1,\ell,m}\right|^2 - (\frac{M}{3})^2(\ell^2+\ell+\frac{1}{4})\left|X^{a''}_{\ell m}\right|^2 - (\frac{M}{3})^2(\ell^2+\ell+1)\left|X^8_{\ell m}\right|^2\right\}, \tag{6.21}$$

where the overdot (prime) denotes a $\tau$ ($\sigma$) derivative.

Clearly, the fields $X_{\ell-1,\ell,m}$, $X^{a''}_{\ell m}$ and $X_{\ell+1,\ell,m}$ have respective masses $\frac{M}{3}\ell$, $\frac{M}{3}(\ell+\frac{1}{2})$, and $\frac{M}{3}(\ell+1)$. To diagonalize the action for the remaining fields $X^8_{\ell m}, X_{\ell\ell m}$ and $A_{\mu\ell m}$, the first step is to introduce auxiliary (scalar) fields $\phi_{\ell m} = (-1)^m \phi^*_{\ell m}$, as was done in Sec. V B 1

$$S = \frac{N}{2}\sum_{\ell m}\int d^2\sigma \left\{ \left| g_s \dot{A}_{\sigma\ell m} - g_s A'_{\tau\ell m} - \frac{M}{3}X^8_{\ell m}\right|^2 + \left| \dot{X}_{\ell\ell m} - i\frac{Mg_s}{3}\sqrt{\ell(\ell+1)}A_{\tau\ell m}\right|^2 \right.$$
$$- \left| X'_{\ell\ell m} - i\frac{Mg_s}{3}\sqrt{\ell(\ell+1)}A_{\sigma\ell m}\right|^2 - \left| w_\ell \phi_{\ell m} + g_s\dot{A}_{\sigma\ell m} - g_s A'_{\tau\ell m} - \frac{M}{3}X^8_{\ell m}\right|^2$$
$$\left. + \left|\dot{X}^8_{\ell m}\right|^2 - \left|X'^8_{\ell m}\right|^2 - (\frac{M}{3})^2(\ell^2+\ell+1)\left|X^8_{\ell m}\right|^2 + \mathcal{L}(X^{a''}, X_{\ell\pm 1,\ell,m})\right\}, \quad (6.22)$$

where $\mathcal{L}(X^{a''}, X_{\ell\pm1,\ell,m})$ is the Lagrangian density for those massive fields—we will drop this in the following—and $w_\ell$ is an arbitrary normalization. Obviously, integrating $\phi_{\ell m}$ out of the action (6.22) gives back the action (6.21). However, after integrating by parts, the action (6.22) is independent of any derivatives of the gauge field.

In fact, $A_{\mu 00}$ only appears linearly in the action; thus integrating out $A_{\mu 00}$ produces the constraints $\partial_\mu \phi_{00} = 0$. The normalizations $w_\ell$ are arbitrary. However it is convenient to choose $w_0 = \frac{M}{3}$. Then the action for the $\ell = 0$ fields becomes

$$S_{\ell=0} = \frac{N}{2}\int d^2\sigma \left\{ \left(\dot{X}^8_{00}\right)^2 - \left(X'^8_{00}\right)^2 - (\frac{M}{3})^2\left(X^8_{00} - \phi_{00}\right)^2\right\}. \quad (6.23)$$

Since $\phi_{00}$ is an arbitrary constant, this shows that, at $\ell = 0$, there is a single degree of freedom with mass $\frac{M}{3}$. Nevertheless, as was also seen in Sec. V, the constant mode of $X^8_{00}$ is arbitrary. Note that the counting of degrees of freedom at $\ell = 0$ is correct; there is no $X_{\ell\ell m}$ for $\ell = 0$, so the only fields considered here are the gauge field, which contributes no degrees of freedom in two dimensions, and $X^8_{00}$ which has one degree of freedom. Also, the specific choice of $w_0$ is not essential. For some arbitrary $w_0$ the dynamical combination which appears in the action is $(w_0\phi_{00} - \frac{M}{3}X^8_{00})$ and the spectrum is of course unchanged.

For $\ell \neq 0$, we can still integrate out the gauge field. Now the convenient choice is $w_\ell = \frac{M}{3}\sqrt{\ell(\ell+1)}$. After an integration by parts, the resulting action is

$$S_{\ell\neq 0} = \frac{N}{2}\int d^2\sigma \left\{ \left|\dot{X}^8_{\ell m}\right|^2 - \left|X'^8_{\ell m}\right|^2 + \left|\dot{\phi}_{\ell m}\right|^2 - |\phi'_{\ell m}|^2 \right.$$
$$\left. -(\frac{M}{3})^2(\ell^2+\ell+1)\left|X^8_{\ell m}\right|^2 - (\frac{M}{3})^2\ell(\ell+1)\left|\phi_{\ell m}\right|^2 + 2(\frac{M}{3})^2\sqrt{\ell(\ell+1)}\,\mathrm{Re}\,X^{8*}_{\ell m}\phi_{\ell m}\right\}. \quad (6.24)$$

Diagonalizing the mass matrix in this action (6.24) gives masses $\frac{M}{3}\ell$ and $\frac{M}{3}(\ell+1)$, which is precisely the spectrum of $X_{\ell\pm1,\ell,m}$! Once again one could have chosen any other $w_\ell$.

Counting, for each $\ell$ we have $4(\ell+1)$ bosons of mass-squared $\left(\frac{M}{3}\right)^2(\ell+1)^2$; $4\ell$ bosons of mass-squared $\left(\frac{M}{3}\right)^2\ell^2$; and (from $X^{a''}$) $4(2\ell+1)$ bosons of mass-squared $\left(\frac{M}{3}\right)^2(\ell+\frac{1}{2})^2$. This is summarized in table I.

This result for the spectrum is not surprising, as we expected $X^8$ and $X^a$ to be in the same supermultiplet, and to therefore have related masses. What is more surprising is that

| origin | degeneracy | mass$^2$ | range |
|---|---|---|---|
| $X^a, X^8, A_\mu$ | $4(\ell+1)$ | $\left(\frac{M}{3}\right)^2(\ell+1)^2$ | $0 \le \ell \le N-1$ |
| | $4\ell$ | $\left(\frac{M}{3}\right)^2\ell^2$ | $1 \le \ell \le N-1$ |
| $X^{a''}$ | $4(2\ell+1)$ | $\left(\frac{M}{3}\right)^2(\ell+\frac{1}{2})^2$ | $0 \le \ell \le N-1$ |

TABLE I: The bosonic spectrum for the irreducible vacuum.

there is, nevertheless a splitting within this supermultiplet. Since the linearly realized (4,4) worldsheet supersymmetries commute with the Hamiltonian, one might have expected the spectrum to be degenerate. In Sec. V A, we examined the superalgebra and observed that the angular momentum generators appear in the square of the supercharge in such a way as to account for this splitting. This is similar to the M-theory BPS analysis of [5, 6]. (In the case of the M-theory Matrix quantum mechanics, $X_{\ell\pm1,\ell m}$ had an identical split spectrum, but the supercharges did not commute with the Hamiltonian in precisely a way that produced the splitting.)

Observe that the field $X_{\ell\ell m}$ is not dynamical and can be removed by a gauge transformation. That is, to linearized order, $X_{\ell\ell m}$ does not appear in the action (6.24) because, to this order, $X_{\ell\ell m}$ is a pure gauge mode. A gauge transformation is generated by the scalar $\Lambda = \sum_{\ell m} \Lambda_{\ell m} Y_{\ell m}$. Under a gauge transformation, $X^i \to X^i + i\,[\Lambda, X^i]$. Because, to zeroth order, $X^a = \frac{Mg_s}{3} J^a$,

$$X_{\ell\ell m} \to X_{\ell\ell m} - i\frac{Mg_s}{3}\sqrt{\ell(\ell+1)}\Lambda_{\ell m} + \mathcal{O}\left((Mg_s)^0\right), \tag{6.25}$$

So, to leading order in perturbation theory[11] in $\frac{1}{Mg_s}$, we see that by setting $\Lambda_{\ell m} = \frac{i}{\left(\frac{Mg_s}{3}\right)\sqrt{\ell(\ell+1)}}\left(X^{(\text{desired})}_{\ell\ell m} - X^{(\text{old})}_{\ell\ell m}\right)$, we can set $X_{\ell\ell m}$ to any convenient value. Recall that $X_{\ell\ell m}$ exist only for $\ell \ge 1$; thus this gauge choice does not fix the U(1) degree of freedom. Otherwise, this fixes $\Lambda$ and so there are no residual gauge transformations other than the U(1). The discussion above would have been simplified very slightly by taking $X_{\ell\ell m} = 0$, but the more general presentation is instructive.

For example, we could alternatively choose a gauge such that $X^3 = J^3 + (\text{fluctuations})$ is diagonal. This is achieved by first choosing to use standard $\mathfrak{su}(2)$ representations, so that $J^3$ is diagonal, and then only allowing the diagonal elements of $X^3$ to fluctuate. In other words, we demand that

$$\sum_{j,\ell,m} X_{j\ell m} Y^3_{j\ell m} = \text{diagonal}. \tag{6.26}$$

Using the explicit form for the vector spherical harmonics in terms of scalar spherical harmonics, eq. (A.33), this is

$$\sum_{\ell,m}\left[\sqrt{\frac{(\ell+1)^2-m^2}{(\ell+1)(2\ell+1)}}X_{\ell+1,\ell,m} + \frac{m}{\sqrt{\ell(\ell+1)}}X_{\ell\ell m} - \sqrt{\frac{\ell^2-m^2}{\ell(2\ell+1)}}X_{\ell-1,\ell,m}\right]Y_{\ell m} = \text{diagonal}. \tag{6.27}$$

[11] The subleading term is bilinear in $\Lambda$ and $X_{j\ell m}$, with coefficients extracted from eq. (A.41).

Since the $Y_{\ell m}$'s are linearly independent, and diagonal precisely for $m=0$, this is solved by setting

$$X_{\ell\ell m} = \frac{1}{m}\sqrt{\frac{(\ell+1)(\ell^2-m^2)}{2\ell+1}}X_{\ell-1,\ell,m} - \frac{1}{m}\sqrt{\frac{\ell[(\ell+1)^2-m^2]}{2\ell+1}}X_{\ell+1,\ell,m}, \quad m\neq 0. \tag{6.28}$$

The gauge freedom (6.25) allows this, and further allows us to set the otherwise unconstrained

$$X_{\ell\ell 0} = 0. \tag{6.29}$$

The overall U(1) is fixed via $A_{\tau 00}=0$.

### C. The Fermionic Spectrum in the Irreducible Vacuum

Now let us look at the fermions. Heretofore, we have implicitly worked in a Majorana basis so that $\Psi=\Psi^*$ and the $\Gamma$-matrices were real and symmetric. It is now convenient to abandon this basis in favour of one in which the $3+4+1=\{a,a'',8\}$ splitting of the $\Gamma$-matrices is more transparent. In terms of the Pauli matrices $\sigma^a$ and the 4-dimensional $\gamma$-matrices, we write

$$\Gamma^a = \sigma^a\otimes 1\!\!1\otimes\sigma^3,\quad \Gamma^{a''} = 1\!\!1\otimes\gamma^{(a''-3)}\otimes\sigma^1,\quad \Gamma^8 = 1\!\!1\otimes\gamma^5\otimes\sigma^1,\quad \Gamma^9 = -1\!\!1\otimes 1\!\!1\otimes\sigma^2. \tag{6.30}$$

Here $\gamma^5=\gamma^{1234}$ and so indeed $\Gamma^9=\Gamma^{12345678}$. In this basis, the charge conjugation matrix $C$ is

$$C = \sigma^2\otimes C\otimes\sigma^1 \tag{6.31}$$

where we have also called the 4-dimensional charge conjugation matrix $C$. As

$$\Gamma^{123} = i1\!\!1\otimes 1\!\!1\otimes\sigma^3, \qquad \Gamma^{89} = -i1\!\!1\otimes\gamma^5\otimes\sigma^3, \tag{6.32}$$

the Weyl representation for the 4-dimensional $\gamma$-matrices is convenient. In particular, we will use a representation in which $\gamma^5=\mathrm{diag}(-1,1,-1,1)$.

