

Penrose limit, Spontaneous Symmetry Breaking and Holography in PP-Wave Background ¹

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abstract

We argue that the gauge theory dual to the Type IIB string theory in ten-dimensional pp-wave background can be thought to ‘live’ on an *Euclidean* subspace spanning four of the eight transverse coordinates. We then show that light-cone time evolution of the string is identifiable as the RG flow of the gauge theory — a relation facilitating ‘holography’ of the pp-wave background. The ‘holography’ reorganizes the dual gauge theory into theories defined over Hilbert subspaces of fixed R-charge. The reorganization breaks the $SO(4,2) \times SO(6)$ symmetry to a maximal subgroup $SO(4) \times SO(4)$ spontaneously. We argue that the low-energy string modes may be regarded as Goldstone modes resulting from such symmetry breaking pattern.

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1 Overview

It was known for some time that there is a certain limit, the so-called Penrose limit, in which any spacetime which solves the Einstein's field equation is reduced to a plane-wave background [1]. Roughly speaking, the plane-wave background refers to spacetime close to a null geodesic. This assertion was extended to supergravity backgrounds [2], involving, in addition to the metric, dilaton, p-form gauge fields, and fermionic superpartners. It was also realized [3] that maximally supersymmetric pp-wave backgrounds [4, 5, 6, 7, 8] are obtainable as the Penrose limits of the $AdS_p \times S^q$ backgrounds in ten-dimensional IIB supergravity and eleven-dimensional supergravity. Remarkably, the first-quantized superstring is exactly solvable in the pp-wave background [9, 10], as the Green-Schwarz string action is quadratic in the worldsheet variables.

Recently, Berenstein, Maldacena and Nastase (BMN) [11] argued that IIB string theory on such a pp-wave background with eight transverse directions is dual to the large R-charge sector of $\mathcal{N} = 4$ supersymmetric gauge theory in the large N limit. They identified a certain class of long supermultiplet operators in the gauge theory with various string states. By summing over a class of Feynman diagrams, they claimed that anomalous contributions to the scaling dimension of these operators indeed reproduce the dispersion relations predicted by the light-cone quantization. More significantly, they proposed a concrete construction of the ten-dimensional string in terms of the four-dimensional gauge theory variables. If correct, the construction marks a significant progress beyond the AdS/CFT correspondence [15], as it provides a dictionary for associating the gauge theory operators for not just supergravity modes, but for higher string modes as well. The BMN proposal has been also extended to backgrounds with less supersymmetry [12, 13, 14].

In this paper, we substantiate aspects of the BMN proposal. Specifically, we clarify holographic relations between the bulk string states and the boundary gauge theory operators. In doing so, we emphasize the crucial role played by the choice of the gauge theory vacuum, on which both the superconformal symmetry and the R-symmetry are spontaneously broken. In section 2, we contrast the bulk-boundary relations displayed in the AdS/CFT correspondence and those in the pp-wave / gauge theory correspondence. In section 3, we illustrate this by working out the profile of the supergravity modes in the pp-wave background. In section 4, we elaborate the pattern of the aforementioned spontaneous breaking of conformal and R-symmetries. We emphasize that the dual gauge theory can be thought to be defined on a Euclidean four-dimensional space. We argue that holography relates the light-cone time in the pp-wave background to the renormalization group scale in the dual gauge theory. We show that this newly identified holography facilitates the nature of the string in terms of the dual gauge theory. In section 5, we discuss aspects of the enhanced supersymmetry in the dual gauge theory. We conclude with remarks in section 6.

2 AdS/CFT versus PP-Wave/Yang-Mills

In the AdS/CFT correspondence, the dual conformal field theory resides on the boundary of the AdS-space [16, 17], and the radial direction of the AdS-space plays the role of scale of the boundary theory [18, 19, 20, 21, 22, 23]. Consider the global coordinates in $AdS_{d+1} \times S^{\bar{d}+1}$ space with metric

$$ds^2 = R^2 \left[- (1 + \bar{r}^2) d\bar{t}^2 + \frac{d\bar{r}^2}{(1+\bar{r}^2)} + \bar{r}^2 d\Omega_{d-1}^2 + (1 - \bar{\rho}^2) d\bar{\theta}^2 + \frac{d\bar{\rho}^2}{(1-\bar{\rho}^2)} + \bar{\rho}^2 d\Omega_{\bar{d}-1}^2 \right], \quad (1)$$