We now expand $\Psi$ using the spinor spherical harmonics (see Appendix A) as

$$\Psi = \sum_{\ell,m}\sum_{I=1}^{8}\sum_{\Lambda=1}^{2}\Big[\psi^{I\Lambda}_{\ell+\frac12,\ell,m}\boldsymbol{S}_{\ell+\frac12,\ell,m}\otimes\boldsymbol{b}_I\otimes\boldsymbol{b}_\Lambda + \psi^{I\Lambda}_{\ell-\frac12,\ell,m}\boldsymbol{S}_{\ell-\frac12,\ell,m}\otimes\boldsymbol{b}_I\otimes\boldsymbol{b}_\Lambda\Big], \tag{6.33a}$$

$$(-1)^m C_{IJ}\sigma^1_{\Lambda\Sigma}\psi^{J\Sigma *}_{\ell+\frac12,\ell,-m} = \psi^{I\Lambda}_{\ell+\frac12,\ell,m}, \qquad (-1)^{m+1}C_{IJ}\sigma^1_{\Lambda\Sigma}\psi^{J\Sigma *}_{\ell-\frac12,\ell,-m} = \psi^{I\Lambda}_{\ell-\frac12,\ell,m}. \tag{6.33b}$$

Here $\boldsymbol{b}_I$ and $\boldsymbol{b}_\Lambda$ are (commutative) orthonormal basis spinors in their respective subspaces, namely $b^J_I=\delta^J_I$ and $b^\Sigma_\Lambda=\delta^\Sigma_\Lambda$. The reality condition on the (Grassmann) coefficients is the Majorana condition $\Psi=C\Psi^*$, where the complex conjugation includes Hermitian conjugation of the U($N$) matrix.

The fermionic quadratic action is now reasonably simple,

$$\begin{aligned} S = N\sum_{\ell,m,I,\Lambda}\Big[ & i\psi^{I\Lambda *}_{\ell+\frac12,\ell,m}\dot\psi^{I\Lambda}_{\ell+\frac12,\ell,m} + i\psi^{I\Lambda *}_{\ell-\frac12,\ell,m}\dot\psi^{I\Lambda}_{\ell-\frac12,\ell,m} + i\psi^{I\Lambda *}_{\ell+\frac12,\ell,m}\sigma^2_{\Lambda\Sigma}\psi'^{I\Sigma}_{\ell+\frac12,\ell,m} \\ & + i\psi^{I\Lambda *}_{\ell-\frac12,\ell,m}\sigma^2_{\Lambda\Sigma}\psi'^{I\Sigma}_{\ell-\frac12,\ell,m} - \frac{M}{4}(-1)^\Lambda[1+\frac13(-1)^I]\psi^{I\Lambda *}_{\ell+\frac12,\ell,m}\psi^{I\Lambda}_{\ell+\frac12,\ell,m} \\ & -\frac{M}{4}(-1)^\Lambda[1+\frac13(-1)^I]\psi^{I\Lambda *}_{\ell-\frac12,\ell,m}\psi^{I\Lambda}_{\ell-\frac12,\ell,m} - \frac{M}{3}(-1)^\Lambda\ell\psi^{I\Lambda *}_{\ell+\frac12,\ell,m}\psi^{I\Lambda}_{\ell+\frac12,\ell,m} \\ & +\frac{M}{3}(-1)^\Lambda(\ell+1)\psi^{I\Lambda *}_{\ell-\frac12,\ell,m}\psi^{I\Lambda}_{\ell-\frac12,\ell,m}\Big]. \end{aligned} \tag{6.34}$$

| origin | degeneracy | mass$^2$ | range |
|---|---|---|---|
| $\frac{\mathbb{1}+\Gamma^{12389}}{2}\Psi$ | $8(\ell+1)$<br>$8\ell$ | $\left(\frac{M}{3}\right)^2(\ell+1)^2$<br>$\left(\frac{M}{3}\right)^2\ell^2$ | $0 \leq \ell \leq N-1$<br>$1 \leq \ell \leq N-1$ |
| $\frac{\mathbb{1}-\Gamma^{12389}}{2}\Psi$ | $8(2\ell+1)$ | $\left(\frac{M}{3}\right)^2(\ell+\frac{1}{2})^2$ | $0 \leq \ell \leq N-1$ |

TABLE II: The fermionic spectrum for the irreducible vacuum. The degeneracy of the fermions is twice that of the bosons, because we include the number of components of the fermions.

It is easy to read off the mass-squareds of the fermions, namely

$$\begin{aligned} j = \ell + \frac{1}{2}, I \text{ even} : &\left(\frac{M}{3}\right)^2(\ell+1)^2, \\ j = \ell \pm \frac{1}{2}, I \text{ odd} : &\left(\frac{M}{3}\right)^2(\ell+\frac{1}{2})^2, \\ j = \ell - \frac{1}{2}, I \text{ even} : &\left(\frac{M}{3}\right)^2\ell^2. \end{aligned} \tag{6.35}$$

This is precisely the spectrum of the bosons. In particular, for each $\ell$, there are $8(2\ell+1)$ fermions of mass-squared $\left(\frac{M}{3}\right)^2(\ell+\frac{1}{2})^2$; $8\ell$ fermions of mass-squared $\left(\frac{M}{3}\right)^2\ell^2$; and $8(\ell+1)$ fermions of mass-squared $\left(\frac{M}{3}\right)^2(\ell+1)^2$. This is precisely (double) the spectrum of the bosons!

Also, recall that whether $I$ is even or odd is correlated with the $\gamma^5$-chirality. Furthermore,

$$\Gamma^{12389} = \mathbb{1} \otimes \gamma^5 \otimes \mathbb{1}. \tag{6.36}$$

Therefore, we see that all fermions of negative $\Gamma^{12389}$ chirality have the same mass as $X^{a''}$, and that all fermions of positive $\Gamma^{12389}$ have a mass splitting, but with masses identical to those of the remaining bosons. This has been tabulated in table II.

### D. Reducible Vacua

For the simplest reducible vacua,

$$X^a = X_0^a = \begin{pmatrix} J^a_{N_1} & & 0 \\ & J^a_{N_2} & \\ 0 & & \ddots \end{pmatrix}, \tag{6.37}$$

with all other fields vanishing, the fluctuations are again written in terms of spherical harmonics. Explicitly, for example,

$$X^a = X_0^a + \begin{pmatrix} \sum X^{(1,1)}_{j\ell m} Y^{(1,1)a}_{j\ell m} & \sum X^{(1,2)}_{j\ell m} Y^{(1,2)a}_{j\ell m} & \dots \\ \sum X^{(1,2)*}_{j\ell m} Y^{(1,2)a\dagger}_{j\ell m} & \sum X^{(2,2)}_{j\ell m} Y^{(2,2)a}_{j\ell m} & \dots \\ \vdots & \ddots & \end{pmatrix}. \tag{6.38}$$

For notational purposes, we write

$$
\begin{aligned}
X^i &= \begin{pmatrix} X^{(1,1)i} & X^{(1,2)i} & \dots \\ X^{(2,1)i} & X^{(2,2)i} & \dots \\ \vdots & \ddots & \end{pmatrix}, \quad X^{(x,y)i\dagger} = X^{(y,x)i}, \\
A_\mu &= \begin{pmatrix} A_\mu^{(1,1)} & A_\mu^{(1,2)} & \dots \\ A_\mu^{(2,1)} & A_\mu^{(2,2)} & \dots \\ \vdots & \ddots & \end{pmatrix}, \qquad A_\mu^{(x,y)\dagger} = A_\mu^{(y,x)},
\end{aligned}
\tag{6.39}
$$

where $x, y$ run over the blocks which have dimension $N_x, N_y, \sum_x N_x = N$. Then,

$$
X^{(x,y)a''} = \sum_{j=\frac{|N_x-N_y|}{2}}^{\frac{N_x+N_y}{2}-1} \sum_{m=-j}^{j} X_{jm}^{(x,y)a''} Y_{jm}^{(N_x,N_y)}, \qquad X_{jm}^{(x,y)a''*} = (-1)^{m-\frac{N_x-N_y}{2}} X_{j,-m}^{(y,x)a''}, \tag{6.40a}
$$

$$
X^{(x,y)8} = \sum_{j=\frac{|N_x-N_y|}{2}}^{\frac{N_x+N_y}{2}-1} \sum_{m=-j}^{j} X_{jm}^{(x,y)8} Y_{jm}^{(N_x,N_y)}, \qquad X_{jm}^{(x,y)8*} = (-1)^{m-\frac{N_x-N_y}{2}} X_{j,-m}^{(y,x)8}, \tag{6.40b}
$$

$$
A_\mu^{(x,y)} = \sum_{j=\frac{|N_x-N_y|}{2}}^{\frac{N_x+N_y}{2}-1} \sum_{m=-j}^{j} A_{\mu jm}^{(x,y)} Y_{jm}^{(N_x,N_y)}, \qquad A_{\mu jm}^{(x,y)*} = (-1)^{m-\frac{N_x-N_y}{2}} A_{\mu j,-m}^{(y,x)}, \tag{6.40c}
$$

$$
X^{(x,y)a} = \frac{Mg_s}{3}\delta_{xy} J_{N_x}^a + \sum_{\ell=\frac{|N_x-N_y|}{2}}^{\frac{N_x+N_y}{2}-1} \sum_{j=\ell-1}^{\ell+1} \sum_{m=-\ell}^{\ell} X_{j\ell m}^{(x,y)} Y_{j\ell m}^{(N_x,N_y)a},
$$

$$
X_{j\ell m}^{(x,y)*} = (-1)^{\ell+m+1-j-\frac{N_x-N_y}{2}} X_{j,\ell,-m}^{(y,x)}. \tag{6.40d}
$$

Because the properties [eqs. (A.26)–(A.29), (A.34)–(A.38) and (A.44)–(A.47)] of the spherical harmonics are essentially independent of the dimension—or even squareness—of the blocks, we can immediately copy the results of the spectrum of the irreducible vacuum, and apply it to the reducible vacua, albeit for a different range of $\ell$s. That is, the spectrum is:

$$
\begin{aligned}
X^a, X^8 &: \text{mass}^2 = (\frac{M}{3})^2 j^2, && \text{degeneracy } 4j, && \text{with } \frac{|N_x-N_y|}{2} \le j \le \frac{N_x+N_y}{2} - 1. \\
X^a, X^8 &: \text{mass}^2 = (\frac{M}{3})^2 (j+1)^2, && \text{degeneracy } 4(j+1), && \text{with } \frac{|N_x-N_y|}{2} \le j \le \frac{N_x+N_y}{2} - 1. \\
X^{a''} &: \text{mass}^2 = (\frac{M}{3})^2 (j+\frac{1}{2})^2, && \text{degeneracy } 4(2j+1), && \text{with } \frac{|N_x-N_y|}{2} \le j \le \frac{N_x+N_y}{2} - 1.
\end{aligned}
\tag{6.41}
$$

with an identical spectrum for the fermions.

However, an analysis in Sec. V A showed that more general supersymmetric vacua exist with constant $X^8$ subject to the condition

$$
\left[X^a, X^8\right] = 0. \tag{6.42}
$$

For the irreducible vacuum, this requires that $X^8$ be proportional to the identity; the constant is then effectively just a $\theta$-angle for the U(1) part of the gauge group. However, for a reducible vacuum consisting of $r > 1$ irreducible representations, the story is more complicated. By eq. (6.42), $X^8$ and $X^3$ are simultaneously diagonalizable; thus, by Schur's lemma, $X^8$ consists (up to a similarity transformation) of $r$ blocks, each proportional to the identity matrix. Thus, eq. (6.40b) is generalized to

$$X^{(x,y)8} = X_0^{(x)8}\delta_{xy}\mathbb{1}_{N_x} + \sum_{j,m} X_{jm}^{(x,y)8} Y_{jm}^{(N_x,N_y)}, \tag{6.43}$$

where $X_0^{(x)8}$ is real.

In Sec. V A we explained that these solutions arise from time-dependent, 1/2 supersymmetric solutions of the 11-dimensional matrix quantum mechanics (4.1). There are similarly other solutions that correspond to rotating fuzzy spheres, and are 1/2 supersymmetric in both the matrix quantum mechanics and the matrix string theory. (For example, in matrix string theory, a rotation in the $X^6$-$X^7$ plane preserves the 4 supersymmetries preserved by $\Gamma^{12389}\epsilon = \epsilon = \Gamma^{12367}\epsilon$. This lifts to a Matrix Theory solution that preserves the 8 supersymmetries preserved by $\Gamma^{12367}\epsilon = \epsilon$.) However, we do not discuss these further.

Rather than work with the action, let us work directly with the equations of motion.[12] The exact equations of motion which follow from the action (6.19) are

$$g_s D^\nu \left( g_s F_{\mu\nu} - \frac{M}{3}\epsilon_{\mu\nu} X^8 \right) + i\left[X^i, D_\mu X^i\right] = 0, \tag{6.44a}$$

$$\begin{aligned} -D^2 X^a &+ \frac{M}{3}\left(\frac{M}{3}X^a + \frac{i}{2g_s}\epsilon_{abc}\left[X^b, X^c\right]\right) + \frac{i}{g_s}\epsilon_{abc}\left[X^b, \left(\frac{M}{3}X^c + \frac{i}{2g_s}\epsilon_{cde}\left[X^d, X^e\right]\right)\right] \\ &+ \frac{1}{g_s^2}\left[X^{a''}, \left[X^{a''}, X^a\right]\right] + \frac{1}{g_s^2}\left[X^8, \left[X^8, X^a\right]\right] = 0, \end{aligned} \tag{6.44b}$$

$$\begin{aligned} -D^2 X^{a''} &+ \left(\frac{M}{6}\right)^2 X^{a''} + \frac{1}{g_s^2}\left[X^a, \left[X^a, X^{a''}\right]\right] + \frac{1}{g_s^2}\left[X^8, \left[X^8, X^{a''}\right]\right] \\ &+ \frac{1}{g_s^2}\left[X^{b''}, \left[X^{b''}, X^{a''}\right]\right] = 0, \end{aligned} \tag{6.44c}$$

$$-D^2 X^8 + \frac{M}{3} g_s F_{\tau\sigma} + \frac{1}{g_s^2}\left[X^a, \left[X^a, X^8\right]\right] + \frac{1}{g_s^2}\left[X^{a''}, \left[X^{a''}, X^8\right]\right] = 0. \tag{6.44d}$$

We note that the mass term for $X^8$ was replaced by $F_{\tau\sigma}$. Moreover, the equation of motion for the gauge field, (6.44a), only requires (assuming the current vanishes) $g_s F_{\tau\sigma} - \frac{M}{3}X^8 = \text{constant}$; this is why we can set $X^8$ to a constant matrix and $F_{\tau\sigma} = 0$ while still obeying the equations of motion, though taking the interaction terms into account shows that $X^8$ should commute with $X^a$, in agreement with the supersymmetry conditions.

[12] It is possible to obtain the spectrum using the same auxiliary fields as in Sec. VI B, but the algebra is more complicated.

Plugging the expansion (6.40) into the equations of motion, and linearizing, gives

$$0 = g_s\partial_\mu\epsilon^{\nu\rho}\partial_\nu A^{(x,y)}_{\rho jm} - \left[(\frac{M}{3})^2 g_s j(j+1) + \frac{1}{g_s}(X_0^{(x)8} - X_0^{(y)8})^2\right]\epsilon_\mu{}^\nu A^{(x,y)}_{\nu jm} + \frac{M}{3}\partial_\mu X^{(x,y)8}_{jm}$$
$$- i\frac{M}{3}(X_0^{(x)8} - X_0^{(y)8})A^{(x,y)}_{\mu jm} - \frac{i}{g_s}(X_0^{(x)8} - X_0^{(y)8})\epsilon_\mu{}^\nu\partial_\nu X^{(x,y)8}_{jm} - i\frac{M}{3}\sqrt{j(j+1)}\epsilon_\mu{}^\nu\partial_\nu X^{(x,y)}_{jjm}, \tag{6.45a}$$

$$0 = \left[-\partial^2 + \frac{1}{g_s^2}(X_0^{(x)8} - X_0^{(y)8})^2\right] X^{(x,y)}_{jjm} + i\frac{Mg_s}{3}\sqrt{j(j+1)}\partial^\mu A^{(x,y)}_{\mu jm}$$
$$- \frac{M}{3}\frac{1}{g_s}(X_0^{(x)8} - X_0^{(y)8})\sqrt{j(j+1)}X^{(x,y)8}_{jm}, \tag{6.45b}$$

$$0 = \left[-\partial^2 + (\frac{M}{3})^2 j(j+1)\right] X^{(x,y)8}_{jm} + i(X_0^{(x)8} - X_0^{(y)8})\partial^\mu A^{(x,y)}_{\mu jm} - \frac{Mg_s}{3}\epsilon^{\mu\nu}\partial_\mu A^{(x,y)}_{\nu jm}$$
$$- \frac{M}{3}\frac{1}{g_s}\sqrt{j(j+1)}(X_0^{(x)8} - X_0^{(y)8})X^{(x,y)}_{jjm}, \tag{6.45c}$$

$$0 = \left[-\partial^2 + (\frac{M}{3})^2(j+1)^2 + \frac{1}{g_s^2}(X_0^{(x)8} - X_0^{(y)8})^2\right] X^{(x,y)}_{j+1,j,m}, \tag{6.45d}$$

$$0 = \left[-\partial^2 + (\frac{M}{3})^2 j^2 + \frac{1}{g_s^2}(X_0^{(x)8} - X_0^{(y)8})^2\right] X^{(x,y)}_{j-1,j,m}, \tag{6.45e}$$

$$0 = \left[-\partial^2 + (\frac{M}{3})^2(j+\frac{1}{2})^2 + \frac{1}{g_s^2}(X_0^{(x)8} - X_0^{(y)8})^2\right] X^{(x,y)a''}_{jm}. \tag{6.45f}$$

The last three equations show that the effect of adding a constant to $X^8$ is to shift the spectrum by $\frac{1}{g_s^2}(X_0^{(x)8} - X_0^{(y)8})^2$. In particular, since the shift is by a difference of $X_0^8$'s, the spectrum for the irreducible vacuum—and also the fluctuations in blocks on the diagonal—is unaffected.

Let us confirm this result by examining the remainder of the spectrum from the gauge field and $X^8$. For now let us choose the gauge $X^{(x,y)}_{jjm} = 0$; as in Sec. VI B, the precise value of $X^{(x,y)}_{jjm}$ eventually drops out anyway. Then eq. (6.45b) becomes the constraint equation

$$\partial^\mu A^{(x,y)}_{\mu jm} = -\frac{i}{g_s^2}(X_0^{(x)8} - X_0^{(y)8})X^{(x,y)8}_{jm}, \quad j \neq 0, \tag{6.46}$$

which generalizes the Lorentz gauge condition.[13] Next take the divergence of the gauge field equation of motion (6.45a). Using the constraint (6.46) we find

$$g_s\epsilon^{\mu\nu}\partial_\mu A^{(x,y)}_{\nu jm} = -\frac{M}{3}\frac{-\partial^2 + \frac{1}{g_s^2}(X_0^{(x)8} - X_0^{(y)8})^2}{-\partial^2 + \left(\frac{M}{3}\right)^2 j(j+1) + \frac{1}{g_s^2}(X_0^{(x)8} - X_0^{(y)8})^2}X^{(x,y)8}_{jm}. \tag{6.47}$$

Eqs. (6.46) and (6.47) completely determine the gauge field in terms of $X^8$.

[13] Thus, fixing a gauge fixes a gauge!

Inserting eqs. (6.46) and (6.47) into the $X^8$ equation of motion (6.45c) gives

$$\frac{\left[-\partial^2+\left(\frac{M}{3}\right)^2 j^2+\frac{1}{g_s^2}(X_0^{(x)8}-X_0^{(y)8})^2\right]\left[-\partial^2+\left(\frac{M}{3}\right)^2(j+1)^2+\frac{1}{g_s^2}(X_0^{(x)8}-X_0^{(y)8})^2\right]}{\left[-\partial^2+\left(\frac{M}{3}\right)^2 j(j+1)^2+\frac{1}{g_s^2}(X_0^{(x)8}-X_0^{(y)8})^2\right]}X_{jm}^{(x,y)8} = 0, \quad (6.48)$$

from which we read off the same mass spectrum as $X^{(x,y)}_{j\pm1,j,m}$. In particular, the two higher derivatives in the numerator of eq. (6.48) imply that there is one extra degree of freedom in $X^8$. As we started with a two-dimensional gauge field, $X^8$ and one-third of $X^a$ (the last is $X_{\ell\ell m}$) for two bosonic degrees of freedom, this is precisely correct.

This derivation is, strictly, only true for $j\neq 0$; for $j=0$, $Y^a_{000}=0$ and so its coefficient in eq. (6.44b)—namely eq. (6.45b)—need not vanish. This is not an issue unless $N_x=N_y$. Moreover, in that case we can still fix eq. (6.46) as our gauge choice; then the first factor of eq. (6.48) cancels the denominator and we find a single massive mode. Alternatively, we can choose the gauge $A^{(x,y)}_{\tau 00}=0$; an analysis of the equations of motion again yields a single massive mode, with mass-squared $\left(\frac{M}{3}\right)^2+\frac{1}{g_s^2}(X_0^{(x)8}-X_0^{(y)8})^2$, as given by eq. (6.48), although in this case, it is convenient to assign this mass to $A^{(x,y)}_{\sigma 00}$. It is somewhat surprising that, although $X_0^{(x)8}$ can have arbitrary values, there are no massless modes associated with them; instead the would-be massless mode has been eaten by the now-massive gauge field. In fact, as we discussed in Sec. V B 1, the massless modes required by Goldstone's theorem are gauge-variant and so do not appear in the physical spectrum [39].

We should also note that the shift in the spectrum implies that $X_0^{(x)8}$ are not periodic. Indeed, although we have occasionally referred to them as $\theta$-angles, $X_0^{(x)8}$ are not periodic because there are no fundamental charges in the theory [52]. In fact, we have already commented that $X_0^{(x)8}$ is simply the location of the $x^{\text{th}}$ fuzzy sphere, in the $X^8$-direction, which is noncompact.

Incidentally, there is a sense in which $X^8$ can be compactified, with arbitrary radius. In the full M-theory solution (2.3), this corresponds to an additional compactification along the Killing vector

$$-\frac{\mu}{3}\hat{x}^8\cos\frac{\mu}{3}\hat{x}^+\frac{\partial}{\partial\hat{x}^-}+\sin\frac{\mu}{3}\hat{x}^+\frac{\partial}{\partial\hat{x}^8}-\cos\frac{\mu}{3}\hat{x}^+\frac{\partial}{\partial\hat{x}^9}. \quad (6.49)$$

In the matrix string theory, this corresponds to allowing for a *time-dependent* identification of $X^8$, $X^8\sim X^8+2\pi R_8\sin\frac{M}{3}\tau$. For example, formally, the configuration

$$\begin{aligned} g_s A_\sigma &= -wR_8\,\sigma\cos\frac{M}{3}\tau, \\ X^8 &= wR_8\,\sigma\sin\frac{M}{3}\tau, \end{aligned} \qquad w\in\mathbb{Z}, \quad (6.50)$$

is a solution to the matrix string theory equations of motion. However, an analysis following [9] shows that such a compactification of $X^8$ breaks all the linearly realized supersymmetries of the matrix string theory.