where the first and the second parts express the AdS_{d+1} and the $S^{\bar{d}+1}$ subspace, respectively. A bulk single particle state of a given mass and spin, satisfying the classical field equation, is specified by several ‘‘momenta’’: angular momentum quantum numbers $(\ell, m_1, \dots, m_{d-2})$ for the S^{d-1} part in AdS_{d+1} space and $(\bar{\ell}, \bar{m}_1, \dots, \bar{m}_{\bar{d}})$ for the $S^{\bar{d}+1}$, respectively, and a principal quantum number n for the remaining radial coordinate \bar{r} in AdS_{d+1} space. The bulk energy ω is then given in terms of these quantum numbers by a dispersion relation. In the dual conformal field theory, we have composite operators $\mathcal{O}_{\{\bar{\ell}, \bar{m}\}}(\bar{t}, \phi_1, \dots, \phi_{d-1})$, where $(\bar{t}, \phi_1, \dots, \phi_{d-1})$ denote coordinates of $\mathbf{R} \times S^{d-1}$. These operators are decomposable into Fourier modes with a given energy ω and S^{d-1} spherical harmonics (l, m_1, \dots, m_{d-2}) . The remaining quantum numbers $(\bar{\ell}, \bar{m}_1, \dots, \bar{m}_{\bar{d}})$ are encoded in the structure of the operators. For instance, in $AdS_5 \times S^5$, the S^5 quantum numbers are encoded in the manner the six Higgs fields Φ^1, \dots, Φ^6 of the $\mathcal{N} = 4$ gauge theory appear in the operator. As a concrete example, a bulk dilaton mode in $AdS_5 \times S^5$ with S^5 angular momentum $(\bar{\ell}, \bar{m}_1, \dots, \bar{m}_{\bar{d}})$ is described by a set of chiral primary operators whose bosonic component is given by

$$\text{Tr} \left[F_{mn} F^{mn} \Phi^{(i_1} \dots \Phi^{i_{\bar{\ell}})} \right] (\bar{t}, \phi_i, \dots, \phi_{d-1}), \quad (2)$$

in which the indices $i_1 \dots i_{\bar{\ell}}$ are decomposed into irreducible representations of $\text{SO}(\bar{d})$. F_{mn} denotes the gauge field strength. Likewise, chiral primary operators

$$\text{Tr} \left[\Phi^{(i_1} \dots \Phi^{i_{\bar{\ell}})} \right] (\bar{t}, \phi_i, \dots, \phi_{d-1}) \quad (3)$$

describe modes of a linear combination of the four-form self-dual potential and the trace of the longitudinal graviton in ten-dimensions.

One can obtain the Penrose limit of Eq.(1) along a generic null geodesic as follows. Boost along the two isometry directions:

$$\begin{aligned} t &= \cosh \alpha \bar{t} - \sinh \alpha \bar{\theta} \\ \theta &= -\sinh \alpha \bar{t} + \cosh \alpha \bar{\theta}, \end{aligned} \quad (4)$$

and rescale two ‘radial’ and light-cone coordinates:

$$r = R\bar{r}, \quad \rho = R\bar{\rho} \quad \text{and} \quad x^\pm = \frac{R}{\sqrt{2}}(\theta \pm t). \quad (5)$$

Then, take the limit

$$R \rightarrow \infty \quad \text{and} \quad \alpha \rightarrow \infty, \quad (6)$$

while holding

$$x^\pm, r, \rho = \text{fixed} \quad \text{and} \quad \frac{e^\alpha}{\sqrt{2}R} \equiv \mu = \text{fixed}. \quad (7)$$

The resulting spacetime is then reduced to

$$\begin{aligned} ds^2 &= 2dx^+ dx^- - \mu^2(r^2 + \rho^2)(dx^+)^2 + dr^2 + r^2 d\Omega_{d-1}^2 + d\rho^2 + \rho^2 d\Omega_{\bar{d}-1}^2 \\ &= 2dx^+ dx^- - \mu^2(\mathbf{x}^2 + \mathbf{y}^2)(dx^+)^2 + d\mathbf{x} \cdot d\mathbf{x} + d\mathbf{y} \cdot d\mathbf{y}, \end{aligned} \quad (8)$$

where we have defined transverse coordinates \mathbf{x}, \mathbf{y} which describe the \mathbf{R}^d made out of r and S^{d-1} , and $\mathbf{R}^{\bar{d}}$ made out of ρ and $S^{\bar{d}-1}$, respectively. Even though the metric exhibits $\text{SO}(d+\bar{d})$ isometry, it turns out the RR five-form field strengths break it to $\text{SO}(d) \times \text{SO}(\bar{d})$.