The fermions are decomposed in blocks of spinor spherical harmonics. The analysis is then virtually identical to both that for the irreducible vacuum and that for the reducible

| origin | degeneracy | mass$^2$ | range |
|---|---|---|---|
| $X^a, X^8, A_\mu$ | $4(j+1)$ | $\left(\frac{M}{3}\right)^2(j+1)^2+\frac{1}{g_s^2}\left(X_0^{(x)8}-X_0^{(y)8}\right)^2$ | $\frac{\lvert N_x-N_y\rvert}{2}\leq j\leq\frac{N_x+N_y}{2}-1$ |
| | $4j$ | $\left(\frac{M}{3}\right)^2 j^2+\frac{1}{g_s^2}\left(X_0^{(x)8}-X_0^{(y)8}\right)^2$ | $\frac{\lvert N_x-N_y\rvert}{2}\leq j\leq\frac{N_x+N_y}{2}-1$ |
| $X^{a''}$ | $4(2j+1)$ | $\left(\frac{M}{3}\right)^2(j+\frac{1}{2})^2+\frac{1}{g_s^2}\left(X_0^{(x)8}-X_0^{(y)8}\right)^2$ | $\frac{\lvert N_x-N_y\rvert}{2}\leq j\leq\frac{N_x+N_y}{2}-1$ |
| $\frac{1\!\!1+\Gamma^{12389}}{2}\Psi$ | $8(j+1)$ | $\left(\frac{M}{3}\right)^2(j+1)^2+\frac{1}{g_s^2}\left(X_0^{(x)8}-X_0^{(y)8}\right)^2$ | $\frac{\lvert N_x-N_y\rvert}{2}\leq j\leq\frac{N_x+N_y}{2}-1$ |
| | $8j$ | $\left(\frac{M}{3}\right)^2 j^2+\frac{1}{g_s^2}\left(X_0^{(x)8}-X_0^{(y)8}\right)^2$ | $\frac{\lvert N_x-N_y\rvert}{2}\leq j\leq\frac{N_x+N_y}{2}-1$ |
| $\frac{1\!\!1-\Gamma^{12389}}{2}\Psi$ | $8(2j+1)$ | $\left(\frac{M}{3}\right)^2(j+\frac{1}{2})^2+\frac{1}{g_s^2}\left(X_0^{(x)8}-X_0^{(y)8}\right)^2$ | $\frac{\lvert N_x-N_y\rvert}{2}\leq j\leq\frac{N_x+N_y}{2}-1$ |

TABLE III: The spectrum for the general vacuum. The indices $x, y$ run over all irreducible representations in the vacuum. The degeneracy of the fermions is twice that for the bosons, because we count the components of the fermions. The number of degrees of freedom is the same between bosons and fermions.

bosons; the additional contribution to the spectrum from having non-zero $X_0^{(x)8}$ comes from the term

$$\frac{1}{g_s}\Psi^{\mathsf{T}}\Gamma^8\left[X^8,\Psi\right]=\sum_{x,y}\sqrt{N_xN_y}\sum_{j\ell m}\sum_{I,\Lambda,\Sigma}(-1)^I\frac{1}{g_s}(X_0^{(x)8}-X_0^{(y)8})\psi_{j\ell m}^{(x,y)I\Lambda*}(\sigma^1)_{\Lambda\Sigma}\psi_{j\ell m}^{(x,y)I\Sigma}. \tag{6.51}$$

The $\sigma^1_{\Lambda\Sigma}$ in eq. (6.51) anticommutes with the $\sigma^3_{\Lambda\Sigma}=(-1)^{\Lambda+1}\delta_{\Lambda\Sigma}$ in eq. (6.34) and so we see that, compared to eq. (6.35), the fermion spectrum is also shifted by $\frac{1}{g_s^2}(X_0^{(x)8}-X_0^{(y)8})^2$. The spectrum has been summarized in Table III.

### E. Twisted Strings

We have now discussed the nontrivial (reducible) vacua. As we reviewed in Sec. VI A, in flat space, the only relevant degrees of freedom for the matrix string, in the $g_s\to 0$ limit, are those that live in the Cartan subalgebra [10, 11, 12]. By a gauge choice we can take the degrees of freedom to be the diagonal elements of the matrix; then, the residual gauge transformations are the elements of the Weyl group $S_N$. Or equivalently, as presented in Sec. VI A, the boundary condition can be diagonalized, by a constant gauge transformation, but then the Cartan subalgebra consists of off-diagonal matrices that are permuted by the diagonal boundary conditions. Thus the boundary conditions are[14]

$$X(\tau,\sigma+2\pi)=U_PX(\tau,\sigma)U_P^{-1},\quad Mg_s\ll 1 \tag{6.52}$$

where $P$ is the element of the Weyl group, and $U_P$ is the matrix in the regular representation. That is, the fields are periodic up to a residual gauge transformation.

However, for the *pp* wave, we have seen that all matrix elements of the fields are important, and there is no restriction to the Cartan subalgebra. An expansion around the vacuum is an expansion about $D_\mu X^i=0$ and $F_{\tau\sigma}=0$. The vacua are representations of

[14] Similar expressions hold for the gauge fields and fermions; we use "$X$" to denote a generic field here.

SU(2), and we choose the gauge in which the SU(2) representations are manifestly reduced, decomposed into the standard irreducible representations such that shortest representations are top-leftmost. Within a given SU(2) representation, one can further order the representations with respect to the value of $X_0^{(x)8}$. Since the representations are constant matrices, $D_\mu X^i = 0$ implies that $[A_\mu, X^a] = 0$. So, $A_\mu$ is pure gauge and an element of $\oplus_n \mathfrak{u}(N_n)$. So, without loss of generality, we can take $\langle A_\mu \rangle = 0$ and the boundary condition is

$$\begin{aligned} A_\mu(\tau,\sigma+2\pi) &= U_P A_\mu(\tau,\sigma) U_P^{-1}, \\ X(\tau,\sigma+2\pi) &= U_P X(\tau,\sigma) U_P^{-1}, \end{aligned} \qquad M g_s \gg 1, \tag{6.53}$$

but now, to be consistent with the form of $X^a$, $P$ is an element of the group $\prod_{n,x} \mathrm{U}(N_{n,x})$, where $N_{n,x}$ is the number of $n$-dimensional SU(2) representations in the vacuum, for which $X_0^8 = x$. Clearly, $\sum_{n,x} n N_{n,x} = N$. We therefore write

$$P \equiv \bigotimes_{n=1}^{N} \bigotimes_x P_{n,x} \in \prod_{n=1}^{N} \prod_x \mathrm{U}(N_{n,x}). \tag{6.54}$$

Turned around, generic boundary conditions permit only the trivial vacua; for example, the irreducible representation exists only in the sector for which the boundary conditions are trivial ($U$ a matrix proportional to the identity). Moreover, $U$ can be diagonalized without destroying the gauge, and so we only need to consider $U$ diagonal, proportional to the identity in blocks, and constant. Thus, in terms of the blocks, the boundary conditions are

$$\begin{aligned} X^{(x,y)i}(\tau,\sigma+2\pi) &= e^{i(\phi^{(x)}-\phi^{(y)})} X^{(x,y)i}(\tau,\sigma), \\ A_\mu^{(x,y)}(\tau,\sigma+2\pi) &= e^{i(\phi^{(x)}-\phi^{(y)})} A_\mu^{(x,y)}(\tau,\sigma), \\ \Psi^{(x,y)}(\tau,\sigma+2\pi) &= e^{i(\phi^{(x)}-\phi^{(y)})} \Psi^{(x,y)i}(\tau,\sigma). \end{aligned} \tag{6.55}$$

There is still further gauge freedom—namely, the unbroken $\prod_{n,x} \mathrm{U}(N_{n,x})$ which allows us to, for example, either set $X_{jjm}^{(x,y)}$ to a convenient value or keep $X^3$ diagonal. Setting $X_{jjm}^{(x,y)}$ to a convenient value requires an infinitesimal gauge transformation of a generic fluctuation, but rotating $X^3$ to make it diagonal generically requires a finite gauge transformation. Explicitly, achieving the gauge for which $X^3$ is diagonal requires setting $X^{(x,y)3} = (\text{diagonal})\delta_{x,y}$. It is clear that the analog of eq. (6.28) is

$$X_{jjm}^{(x,y)} = \frac{1}{m}\sqrt{\frac{(j+1)(j^2-m^2)}{2j+1}} X_{j-1,j,m}^{(x,y)} - \frac{1}{m}\sqrt{\frac{j[(j+1)^2-m^2]}{2j+1}} X_{j+1,j,m}^{(x,y)}, \quad m \neq 0, \tag{6.56}$$

and—when $j$ takes integer values—we can still further declare

$$X_{jj0}^{(x,y)} = 0. \tag{6.57}$$

This uses up all but some U(1) degrees of gauge freedom, but results only in

$$X^{(x,y)3} = \begin{cases} \sum_j \left[ \sqrt{\frac{j+1}{2j+1}} X_{j+1,j,0}^{(x,y)} - \sqrt{\frac{j}{2j+1}} X_{j-1,j,0}^{(x,y)} \right] Y_{j0}^{(N_x,N_y)}, & N_x = N_y \bmod 2, \\ 0, & \text{otherwise.} \end{cases} \tag{6.58}$$

Making $X^3$ purely diagonal would require a relation between the fields $X^{(x,y)}_{j\pm1,j,0}$, which requires the higher order terms in the gauge transformation. Thus the $X^3$-diagonal gauge is incompatible with $\langle A_\mu \rangle = 0$. So although the gauge in which $X^3$ is diagonal appears to reduce the allowed boundary conditions to $\prod_{n,x} S_{N_{n,x}}$, in fact, the full $\prod_{n,x} \mathrm{U}(N_{n,x})$ has simply been moved into Wilson lines.

Moreover, the $X^3$-diagonal gauge is calculationally inconvenient. To achieve it required a relation between $X^{(x,y)}_{j\pm1,j,0}$, and these have different masses. Thus, although $X^3$ is Hermitian, and so, of course, can be chosen to be diagonal, such a choice does not appear to be compatible with diagonalization of the action. At best we can only make each $N_x \times N_y$ block matrix in $X^3$ "diagonal". Of course, this is just a manifestation that in this gauge the gauge field is generically nontrivial, and so the linearization of the gauge field was unjustified. In other words, finding the spectrum in this gauge is hard.

Finally, one might object, with regards to our argument that we only have to consider $\prod_{n,x} \mathrm{U}(N_{n,x})$ instead of $\mathrm{U}(N)$ or $S_N$, that the ordering of vacua was also possible in flat space; on ordering the eigenvalues from smallest to largest, the residual gauge transformations appear to be much smaller than $S_N$ (in fact, appear to be nontrivial only on a set of vacua of measure zero!). However, this neglects the fact that the eigenvalues can vary continuously with $\sigma$. Therefore, although one might order the eigenvalues at $\sigma = 0$, by $\sigma = 2\pi$ they need not be ordered; the transformation $U_P \in S_N$ is then necessary to reorder them, and conversely, for any element of $U_P \in S_N$, one can always arrange for $U_P$ to be the necessary transformation. For the *pp* wave, however, the vacua are discrete and cannot vary continuously with $\sigma$. This is true even for values of $X_0^{(x)8}$, which, as we discussed in sections V B 1 and VI D, define superselection sectors. Thus, our discussion is sensible.

We conclude that the boundary conditions are given by eq. (6.55). Thus the spectrum of the diagonal blocks strings is unaffected, and the off-diagonal blocks are "twisted". That is, the moding of the $\sigma$-momentum is fractional ($n + \frac{\phi^{(x)} - \phi^{(y)}}{2\pi}$, where $n$ is integer), and generically the fields have no periodicity.

## VII. THERMODYNAMICS

In this section we make some qualitative speculations about the thermodynamics of the *pp* matrix string theory. More rigorous statements, along with calculational details, are deferred to [53].

We have seen that for $Mg_s \gg 1$, all of the matrix elements are important, and not just the diagonal ones. Thus, there are $N^2$ degrees of freedom, even for the trivial vacuum. One might think that this renders the partition function for the canonical ensemble divergent, for any temperature. Specifically, the extra factor of $N$ degrees of freedom makes the density of states appear to have, roughly, $e^N$ behaviour instead of the $e^{\sqrt{N}}$ behaviour of Cardy's formula. If true, the temperature is always above the Hagedorn temperature, and the canonical ensemble is ill-defined for $Mg_s \gg 1$.

However, the boundary condition (6.55) modifies this. Including the thermal circle in the

$\tau$-direction, the boundary conditions are

$$\begin{aligned}
X^{(x,y)i}(\tau,\sigma+2\pi) &= e^{i(\phi^{(x)}-\phi^{(y)})}X^{(x,y)i}(\tau,\sigma), & X^{(x,y)i}(\tau,\sigma+2\pi) &= e^{i(\theta^{(x)}-\theta^{(y)})}X^{(x,y)i}(\tau,\sigma), \\
A_\mu^{(x,y)}(\tau,\sigma+2\pi) &= e^{i(\phi^{(x)}-\phi^{(y)})}A_\mu^{(x,y)}(\tau,\sigma), & A_\mu^{(x,y)}(\tau,\sigma+2\pi) &= e^{i(\theta^{(x)}-\theta^{(y)})}A_\mu^{(x,y)}(\tau,\sigma), \\
\Psi^{(x,y)}(\tau,\sigma+2\pi) &= e^{i(\phi^{(x)}-\phi^{(y)})}\Psi^{(x,y)i}(\tau,\sigma), & \Psi^{(x,y)}(\tau,\sigma+2\pi) &= -e^{i(\theta^{(x)}-\theta^{(y)})}\Psi^{(x,y)i}(\tau,\sigma).
\end{aligned} \tag{7.1}$$

In principle the "$U_Q$" for the thermal circle could be an arbitrary U($N$) matrix; however, consistency at $(\sigma+2\pi,\tau+2\pi)$ requires $U_P$ and $U_Q$ to commute and therefore be simultaneously diagonalizable.

The partition function includes an integral over the boundary conditions $\phi^{(x)},\theta^{(x)}$; as explained in Sec. VI A, this is equivalent to summing over Wilson lines. It turns out that this integral over phases for the off-diagonal matrix elements suppresses their contribution to the partition function. So, in fact, the Hagedorn behaviour appears to persist, even in the presence of fuzzy spheres.

## VIII. CONCLUSIONS

In this paper we have used a matrix theory formulation to study aspects of the nonperturbative behaviour of strings in a *pp* wave background with a compact lightlike direction. The matrix string theory may be characterized by a dimensionless parameter $Mg_s$. For small $Mg_s$ one has a strongly coupled two-dimensional Yang-Mills theory whose IR limit consists of $N$ degrees of freedom and reproduces the standard perturbative string in the appropriate *pp* wave background, in a way entirely analogous to flat space.

However when $Mg_s \gg 1$ the Yang-Mills theory is essentially perturbative. The underlying string theory is at *finite* but small string coupling. Nevertheless, this regime probes essentially nonperturbative properties. In particular there are degenerate vacua of the light cone Hamiltonian corresponding to BPS states representing (multiple) fuzzy spheres of various sizes.

Perhaps the most significant result of our analysis is that the physical fluctuations around the fuzzy sphere vacua consist of $N^2$ degrees of freedom rather than $N$ degrees of freedom which appear at small $Mg_s$ limit. The mass spectra of these fluctuations have spacings which are independent of the sizes of the fuzzy spheres. Furthermore, in a gauge where all the fields are periodic along the string, the Wilson line degrees of freedom are now characterized by continuous rather than discrete parameters. Equivalently if one transforms to a description where the Wilson lines are trivial, the "matter" degrees of freedom generically do not return to their original values as one goes around the string finite number of times. Thus as we change the parameters of the theory, strings cease to be stringlike.

We expect these results to have non-trivial consequences for thermodynamics. In particular the Hagedorn behaviour should be quantitatively, though perhaps not qualitatively, modified. We hope to report on these aspects soon.

It would be interesting to construct and study matrix string theories in other types of *pp* wave backgrounds with different amounts of supersymmetry. For example, matrix string theory for *pp* waves in IIB theory should allow us to study the nonperturbative behaviour of large angular momentum states of string theory in the underlying $AdS$ background.

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## APPENDIX A: (FUZZY) SPHERICAL AND VECTOR SPHERICAL HARMONICS

### 1. Ordinary Spherical Harmonics

We start by reviewing some facts about ordinary spherical harmonics on $S^2$.

#### a. *Scalar Spherical Harmonics*

The scalar spherical harmonics, $Y_{\ell m}$, are well-understood. The property we wish to emphasize, is the one-to-one correspondence between scalar spherical harmonics and homogeneous, harmonic polynomials on $\mathbb{R}^3$. Specifically,

$$H_{\ell m}(\boldsymbol{X}) = r^{\ell} Y_{\ell m}(\theta, \phi) \tag{A.1}$$

are homogeneous, harmonic polynomials of degree $\ell$:

$$\boldsymbol{X} \cdot \boldsymbol{\partial} H_{\ell m}(\boldsymbol{X}) = \ell H_{\ell m}(\boldsymbol{X}), \qquad\qquad \boldsymbol{\partial}^2 H_{\ell m}(\boldsymbol{X}) = 0. \tag{A.2}$$

It is straightforward to see that this is equivalent to the well-known property $-\boldsymbol{\nabla}^2 Y_{\ell m}(\theta, \phi) = \ell(\ell+1) Y_{\ell m}(\theta, \phi)$. Furthermore, one can show (*e.g.* [54]) that the number of linearly independent harmonic polynomials that are homogeneous of degree $\ell$ is precisely $2\ell + 1$.

#### b. *Vector Spherical Harmonics*

Vector spherical harmonics can be defined by [55]

$$\boldsymbol{Y}_{j\ell m}(y) = \sum_{m'=-1}^{1} \left( \begin{smallmatrix} \ell & 1 \\ m-m' & m' \end{smallmatrix} \middle| \begin{smallmatrix} j \\ m \end{smallmatrix} \right) Y_{\ell, m-m'} \hat{\boldsymbol{e}}_{m'}, \tag{A.3}$$

where $\left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\middle|\begin{smallmatrix} j_3 \\ m_3 \end{smallmatrix}\right)$ are Clebsch-Gordan coefficients[15] and

$$\hat{\boldsymbol{e}}_{-1} = \frac{\hat{\boldsymbol{X}} - i\hat{\boldsymbol{Y}}}{\sqrt{2}} \qquad \hat{\boldsymbol{e}}_0 = \hat{\boldsymbol{Z}} \qquad \hat{\boldsymbol{e}}_1 = -\frac{\hat{\boldsymbol{X}} + i\hat{\boldsymbol{Y}}}{\sqrt{2}}; \tag{A.4}$$

compare eq. (A.4) with expressions for $rY_{1\beta}(\boldsymbol{X})$. As the Clebsch-Gordan coefficients vanish unless $j = \ell$ or $j = \ell \pm 1$, there are three families of vector spherical harmonics. These are often written [56]

$$\boldsymbol{V}_{\ell m} = -\boldsymbol{Y}_{\ell,\ell+1,m}, \qquad \boldsymbol{X}_{\ell m} = \boldsymbol{Y}_{\ell,\ell,m}, \qquad \boldsymbol{W}_{\ell m} = \boldsymbol{Y}_{\ell,\ell-1,m}. \tag{A.5}$$

We will use both notations here, but in the main text we use $\boldsymbol{Y}_{j\ell m}$ to avoid confusing the matrix string field with a vector spherical harmonic. They obey the reality property

$$\boldsymbol{Y}^*_{j\ell m} = (-1)^{m+\ell+1-j}\boldsymbol{Y}_{j,\ell,-m} \tag{A.6}$$

and are orthonormal

$$\int d\Omega_2 \boldsymbol{Y}_{j\ell m} \cdot \boldsymbol{Y}^*_{j'\ell' m'} = \delta_{jj'}\delta_{\ell\ell'}\delta_{mm'}. \tag{A.7}$$

Alternatively, there are essentially three "natural" ways of generating vectors out of the scalar spherical harmonics, namely

$$\boldsymbol{J}H_{\ell m}, \boldsymbol{\partial}H_{\ell m}, \hat{\boldsymbol{r}}H_{\ell m}. \tag{A.8}$$

These are related to the vector spherical harmonics via [56][16]

$$\boldsymbol{J}H_{\ell m} = \sqrt{\ell(\ell+1)}r^\ell \boldsymbol{X}_{\ell m}, \tag{A.9a}$$

$$\boldsymbol{\partial}H_{\ell m} = \sqrt{\ell(2\ell+1)}r^{\ell-1}\boldsymbol{W}_{\ell m}, \tag{A.9b}$$

$$\boldsymbol{X}H_{\ell m} = -\sqrt{\frac{\ell+1}{2\ell+1}}r^{\ell+1}\boldsymbol{V}_{\ell m} + \sqrt{\frac{\ell}{2\ell+1}}r^{\ell+1}\boldsymbol{W}_{\ell m} = -\sqrt{\frac{\ell+1}{2\ell+1}}r^{\ell+1}\boldsymbol{V}_{\ell m} + \frac{r^2}{2\ell+1}\boldsymbol{\partial}H_{\ell m}. \tag{A.9c}$$

From either (A.9) or (A.3), we can compute $\boldsymbol{V}_{\ell m}$ and $\boldsymbol{W}_{\ell m}$ explicitly. We find,

$$\begin{aligned}\boldsymbol{W}_{\ell m} = \frac{1}{\sqrt{\ell(2\ell-1)}}\Bigg\{ & -\boldsymbol{e}_{-1}\sqrt{\frac{(\ell-m)(\ell-m-1)}{2}}Y_{\ell-1,m+1} \\ & -\boldsymbol{e}_1\sqrt{\frac{(\ell+m)(\ell+m-1)}{2}}Y_{\ell-1,m-1} + \boldsymbol{e}_0\sqrt{\ell^2-m^2}Y_{\ell-1,m}\Bigg\},\end{aligned} \tag{A.10}$$

$$\begin{aligned}\boldsymbol{V}_{\ell m} = \frac{1}{\sqrt{(\ell+1)(2\ell+3)}}\Bigg\{ & \boldsymbol{e}_{-1}\sqrt{\frac{(\ell+m+1)(\ell+m+2)}{2}}Y_{\ell+1,m+1} \\ & +\boldsymbol{e}_1\sqrt{\frac{(\ell+m)(\ell+m-1)}{2}}Y_{\ell+1,m-1} + \boldsymbol{e}_0\sqrt{(\ell+1)^2-m^2}Y_{\ell+1,m}\Bigg\}.\end{aligned} \tag{A.11}$$

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[15] We use a notation in which Clebsch-Gordan coefficients could be easily confused with the more symmetric 3-$j$ symbols $\left(\begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix}\right) \equiv \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}}\left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\middle|\begin{smallmatrix} j_3 \\ -m_3 \end{smallmatrix}\right)$, but the presence or absence of the vertical bar should alleviate the confusion.

[16] We have included a radial factor for later convenience.