A novel feature of the pp-wave background is that the single particle bulk states are now given in terms of certain harmonic oscillator quantum numbers $(n_1 \cdots n_d)$ and $(m_1 \cdots m_{\bar{d}})$ for a given value of the momentum conjugate to x^- , which we call $p_- \equiv 2p^+$. The light-cone energy $p_+ \equiv 2p^-$ is then given by a dispersion relation. We will illustrate this later in this section. According to the BMN proposal, with these harmonic oscillator quantum numbers, the chiral primary operators dual to a single-particle bulk state with the lowest light-cone energy, which turns out to be a linear combination of the self-dual RR four-form potential and trace of the graviton, take the form of ‘Z-string’:

$$\sum \text{Tr} [Z \cdots ZZ (D_{i_1} Z) ZZ \cdots ZZ \Phi^{a_1} ZZ \cdots ZZ (D_{i_2} Z) ZZ \cdots ZZ \Phi^{a_2} ZZ \cdots]. \quad (9)$$

Here, along a string of J factors of $Z \equiv (\Phi^5 + i\Phi^6)$, one distributes n_i insertions of $(D_i Z)$ and m_a insertions of ‘transverse’ Higgs fields Φ^a ($a = 7, \cdots, 10$). Then, Φ^5 and Φ^6 are the two remaining, ‘longitudinal’ Higgs fields in the $\mathcal{N} = 4$ gauge theory. The sum is over all distinct (up to cyclic permutation) locations of the operators $D_i Z$ and Φ^a in the string of Z ’s. The light cone momentum $p_- = -i\frac{\partial}{\partial x^-}$ and the light cone energy $p_+ = i\frac{\partial}{\partial x^+}$ are related to the dimension Δ and J of the operator Eq.(9) by the relations

$$\begin{aligned} p_- &= \frac{1}{2\mu R^2}(\Delta + J) = \frac{1}{2\mu R^2} \left(2J + \sum_{i=1}^d n_i + \sum_{a=1}^{\bar{d}} m_a \right) \\ p_+ &= \mu(\Delta - J) = \mu \left(\sum_{i=1}^d n_i + \sum_{a=1}^{\bar{d}} m_a \right). \end{aligned} \quad (10)$$

On the other hand, the coordinate transformations, Eqs.(4-7), imply

$$\begin{aligned} p_- &= \frac{1}{2\mu R^2} \left(i \frac{\partial}{\partial \bar{t}} - i \frac{\partial}{\partial \bar{\theta}} \right) \\ p_+ &= \mu \left(i \frac{\partial}{\partial \bar{t}} + i \frac{\partial}{\partial \bar{\theta}} \right). \end{aligned} \tag{11}$$

As $J = -i \frac{\partial}{\partial \bar{\theta}}$, Eq.(11) is consistent with Eq.(10) since $\Delta = i \frac{\partial}{\partial \bar{t}}$ ².

For other single-particle supergravity states such as the dilaton, one needs to insert an operator $F_{mn} F^{mn}$ inside the Z -string. For higher string-mode states, each term in the sum of Eq.(9) should be weighted by a phase-factor, which depends on the locations of various operators in the Z -string.

Note that *all* the bulk quantum numbers appear in the structure of the dual gauge theory operators, Eq.(9). This is in sharp contrast to the AdS/CFT correspondence, where only *half* of the quantum numbers reside in the operator structure. There, the remaining *half* were encoded as dependence of the operator on coordinates of the four-dimensional spacetime, the boundary of AdS_5 , on which the dual gauge theory resides. Evidently, the operators in Eq.(9) cannot be regarded as functions of the coordinates of the four-dimensional spacetime, as that would result in more quantum numbers than needed for specifying a given single particle supergravity/string state in the bulk.

In subsequent sections, we will argue that the gauge theory dual to the pp-wave background Eq.(9) can be thought of as “living” in a Euclidean four-dimensional space \mathbf{R}^4 spanned by the \mathbf{x} coordinates. The precise form of the dual gauge theory operators are then given in terms of the Hermite transformation of local operators defined on \mathbf{R}^4 . The fact that this space has to be Euclidean, rather than Minkowski spanned by light-cone coordinates and part of \mathbf{R}^4 , follows from the correspondence between the operators Eq.(9) and the one-particle states of the bulk supergravity/string theory. The latter states are described in terms of $(d + \bar{d})$ set of simple harmonic oscillator operators with indices in a Euclidean space. For a string theory defined in the bulk, these oscillators also carry a label for the level number [9]. As we will see, this observation leads naturally to an interpretation of x^+ as the holographic bulk coordinate in the Penrose limit, so that evolution in x^+ in the bulk generates scale transformation in the dual gauge theory.

We will argue that in the gauge theory selecting a sector with fixed SO(2) charge J is tantamount to a spontaneous breaking of the conformal group SO(4,2) to SO(4) and the R-symmetry group to SO(4) as well. The low-energy fluctuations are then the Goldstone modes of the broken symmetries. The representation of these operators in terms of Hermite transforms then follows automatically.

²Here we assume that (9) means that the operator is evaluated at the origin of a \mathbf{R}^4

3 Supergravity Modes in pp-Wave Backgrounds

Consider a minimally coupled, massless scalar field D whose field equation is given in the global coordinates Eq.(8) as

$$\left[2\partial_+\partial_+ + \mu^2(\mathbf{x}^2 + \mathbf{y}^2)\partial_-^2 + \sum_{i=1}^d \partial_{\mathbf{x}^i}^2 + \sum_{a=1}^{\bar{d}} \partial_{\mathbf{y}^a}^2 \right] D(x^+, x^-, \mathbf{x}, \mathbf{y}) = 0. \quad (12)$$

The normal-modes with $p_- > 0$ are given by

$$\begin{aligned} D_{p_+, p_-, \mathbf{n}, \mathbf{m}}(x^+, x^-, \mathbf{x}, \mathbf{y}) &= e^{-\frac{1}{2}\mu p_-(\mathbf{x}^2 + \mathbf{y}^2)} \prod_{i=1}^d H_{n_i}(\sqrt{\mu p_-} \mathbf{x}^i) \prod_{a=1}^{\bar{d}} H_{m_a}(\sqrt{\mu p_-} \mathbf{y}^a) \\ &\times \exp(ip_- x^- + p_+ x^+), \end{aligned} \quad (13)$$

where $H_{\mathbf{n}}(\mathbf{x}), H_{\mathbf{m}}(\mathbf{y})$ denote the Hermite polynomials³. The states of this scalar field theory are therefore created from the bulk Fock-space vacuum by creation operators $\mathbf{a}^\dagger(\mathbf{n}_i, \mathbf{m}_a, p^+)$ in the light-cone quantization. In a first-quantized theory of particles in this background, states are created by creation operators $\mathbf{c}^{i\dagger}, \mathbf{c}^{a\dagger}$:

$$\begin{aligned} \mathbf{c}^i &= (\mathbf{p}_i + i\mathbf{x}^i)/\sqrt{2} & \text{and} & & \mathbf{c}^{i\dagger} &= (\mathbf{p}_i - i\mathbf{x}^i)/\sqrt{2}, \\ \mathbf{c}^a &= (\mathbf{p}_a + i\mathbf{y}^a)/\sqrt{2} & \text{and} & & \mathbf{c}^{a\dagger} &= (\mathbf{p}_a - i\mathbf{y}^a)/\sqrt{2}, \end{aligned} \quad (14)$$

where the indices i, a refer to the transverse directions along $\mathbf{R}^d \times \mathbf{R}^{\bar{d}}$, spanning a $(d + \bar{d})$ -dimensional transverse space. The bulk dispersion relation is then given by

$$2p^- = p_+ = \mu \left(\sum_{i=1}^d n_i + \sum_{a=1}^{\bar{d}} m_a + \frac{1}{2}(d + \bar{d}) \right). \quad (15)$$

Note that the value of p_+ is independent of the value of p_- . This is because the supergravity modes are massless. For massive, string oscillation fields, the dispersion relation depends explicitly on p_- . The sum over zero-point energies is standard. We will see that, for the explicit example of ten-dimensional pp-wave background, this zero-point energy is precisely what is required for precise correspondence with appropriate operators in the dual gauge theory. We note, for future reference, that the dispersion relation for the mode which is a linear combination of the four form RR potential and the trace of the longitudinal graviton does not contain this zero-point fluctuation.