*c. Spinor Spherical Harmonics*

By analogy with eq. (A.3), we can define spinor spherical harmonics in terms of the scalar spherical harmonics via

$$S^{\alpha}_{j\ell m} = \left(\begin{array}{cc|c} \ell & \frac{1}{2} & j \\ m-\alpha & \alpha & m \end{array}\right) Y_{\ell,m-\alpha}, -j \leq m \leq j. \tag{A.12}$$

Here $\alpha = \pm\frac{1}{2}$ is a spinor index, and clearly $j = \ell \pm \frac{1}{2}$ and $m$ is a half-integer. Explicitly, [57]

$$\boldsymbol{S}_{\ell+\frac{1}{2},\ell,m} = \begin{pmatrix} \sqrt{\frac{\ell+m+\frac{1}{2}}{2\ell+1}} Y_{\ell,m-\frac{1}{2}} \\ \sqrt{\frac{\ell-m+\frac{1}{2}}{2\ell+1}} Y_{\ell,m+\frac{1}{2}} \end{pmatrix}, \qquad \boldsymbol{S}_{\ell-\frac{1}{2},\ell,m} = \begin{pmatrix} -\sqrt{\frac{\ell-m+\frac{1}{2}}{2\ell+1}} Y_{\ell,m-\frac{1}{2}} \\ \sqrt{\frac{\ell+m+\frac{1}{2}}{2\ell+1}} Y_{\ell,m+\frac{1}{2}} \end{pmatrix}. \tag{A.13}$$

These are clearly orthonormal

$$\int d\Omega_2 \boldsymbol{S}^{\dagger}_{j\ell m} \boldsymbol{S}_{j'\ell' m'} = \delta_{jj'}\delta_{\ell\ell'}\delta_{mm'}. \tag{A.14}$$

Also, they obey the reality condition

$$\sigma^2 \boldsymbol{S}^{*}_{j,\ell,m} = -i(-1)^{j+\ell+m} \boldsymbol{S}_{j,\ell,-m}; \tag{A.15}$$

$\sigma^2$ appears as it is the charge conjugation matrix.

### 2. Fuzzy Spherical Harmonics

*a. Scalar Spherical Harmonics*

The coordinates of the fuzzy sphere are $\boldsymbol{J}$. So, while it is difficult to generalize the spherical harmonics themselves, it is trivial to generalize $H_{\ell m}(\boldsymbol{X})$. The only subtlety is ordering, but since the commutator of two $J$'s gives a $J$, it is clear that one should use the symmetric ordering. Up to normalization, this defines the matrix scalar spherical harmonics, which we also denote $Y_{\ell m}$, hopefully without confusion.

Equivalently, in an $N$-dimensional irreducible representation of SU(2), we define $Y_{\ell,m}$ recursively as

$$Y_{\ell m} = \frac{1}{\sqrt{(\ell+m+1)(\ell-m)}} \left[J^-, Y_{\ell,m+1}\right], \quad Y_{\ell\ell} = (-1)^{\ell} \frac{\sqrt{N}}{\ell!} \sqrt{\frac{(2\ell+1)!(N-\ell-1)!}{(N+\ell)!}} (J^+)^{\ell}. \tag{A.16}$$

It is sometimes convenient—especially for multiple commutators—to use the Lie derivative to denote a commutator; we also introduce the normalization constant

$$N_\ell = (-1)^{\ell} \frac{\sqrt{N}}{\ell!} \sqrt{\frac{(2\ell+1)!(N-\ell-1)!}{(N+\ell)!}}, \tag{A.17}$$

so that

$$Y_{\ell m} = \frac{1}{\sqrt{(\ell+m+1)(\ell-m)}} \pounds_{J^-} Y_{\ell,m+1}, \qquad Y_{\ell\ell} = N_\ell (J^+)^{\ell} = (-1)^{\ell} |N_\ell| (J^+)^{\ell}. \tag{A.18}$$

The sign ensures that, written as polynomials in $\boldsymbol{X}$, $Y^{\text{fuzzy}}_{\ell m}$ "=" $H^{\text{classical}}_{\ell m}$.

It is straightforward to check that the definition (A.16) ensures the usual properties,

$$\big[J^{\pm}, Y_{\ell m}\big] = \sqrt{(\ell \mp m)(\ell \pm m + 1)}\, Y_{\ell m\pm 1}, \qquad \big[J^3, Y_{\ell m}\big] = mY_{\ell m},$$
$$[J^a, [J^a, Y_{\ell m}]] = \ell(\ell+1)Y_{\ell m}, \tag{A.19}$$
$$Y^{\dagger}_{\ell m} = (-1)^m Y_{\ell,-m}. \tag{A.20}$$

$N_\ell$ has been chosen for the normalization

$$\operatorname{Tr}\Big[Y^{\dagger}_{\ell m} Y_{\ell' m'}\Big] = N\delta_{\ell\ell'}\delta_{mm'}. \tag{A.21}$$

The consistency of this normalization as $m$ varies is easy to check, and the orthogonality follows from the commutation relations with $J^3$ (for $m$) and $(\pounds_{J^a})^2$ (for $\ell$). In particular, this implies that the $Y_{\ell m}$'s are linearly independent. However, it is worth mentioning that the normalization, $N_\ell$, depends on $N$ as well as on $\ell$ in a nontrivial way.

More generally, since the spherical harmonics are acted on by angular momentum generators, it is natural to consider the $N_1 \times N_2$ matrix whose matrix elements are

$$[Y^{(N_1,N_2)}_{jm}]_{M_1 M_2} = (N_1 N_2)^{1/4}(-1)^{\frac{N_2-1}{2}-M_2}\left(\begin{smallmatrix} \frac{N_1-1}{2} & \frac{N_2-1}{2} \\ M_1 & -M_2 \end{smallmatrix}\,\middle|\,\begin{smallmatrix} j \\ m\end{smallmatrix}\right), \qquad \begin{aligned} &\tfrac{|N_1-N_2|}{2} \le j \le \tfrac{N_1+N_2}{2}+1, \\ &-\tfrac{N_1-1}{2} \le M_1 \le \tfrac{N_1-1}{2}, \\ &-\tfrac{N_2-1}{2} \le M_2 \le \tfrac{N_2-1}{2}. \end{aligned} \tag{A.22}$$

Using the standard representation of the angular momentum generators $J^a$, and the properties of the Clebsch-Gordan coefficients, [57]

$$\left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\,\middle|\,\begin{smallmatrix} j_3 \\ m_3\end{smallmatrix}\right) = \left(\begin{smallmatrix} j_2 & j_1 \\ -m_2 & -m_1 \end{smallmatrix}\,\middle|\,\begin{smallmatrix} j_3 \\ -m_3\end{smallmatrix}\right), \tag{A.23}$$

$$\begin{aligned} \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}\left(\begin{smallmatrix} j_1 & j_2 \\ m_1\pm 1 & m_2 \end{smallmatrix}\,\middle|\,\begin{smallmatrix} j_3 \\ m_3\end{smallmatrix}\right) + \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}\left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2\pm 1 \end{smallmatrix}\,\middle|\,\begin{smallmatrix} j_3 \\ m_3\end{smallmatrix}\right) \\ = \sqrt{(j_3 \pm m_3)(j_3 \mp m_3 + 1)}\left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\,\middle|\,\begin{smallmatrix} j_3 \\ m_3\mp 1\end{smallmatrix}\right), \end{aligned} \tag{A.24}$$

$$\sum_{m_1,m_2} \left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\,\middle|\,\begin{smallmatrix} j_3 \\ m_3\end{smallmatrix}\right)\left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\,\middle|\,\begin{smallmatrix} j_3' \\ m_3'\end{smallmatrix}\right) = \delta_{j_3,j_3'}\delta_{m_3,m_3'}, \tag{A.25}$$

we easily deduce that (here, we loosely write $[J^a, \cdot]$ for the difference between left multiplication and right multiplication with representations of the appropriate dimension)

$$\Big[J^{\pm}, Y^{(N_1,N_2)}_{jm}\Big] = \sqrt{(j \mp m)(j \pm m + 1)}\, Y^{(N_1,N_2)}_{jm\pm 1}, \qquad \Big[J^3, Y^{(N_1,N_2)}_{jm}\Big] = mY^{(N_1,N_2)}_{jm}, \tag{A.26}$$
$$\Big[J^a, \Big[J^a, Y^{(N_1,N_2)}_{jm}\Big]\Big] = j(j+1)Y^{(N_1,N_2)}_{jm}, \tag{A.27}$$
$$Y^{(N_1,N_2)\dagger}_{jm} = (-1)^{m-\frac{N_1-N_2}{2}} Y^{(N_2,N_1)}_{j,-m}, \tag{A.28}$$
$$\operatorname{Tr} Y^{(N_1,N_2)\dagger}_{jm} Y^{(N_1,N_2)}_{j'm'} = \sqrt{N_1 N_2}\,\delta_{jj'}\delta_{mm'}. \tag{A.29}$$

In particular, we see that[17]

$$[Y_{\ell m}]_{MM'} = [Y_{\ell m}^{(N,N)}]_{MM'} = \sqrt{N}(-1)^{\frac{N-1}{2}-M'}\left( \begin{smallmatrix} \frac{N-1}{2} & \frac{N-1}{2} \\ M & -M' \end{smallmatrix} \middle| \begin{smallmatrix} \ell \\ m \end{smallmatrix} \right). \tag{A.30}$$

Whether to use eq. (A.16) or eq. (A.30) as the definition of the fuzzy spherical harmonics is according to taste; however, we have seen that the definition (A.30) immediately generalizes to non-square matrices (A.22). These are used for nontrivial, non-irreducible representations [5].

We can also observe that $J^a$ commutes with $\sum_{m=-\ell}^{\ell} Y_{\ell m} Y_{\ell m}^\dagger$. By Schur's lemma, this implies that

$$\sum_{m=-j}^{j} Y_{jm}^{(N_1,N_2)} Y_{jm}^{(N_1,N_2)\dagger} = (2j+1)\sqrt{\frac{N_2}{N_1}}\mathbb{1}, \tag{A.31}$$

where the proportionality constant follows upon taking the trace and using the aforementioned normalization.[18]

Note an important difference between the classical and fuzzy spherical harmonics. Because $(J^+)^N = 0$, there are only a finite number of fuzzy spherical harmonics, namely $\ell \leq N-1$, for a total of $N^2$. Thus, the fuzzy spherical harmonics form an orthogonal basis for the $N \times N$ matrices. [Alternatively, and more generally, the Clebsch-Gordan coefficients restrict $j$ as quoted in eq. (A.30); thus the number of $Y_{jm}^{(N_1,N_2)}$ is $N_1 N_2$, and they form an orthogonal basis for the $N_1 \times N_2$ matrices.] Note that $Y_{00} = \mathbb{1}$.

For computations involving interactions and higher order terms in the gauge transformations, one needs a formula for the products of spherical harmonics. Since the spherical harmonics form a basis, a product of spherical harmonics is a sum of spherical harmonics. Specifically, we have found

$$\begin{aligned} Y_{j_1 m_1}^{(N_1,N_2)} Y_{j_2 m_2}^{(N_2,N_3)} &= \sqrt{N_2}\sqrt{(2j_1+1)(2j_2+1)}(-1)^{2j_1 - \frac{N_1-N_3}{2} + N_2 - 1} \\ &\quad \times \sum_{j_3} (-1)^{j_3} \left( \begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix} \middle| \begin{smallmatrix} j_3 \\ m_1+m_2 \end{smallmatrix} \right) \left\{ \begin{smallmatrix} j_1 & j_2 & j_3 \\ \frac{N_3-1}{2} & \frac{N_1-1}{2} & \frac{N_2-1}{2} \end{smallmatrix} \right\} Y_{j_3, m_1+m_2}^{(N_1,N_3)}, \end{aligned} \tag{A.32}$$

where $\left\{ \begin{smallmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{smallmatrix} \right\}$ is the 6-$j$ symbol; see *e.g.* [54]. The Clebsch-Gordan coefficient in eq. (A.32) is easy to understand since the coefficients on the right-hand side are given by $(N_1 N_3)^{-1/2} \operatorname{Tr} Y_{j_1 m_1} Y_{j_2 m_2} Y_{j_3 m_3}^\dagger$. Inserting $\mathcal{L}_{J^\pm}$ into the trace, and using the fact that the trace of a commutator is zero, results in recursion relations for these coefficients which are identical to those [eq. (A.24)] for the Clebsch-Gordan coefficients. Thus, the $m_1, m_2$-dependence of the right-hand side is determined, up to a $j_1, j_2, j_3$-dependent normalization. The normalization is most easily found by comparing the explicit trace written using the expression (A.30) for the spherical harmonics, with *e.g.* eq. (6.2.8) of [54] which expresses a sum of three Clebsch-Gordan coefficients in terms of the 6-j symbol. This method, of course, reproduces the Clebsch-Gordan coefficient.