As we will elaborate more, the dual gauge theory has operators which are Hermite transforms of local operators defined on the \mathbf{R}^4 spanned by \mathbf{x} .

$$\mathcal{O}[\mathbf{n}] = \text{H.T.} [\mathcal{O}], \quad (16)$$

³Note that, due to the harmonic potential provided by the second term in Eq.(12), there is no real distinction between normalizable and non-normalizable modes for $p_- > 0$. The modes with $p_- = 0$ are not \mathcal{L}_2 normalizable, but are δ -function normalizable.

where the Hermite transform of a generic operator $\mathcal{O}(\mathbf{x})$ on \mathbf{R}^d is defined as

$$\text{H.T.}[\mathcal{O}] = \frac{1}{\mathcal{N}} \int d\mu[\mathbf{x}] \prod_{i=1}^d H_{n_i}(\sqrt{\mu p_-} x^i) \mathcal{O}(\mathbf{x}),$$

where \mathcal{N} is a normalization factor, and the measure is given by

$$d\mu[\mathbf{x}] := d^d \mathbf{x} e^{-\frac{1}{2}\mu p_- \mathbf{x}^2}.$$

Using the standard recursion relation for Hermite polynomials, the Hermite transform can be reduced to expressions involving derivatives acting on \mathcal{O} and no factors of the Hermite polynomials.

It may be useful to formally define operators on a $\mathbf{R}^d \times \mathbf{R}^{\bar{d}}$ by introducing a set of fiducial coordinates \mathbf{y} for the $\mathbf{R}^{\bar{d}}$. Performing the Hermite transform on this eight-dimensional space,

$$\text{H.T.}[\mathcal{O}] = \frac{1}{\mathcal{N}} \int d\mu[\mathbf{x}] d\mu[\mathbf{y}] \prod_{i=1}^d H_{n_i}(\sqrt{\mu p_-} x^i) \prod_{a=1}^{\bar{d}} H_{m_a}(\sqrt{\mu p_-} y^a) \mathcal{O}(\mathbf{x}, \mathbf{y}).$$

Using recursion relations obeyed by the Hermite polynomials, one can then express the Hermite transform in terms of derivatives with respect to \mathbf{y} , which in turn become commutators with Higgs fields inside the operator.

3.1 Dilaton

In the ten-dimensional pp-wave background, the dilaton field equation take the same form as Eq.(12). Thus, the light-cone energy spectrum of the dilaton state is given by

$$E_{\text{dilaton}} = \mu \left(\sum_{i=1}^4 n_i + \sum_{a=1}^4 m_a + \frac{1}{2}(4+4) \right). \quad (17)$$

According to the BMN proposal, the light-cone energy (measured in unit of μ) ought to match with $(\Delta - J)$ of a gauge theory operator dual to the dilaton. A single insertion of $F_{mn} F^{mn}$, which carries $\Delta = 4$ and $J = 0$, inside the Z -string in Eq.(16) is precisely what we need to match the zero-point light-cone energy. Interestingly, in providing the requisite zero-point energy $(4+4)/2 = 4$, four-dimensionality of the internal space $\mathbf{R}^{\bar{4}}$ has played a crucial role.

The single particle dilaton ground-state corresponds to $\mathbf{n} = \mathbf{m} = 0$. For the states with higher energy, using the recursion relation of Hermite polynomials, we deduce that the corresponding operators are precisely insertions of the ‘transverse’ Higgs fields and covariant derivatives, viz. a set of operators of the form

$$\text{Tr} [F_{mn} F^{mn} Z Z \cdots Z Z \Phi^{a_1} Z Z \cdots Z Z (D_i Z) Z Z \cdots Z Z \Phi^{a_2} Z Z \cdots].$$

3.2 Longitudinal Graviton and Four-Form Potential

The pp-wave background is supported by a homogeneous RR 5-form field strength

$$F_{+1234} = +F_{+5678} = \mu,$$

giving rise to Eq.(1) through the Einstein's field equation. As such, degrees of freedom of the graviton and the four-form RR potentials would mix each other. More precisely, expanding Type IIB supergravity field equations of the metric and the RR four-form potential to linear order fluctuations, $h_{\mu\nu}$, $c_{\mu\nu\alpha\beta}$, and taking the light-cone gauge $h_{\mu-} = 0$, $c_{\mu\nu\alpha-} = 0$, we find that the mixing takes place between the traces of the graviton and the scalars of the RR four-form potential. We thus denote these modes as

$$\begin{aligned} h &:= h_{ij}\delta^{ij} & \text{and} & & c &:= \frac{1}{4!}\epsilon^{ijkl}c_{ijkl} \\ \bar{h} &:= h_{mn}\delta^{mn} & \text{and} & & \bar{c} &:= \frac{1}{4!}\epsilon^{mnpq}c_{mnpq}. \end{aligned}$$

These fields are singlets of the two $\text{SO}(4)$'s on $\mathbf{R}^4 \times \mathbf{R}^{\bar{4}}$, respectively.

Then, the linearized field equations exhibiting the mode mixing are given by

$$\begin{aligned} \Delta_{\text{L}}h - 16\mu\partial_-c &= 0 \\ \nabla^2c - 2\mu\partial_-h &= 0, \end{aligned}$$

where Δ_{L} stands for the Lichnerowicz operator for the spin-2 graviton. Utilizing the fact that $(\Lambda_{\text{L}}h)_{ij} = -\frac{1}{2}\nabla^2h_{ij}$ and diagonalizing the two coupled equations, we obtain scalar-mode field equations

$$\left[\nabla^2 - 8i\mu\partial_-\right](h + 4ic) = 0 \tag{18}$$

and its complex conjugated equation for $(h - 4ic)$. Exactly the same set of equations hold for \bar{h} and \bar{c} as well.

The field equation Eq.(18) is soluble exactly as in the dilaton case. We find that the light-cone spectrum of the $(h + 4ic)$ 'complex scalar' field is given by

$$\begin{aligned} E_{Z\text{-scalar}} &= \mu \left(\sum_{i=1}^4 n_i + \sum_{a=1}^4 m_a + \frac{1}{2}(4+4) \right) - 4\mu \\ &= \mu \left(\sum_{i=1}^4 n_i + \sum_{a=1}^4 m_a \right) \geq 0. \end{aligned}$$

On the right-hand side of the first expression, the first and the second terms are contributions from ∇^2 and $-8i\mu\partial_-$, respectively. Evidently, the zero-point energy arising from fluctuations along the eight transverse directions is cancelled precisely by the classical contribution -4μ to

the light-cone energy. Hence, along with the second set of complex ‘scalar’ field $(\bar{h} + 4i\bar{c})$, we conclude that there are two bulk ‘scalar’ modes yielding the minimum of the light-cone energy to be zero. These bulk ‘scalar’ fields are then identified with the dual gauge theory operators

$$\text{Tr} [ZZ \cdots ZZ \cdots ZZ],$$

viz. the Z-string, first introduced by BMN.

In contrast, the complex-conjugate ‘scalar’ fields $(h - 4ic)$ and $(\bar{h} - 4i\bar{c})$ are subject to the classical contribution $+2\mu$ to the light-cone energy. It implies that the minimum of the light-cone energy is $+8\mu$, instead of 0, rendering the corresponding dual gauge theory operator involving eight powers of Φ^a ’s distributed along the Z-string.

4 Penrose Contraction, Spontaneous Symmetry Breaking & Euclidean Dual Gauge Theory

We now turn to the dual $\mathcal{N} = 4$ supersymmetric gauge theory. This theory is invariant under $\text{SO}(4,2) \otimes \text{SO}(6)$, where $\text{SO}(6)$ refers to the internal R-symmetry. Denote the generators of $\text{SO}(4,2)$ as J_{AB} with $A, B = 1, \dots, 6$, where 5, 6 are the directions with negative signature, and those of $\text{SO}(6)$ as J_{UV} with $U, V = 7, \dots, 12$. In terms of J_{AB} , the generators of the conformal group are

$$J_{ij}, \quad P_i = J_{5i} + J_{6i}, \quad K_i = J_{5i} - J_{6i}, \quad D_1 = J_{56} \quad (19)$$

with $i, j = 1, \dots, 4$. The same can be done for the generators of $\text{SO}(6)$, and we define J_{ab}, P_a, K_a, D_2 accordingly, where $a, b = 7, \dots, 10$ and $D_2 = J_{11,12}$.