[17] The equality follows from linear independence and normalization, but only up to a phase. We can check the phase by examining the sign of the non-zero matrix element $[Y_{\ell\ell}]_{\frac{N-1}{2},\frac{N-1}{2}-\ell} = (-1)^\ell |N_\ell| \sqrt{\frac{(N-1)!\ell!}{(N-1-\ell)!}} \stackrel{?}{=} \sqrt{N}(-1)^\ell \left( \begin{smallmatrix} \frac{N-1}{2} & \frac{N-1}{2} \\ \frac{N_1}{2} & \ell - \frac{N-1}{2} \end{smallmatrix} \middle| \begin{smallmatrix} \ell \\ \ell \end{smallmatrix} \right)$. By definition [57], this Clebsch-Gordan coefficient is real and positive, and so we confirm that the phase is correct.

[18] This is reminiscent of the classical "sum rule" $\sum_{m=-\ell}^{\ell} |Y_{\ell m}(\theta,\phi)|^2 = \frac{2\ell+1}{4\pi}$.

*b. Vector Spherical Harmonics*

We can now define the vector spherical harmonics in parallel to the ordinary sphere. We set, with $-j \leq m \leq j$,

$$\boldsymbol{Y}^{(N_1,N_2)}_{jjm} = \boldsymbol{X}^{(N_1,N_2)}_{jm} = \frac{1}{\sqrt{j(j+1)}}\left[\boldsymbol{J}, Y^{(N_1,N_2)}_{jm}\right], \tfrac{|N_1-N_2|}{2} \leq j \leq \tfrac{N_1+N_2}{2} - 1; j \neq 0, \quad \text{(A.33a)}$$

$$\begin{aligned}\boldsymbol{Y}^{(N_1,N_2)}_{j,j-1,m} = \boldsymbol{W}^{(N_1,N_2)}_{jm} = \frac{1}{\sqrt{j(2j-1)}}\Bigg\{&\boldsymbol{e}_{-1}\sqrt{\frac{(j-m)(j-m-1)}{2}}Y^{(N_1,N_2)}_{j-1,m+1} \\ &+\boldsymbol{e}_1\sqrt{\frac{(j+m)(j+m-1)}{2}}Y^{(N_1,N_2)}_{j-1,m-1} + \boldsymbol{e}_0\sqrt{j^2-m^2}Y^{(N_1,N_2)}_{j-1,m}\Bigg\}, \\ &\tfrac{|N_1-N_2|}{2} + 1 \leq j \leq \tfrac{N_1+N_2}{2},\end{aligned} \quad \text{(A.33b)}$$

$$\begin{aligned}\boldsymbol{Y}^{(N_1,N_2)}_{j,j+1,m} = -\boldsymbol{V}^{(N_1,N_2)}_{jm} = \frac{1}{\sqrt{(j+1)(2j+3)}}\Bigg\{&\boldsymbol{e}_{-1}\sqrt{\frac{(j+m+1)(j+m+2)}{2}}Y^{(N_1,N_2)}_{j+1,m+1} \\ &+\boldsymbol{e}_1\sqrt{\frac{(j-m+1)(j-m+2)}{2}}Y^{(N_1,N_2)}_{j+1,m-1} - \boldsymbol{e}_0\sqrt{(j+1)^2-m^2}Y^{(N_1,N_2)}_{j+1,m}\Bigg\}, \\ &\tfrac{|N_1-N_2|}{2} - 1 \leq j \leq \tfrac{N_1+N_2}{2} - 2; j \geq 0.\end{aligned} \quad \text{(A.33c)}$$

The restrictions on $j$ follow from the range of $j$ in $Y^{(N_1,N_2)}_{jm}$, plus the fact that because $Y_{00} = \mathbb{1}$, it necessarily commutes with $\boldsymbol{J}$. There are therefore $[N_1N_2 - \delta_{N_1,N_2}]$ $\boldsymbol{X}_{jm}$'s, $[\min(N_1,N_2)(\max(N_1,N_2)+2)]$ $\boldsymbol{W}_{jm}$'s and $[\min(N_1,N_2)(\max(N_1,N_2)-2)+\delta_{N_1,N_2}]$ $\boldsymbol{V}_{jm}$'s, for a grand total of $3N_1N_2$, the same as the number of linearly independent triplets of $N_1 \times N_2$ matrices.

Some useful identities are (we drop the dimension-specifying superscripts where they are trivial)

$$\boldsymbol{Y}^{(N_1,N_2)\dagger}_{j\ell m} = (-1)^{j-\ell+m+1-\frac{N_1-N_2}{2}}\boldsymbol{Y}^{(N_2,N_1)}_{j,\ell,-m}, \quad \text{(A.34)}$$

$$\begin{gathered}\epsilon_{abc}\left[J^b, Y^c_{jjm}\right] = iY^a_{jjm}, \qquad \epsilon_{abc}\left[J^b, Y^c_{j-1,jm}\right] = i(j+1)Y^a_{j-1,j,m}, \\ \epsilon_{abc}\left[J^b, Y^c_{j+1,j,m}\right] = -ijY^a_{j+1,j,m},\end{gathered} \quad \text{(A.35)}$$

$$\left[J^a, X^a_{jm}\right] = \sqrt{j(j+1)}Y_{jm}, \qquad \left[J^a, W^a_{jm}\right] = 0, \qquad \left[J^a, V^a_{jm}\right] = 0, \quad \text{(A.36)}$$

$$[J^a, [J^a, \boldsymbol{Y}_{j\ell m}]] = \ell(\ell+1)\boldsymbol{Y}_{j\ell m}, \quad \text{(A.37)}$$

$$\operatorname{Tr}\boldsymbol{Y}^{(N_1,N_2)}_{j\ell m}\cdot\boldsymbol{Y}^{(N_1,N_2)\dagger}_{j'\ell'm'} = \sqrt{N_1N_2}\delta_{jj'}\delta_{\ell\ell'}\delta_{mm'}. \quad \text{(A.38)}$$

In particular, the inner product (A.38) shows that these vector spherical harmonics are linearly independent, and therefore form a complete, orthogonal basis for vectors of $N \times N$ matrices. In the main text we exclusively use the $\boldsymbol{Y}_{j\ell m}$ notation, to avoid confusing the matrix string field with a vector spherical harmonic.

It is useful to be able to write products in terms of the basis. Using eq. (A.32) and the definition (A.3), one finds that

$$\begin{aligned}Y^{(N_1,N_2)}_{j_1m_1}Y^{(N_2,N_3)a}_{j_2\ell_2m_2} &= \sqrt{N_2}\sqrt{(2j_1+1)(2\ell_2+1)(2j_2+1)}(-1)^{j_1-\ell_2-\frac{N_1-N_3}{2}+N_2} \\ &\times\sum_{j_3,\ell_3}(-1)^{j_3-\ell_3}\sqrt{2\ell_3+1}\left(\begin{smallmatrix}j_1 & j_2 \\ m_1 & m_2\end{smallmatrix}\middle|\begin{smallmatrix}j_3 \\ m_1+m_2\end{smallmatrix}\right)\left\{\begin{smallmatrix}j_1 & j_3 & j_2 \\ 1 & \ell_2 & \ell_3\end{smallmatrix}\right\}\left\{\begin{smallmatrix}j_1 & \ell_2 & \ell_3 \\ \frac{N_3-1}{2} & \frac{N_1-1}{2} & \frac{N_2-1}{2}\end{smallmatrix}\right\}Y^{(N_1,N_3)a}_{j_3,\ell_3,m_1+m_2},\end{aligned} \quad \text{(A.39)}$$

and

$$Y^{(N_1,N_2)a}_{j_2\ell_2m_2}Y^{(N_2,N_3)}_{j_1m_1} = \sqrt{N_2}\sqrt{(2j_1+1)(2\ell_2+1)(2j_2+1)}(-1)^{2j_1-\frac{N_1-N_3}{2}+N_2}$$
$$\times\sum_{j_3,\ell_3}(-1)^{j_3}\sqrt{2\ell_3+1}\left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2\end{smallmatrix}\middle|\begin{smallmatrix} j_3 \\ m_1+m_2\end{smallmatrix}\right)\left\{\begin{smallmatrix} j_1 & j_3 & j_2 \\ 1 & \ell_2 & \ell_3\end{smallmatrix}\right\}\left\{\begin{matrix} \ell_2 & j_1 & \ell_3 \\ \frac{N_3-1}{2} & \frac{N_1-1}{2} & \frac{N_2-1}{2}\end{matrix}\right\}Y^{(N_1,N_3)a}_{j_3,\ell_3,m_1+m_2}. \quad (A.40)$$

Therefore, for square matrices,

$$\left[Y_{\ell_1m_1},Y^a_{j_2\ell_2m_2}\right] = \sqrt{N}\sqrt{(2\ell_1+1)(2\ell_2+1)(2j_2+1)}(-1)^{N+1}$$
$$\times\sum_{j_3,\ell_3}(-1)^{j_3}\left[1-(-1)^{\ell_1+\ell_2+\ell_3}\right]\sqrt{2\ell_3+1}\left(\begin{smallmatrix} \ell_1 & j_2 \\ m_1 & m_2\end{smallmatrix}\middle|\begin{smallmatrix} j_3 \\ m_1+m_2\end{smallmatrix}\right)$$
$$\times\left\{\begin{smallmatrix} \ell_1 & j_3 & j_2 \\ 1 & \ell_2 & \ell_3\end{smallmatrix}\right\}\left\{\begin{matrix} \ell_1 & \ell_2 & \ell_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2}\end{matrix}\right\}Y^a_{j_3,\ell_3,m_1+m_2}; \quad (A.41)$$

thus only odd $\ell_1+\ell_2+\ell_3$ contribute to the commutator, but there is no (obvious) restriction on $j_3$. We should note that one of the 6-$j$ symbols in these expressions is obviously that from eq. (A.32) and the presence of the other is reminiscent of the classical formula [55] and could presumably be deduced from recursion relations.

*c. Spinor Spherical Harmonics*

The spinor spherical harmonics are also written in parallel to the ordinary sphere. That is,

$$\boldsymbol{S}^{(N_1,N_2)}_{j+\frac{1}{2},j,m} = \begin{pmatrix} \sqrt{\frac{j+m+\frac{1}{2}}{2j+1}}Y^{(N_1,N_2)}_{j,m-\frac{1}{2}} \\ \sqrt{\frac{j-m+\frac{1}{2}}{2j+1}}Y^{(N_1,N_2)}_{j,m+\frac{1}{2}}\end{pmatrix}, \qquad 0\le j\le N-1, -j-\frac{1}{2}\le m\le j+\frac{1}{2} \quad (A.42)$$

$$\boldsymbol{S}^{(N_1,N_2)}_{j-\frac{1}{2},j,m} = \begin{pmatrix} -\sqrt{\frac{j-m+\frac{1}{2}}{2j+1}}Y^{(N_1,N_2)}_{j,m-\frac{1}{2}} \\ \sqrt{\frac{j+m+\frac{1}{2}}{2j+1}}Y^{(N_1,N_2)}_{j,m+\frac{1}{2}}\end{pmatrix}, \qquad 0\le j\le N-1, -j+\frac{1}{2}\le m\le j-\frac{1}{2}. \quad (A.43)$$

The restriction on $j$ is obvious and the restriction on $m$ follows from the usual restriction on $m$, along with an analysis of the coefficients. We count $[N_1N_2+\min(N_1,N_2)]$ $\boldsymbol{S}^{(N_1,N_2)}_{j+\frac{1}{2},j,m}$'s and $[N_1N_2-\min(N_1,N_2)]$ $\boldsymbol{S}^{(N_1,N_2)}_{j-\frac{1}{2},j,m}$'s, for a total of $2N_1N_2$ spinor spherical harmonics, as expected.

Some useful identities are

$$\sigma^2\boldsymbol{S}^{(N_1,N_2)*}_{j+\frac{1}{2},j,m} = (-1)^{m-\frac{N_1-N_2}{2}}\boldsymbol{S}^{(N_2,N_1)}_{j+\frac{1}{2},j,-m}, \quad \sigma^2\boldsymbol{S}^{(N_1,N_2)*}_{j-\frac{1}{2},j,m} = (-1)^{m+1-\frac{N_1-N_2}{2}}\boldsymbol{S}^{(N_2,N_1)}_{j-\frac{1}{2},j,-m}, \quad (A.44)$$

$$\sigma^a\left[J^a,\boldsymbol{S}_{j+\frac{1}{2},j,m}\right] = j\boldsymbol{S}_{j+\frac{1}{2},j,m}, \qquad \sigma^a\left[J^a,\boldsymbol{S}_{j-\frac{1}{2},j,m}\right] = -(j+1)\boldsymbol{S}_{j-\frac{1}{2},j,m}, \quad (A.45)$$

$$[J^a,[J^a,\boldsymbol{S}_{j\ell m}]] = \ell(\ell+1)\boldsymbol{S}_{j\ell m}, \quad (A.46)$$

$$\mathrm{Tr}\,\boldsymbol{S}^{(N_1,N_2)*}_{j\ell m}\boldsymbol{S}^{(N_1,N_2)}_{j'\ell'm'} = \sqrt{N_1N_2}\delta_{jj'}\delta_{\ell\ell'}\delta_{mm'}, \quad (A.47)$$

where $\sigma^a$ are the Pauli matrices, and the complex conjugation includes Hermitian conjugation of the U($N$) matrices. In particular, we see that the spinor spherical harmonics are linearly independent and therefore form a basis.

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[1] R. R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, Nucl. Phys. **B625**, 70–96 (2002); [hep-th/0112044].

[2] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, Class. Quant. Grav. **19**, L87–L95 (2002); [hep-th/0201081].

[3] D. Berenstein, J. Maldacena and H. Nastase, *Strings in flat space and pp waves from $\mathcal{N}=4$ Super Yang Mills*, J. High Energy Phys. **04**, 013 (2002); [hep-th/0202021].

[4] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M theory as a matrix model: A conjecture*,, Phys. Rev. D **55**, 5112–5128 (1997); [hep-th/9610043].

[5] K. Dasgupta, M. M. Sheikh-Jabbarri and M. Van Raamsdonk, *Matrix Perturbation Theory For M-theory On a PP-Wave*, J. High Energy Phys. **05**, 056 (2002); [hep-th/0205185].

[6] K. Dasgupta, M. M. Sheikh-Jabbarri and M. Van Raamsdonk, *Protected Multiplets of M-Theory on a Plane Wave*, J. High Energy Phys. **09**, 021 (2002); [hep-th/0207050].

[7] N. Kim and J. Plefka, *On the Spectrum of PP-Wave Matrix Theory*, Nucl. Phys. **B643**, 31–48 (2002); [hep-th/0207034].

[8] N. Kim and J.-H. Park, *Superalgebra for M-theory on a pp-wave*, Phys. Rev. D **66**, 106007 (2002); [hep-th/0207061].

[9] J. Michelson, *(Twisted) Toroidal Compactification of pp-Waves*, Phys. Rev. D **66**, 066002 (2002); [hep-th/0203140]

[10] L. Motl, *Proposals on Nonperturbative Superstring Interactions*, HEP-UK-0003, hep-th/9701025.

[11] T. Banks and N. Seiberg, *Strings from Matrices*, Nucl. Phys. **B497**, 41–55 (1997); [hep-th/9702187].

[12] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix String Theory*, Nucl. Phys. **B500**, 43–61 (1997); [hep-th/9703030].

[13] G. Bonelli, *Matrix Strings in pp-wave backgrounds from deformed Super Yang-Mills Theory*, J. High Energy Phys. **08**, 022 (2002); [hep-th/0205213].

[14] K. Sugiyama and K. Yoshida, *Type IIA String and Matrix String on PP-wave*, Nucl. Phys. **B644**, 128–150 (2002); [hep-th/0208029].

[15] S. Hyun and H. Shin, *N=(4,4) Type IIA String Theory on PP-Wave Background*, J. High Energy Phys. **10**, 070 (2002); [hep-th/0208074].

[16] S. Hyun and H. Shin, *Solvable $N=(4,4)$ Type IIA String Theory in Plane-Wave Background and D-Branes*, Nucl. Phys. **B654**, 114–134 (2003); [hep-th/0210158].

[17] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A New Maximally Supersymmetric Background of IIB Superstring Theory*, J. High Energy Phys. **01**, 047 (2002); [hep-th/0110242].

[18] R. Gopakumar, *String interactions in PP-waves*, Phys. Rev. Lett. 892002171601; [hep-th/0205174].

[19] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, *PP-wave string interactions from perturbative Yang-Mills theory*, J. High Energy Phys. **07**, 017 (2002); [hep-th/0205089].

[20] R. C. Myers, *Dielectric-Branes*, J. High Energy Phys. **12**, 022 (1999); [hep-th/9910053].

[21] W. Taylor and M. Van Raamsdonk, *Multiple Dp-branes in Weak Background Fields*, Nucl. Phys. **B573**, 703–734 (2000); [hep-th/9910052].

[22] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the Giant Gravitons from Anti-de Sitter Space*, J. High Energy Phys. **06**, 008 (2000); [hep-th/0003075].

[23] S. R. Das, S. P. Trivedi and S. Vaidya, *Magnetic moments of branes and giant gravitons*, J. High Energy Phys. **10**, 037 (2000); [hep-th/0008203].

[24] D. Brecher, B. Janssen and Y. Lozano, *Dielectric Fundamental Strings in Matrix String Theory*, Nucl. Phys. **B634**, 23–50 (2002); [hep-th/0112180].

[25] B. Janssen and Y. Lozano, *On the Dielectric Effect for Gravitational Waves*, Nucl. Phys. **B643**, 399–430 (2002); [hep-th/0205254].

[26] B. Janssen and Y. Lozano, *A Microscopical Description of Giant Gravitons*, Nucl. Phys. **B658**, 281–299 (2003); [hep-th/0207199].

[27] E. A. Bergshoeff, M. de Roo and A. Sevrin, *On the supersymmetric non-abelian Born-Infeld action*, Fortschr. Phys. **49** 433–440 (2001) and Nucl. Phys. **102** Proc. Suppl. 50–55 (2001); [hep-th/0011264].

[28] S. R. Das, A. Jevicki and S. D. Mathur, *Vibration modes of giant gravitons*, Phys. Rev. D **63**, 024013 (2001); [hep-th/0009019].

[29] V. Balasubramanian, M. Berkooz, A. Naqvi and M. Strassler, *Giant Gravitons in Conformal Field Theory*, J. High Energy Phys. **04**, 034 (2002); [hep-th/0107119].

[30] S. Corley, A. Jevicki and S. Ramgoolam, *Exact Correlators of Giant Gravitons from dual N=4 SYM*, Adv. Theor. Math. Phys. **5**, 809–839 (2002); [hep-th/0111222].

[31] V. Balasubramanian, M.-x. Huang, T. S. Levi and A. Naqvi, *Open Strings from N=4 Super Yang-Mills*, J. High Energy Phys. **08**, 037 (2002); [hep-th/0204196].

[32] H. Takayanagi and T. Takayanagi, *Notes on Giant Gravitons on PP-waves*, J. High Energy Phys. **12**, 018 (2002); [hep-th/0209160].

[33] W. Taylor, *Lectures on D-branes, Gauge Theory and M(atrices)*, in *1997 Summer School in High Energy Physics and Cosmology*, edited by E. Gava *et. al.*, (World Scientific, Singapore, 1998), p. 192–271; [hep-th/9801182].

[34] D. Bak, *Supersymmetric Branes in the Matrix Model of PP Wave Background*, Phys. Rev. D **67**, 045017 (2003); [hep-th/0204033].

[35] G. Bonelli, L. Bonora and F. Nesti, *Matrix String Theory, 2D SYM Instantons and affine Toda systems*, Phys. Lett. B **435**, 303–311 (1998); [hep-th/9805071].

[36] J. D. Lykken, *Introduction to Supersymmetry*, in *Field Strings and Duality, TASI 96*, edited by C. Efthimiou and B. Green, (World Scientific, Singapore, 1997), p. 85–153.

[37] J. Michelson, *A pp-wave with 26 Supercharges*, Class. Quant. Grav. **19**, 5935–5949 (2002); [hep-th/0206204].

[38] J. A. Harvey, G. Moore and A. Strominger, *Reducing S-duality to T-duality*, Phys. Rev. D **52**, 7161–7167 (1995); [hep-th/9501022].

[39] J. B. Kogut and L. Susskind, *How quark confinement solves the $\eta \to 3\pi$ problem*, Phys. Rev. D **11**, 3594–3610 (1975).

[40] G. Grignani and G. W. Semenoff, *Thermodynamic Partition Function of Matrix Superstrings*, Nucl. Phys. **B561**, 243–272 (1999); [hep-th/9903246].

[41] G. Grignani, M. Orselli, G. W. Semenoff and D. Trancanelli, *The superstring Hagedorn temperature in a pp-wave background*, J. High Energy Phys. **06**, 006 (2003); [hep-th/0301186].

[42] S. Hyun, J.-D. Park and S.-H. Yi, *Thermodynamic behavior of IIA string theory on a pp-wave*,

SNU-TP 03-009, hep-th/0304239.

[43] Y. Sugawara, *Thermal Amplitudes in DLCQ Superstrings on PP-Waves*, Nucl. Phys. **B650**, 75–113 (2003); [hep-th/0209145].

[44] Y. Sugawara, *Thermal Partition Function of Superstring on Compactified PP-Wave*, UT-03-01, hep-th/0301035.

[45] B. R. Greene, K. Schalm and G. Shiu, *On the Hagedorn Behaviour of PP-wave Strings and N=4 SYM Theory at Finite R-Charge Density*, Nucl. Phys. **B652**, 105–126 (2003); [hep-th/0208163].

[46] L. A. Pando Zayas and D. Vaman *Strings in RR Plane Wave Background at Finite Temperature*, Phys. Rev. D **67**, 106006 (2003); [hep-th/0208066].

[47] Y. Hosotani, *Dynamics of Non-integrable Phases and Gauge Symmetry Breaking*, Ann. Phys. (NY) **190**, 233–253 (1989).

[48] A. Hashimoto, *Perturbative Dynamics of Fractional Strings on Multiply Wound D-strings*, Int. J. Mod. Phys. **A13**, 903–914 (1998); [hep-th/9610250].

[49] T. Wynter, *Gauge fields and interactions in matrix string theory*, Phys. Lett. B **415**, 349–357 (1997); [hep-th/9709029].

[50] T. Wynter, *Anomalies and large N limits in matrix string theory*, Phys. Lett. B **439**, 37–45 (1998); [hep-th/9806173].

[51] M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, *Topological Reduction of 4-D SYM to 2-D Sigma Models*, Nucl. Phys. **B448**, 166–186 (1995); [hep-th/9501096]

[52] S. Coleman, *More about the Massive Schwinger Model*, Ann. Phys. (NY) **101**, 239–267 (1976).

[53] S. R. Das, J. Michelson and A. D. Shapere, to appear.

[54] A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, (Princeton University Press, Princeton, 1974).

[55] J. Mathews, *Tensor Spherical Harmonics*, (California Institute of Technology, Pasadena, 1981).

[56] E. L. Hill, *The Theory of Vector Spherical Harmonics*, Am. J. Phys. **22**, 211–214 (1954).

[57] E. Merzbacher, *Quantum Mechanics, Second Edition*, (John Wiley & Sons, New York, 1970), p. 389–393.