Let us now assume that there exists a vacuum state, on which the $\text{SO}(4,2) \otimes \text{SO}(6)$ is broken spontaneously to $\text{SO}(3,1) \otimes \text{SO}(4)$, viz. standard symmetry breaking pattern preserving Lorentz plus ‘transverse’ internal symmetries. The number of generators of broken symmetries is eighteen, viz. nine nonlinearly realized symmetries for each product group. The generators of the broken symmetries are $P_i, K_i, P_a, K_a, D_1, D_2$ and the generators of the unbroken symmetries are the J_{ij} for the Lorentz group $\text{SO}(3,1)$ and the J_{ab} for the internal symmetry group $\text{SO}(4)$. One easily finds that generators of the broken symmetries satisfy the following commutation relations

$$[P_i, K_i] = D_1 \quad \text{and} \quad [P_a, K_a] = D_2.$$

These commutation relations are very suggestive. If one were to put aside the fact that D_1 and D_2 do not commute with the P ’s and K ’s, one may try to interpret the previous commutation relations as defining two Heisenberg algebras $\mathfrak{h}(4) \oplus \mathfrak{h}(4)$, each one with eight generators, for which the D_1 and D_2 as the two central extensions. This interpretation, as it stands, is not

viable if one just considers the standard symmetry breaking pattern $SO(4,2) \otimes SO(6)$ to $SO(3,1) \otimes SO(4)$: D_1 and D_2 are not central terms and we can not organize the generators of the broken symmetries in terms of two Heisenberg algebras. It is precisely at this point where the existence of supergravity/string duals and concept of the Penrose contraction can help us to define a different pattern of the symmetry breaking.

4.1 Penrose Contraction

As is well known, the symmetry algebra $SO(4,2) \otimes SO(6)$ of $\mathcal{N} = 4$ gauge theory are realizable as isometries of the $AdS_5 \times S^5$ spacetime. The Penrose limit recapitulated in section 2 preserves the total number of Killing vectors but can change their algebraic relations. In particular, if we perform the Penrose limit on a generic light geodesic in $AdS_5 \times S^5$ the Killing vectors define the algebra $[\mathfrak{h}(4) \oplus \mathfrak{h}(4)] \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(4)$, where the bracket is to emphasize the fact that two Heisenberg algebras share the same central extension. The extra Killing vector defines an outer-automorphism of the Heisenberg algebras. We interpret the Penrose limit as defining a sort of spontaneous symmetry breaking from $SO(4,2) \otimes SO(6)$ to $SO(4) \otimes SO(4)$ with the eighteen generators of the broken symmetries defining the two Heisenberg algebras $\mathfrak{h}(4)$'s and the outer-automorphism.

As the simplest illustration, consider $AdS_2 \times S^2$, relevant for the near-horizon geometry of four-dimensional BPS black holes. In this case, the symmetry group is $SO(1,2) \otimes SO(3)$ with six generators that we will denote $P_1, K_1, P_2, K_2, D_1, D_2$. They satisfy, in particular, $[P_1, K_1] = D_1$ and $[P_2, K_2] = D_2$ ⁴. In the Penrose limit, P_i, K_i become the generators of two Heisenberg algebras and the D_i 's produce the common central term and the outer automorphism. In fact, denoting the Penrose scaling by Ω , we get $D_i(\Omega) = d^{i,0}\Omega^{-2} + d^{i,1} + d^{i,2}\Omega^2 + \dots$ with $d^{1,0} = d^{2,0}$. The central term is defined by $d^{i,0}$ and the outer-automorphism by $(d^{i,1} - d^{i,2})$. Expansion of $D^i(\Omega)$ is then interpretable as a perturbative expansion in powers of the Penrose scaling parameter, Ω .

An important aspect of the Penrose limit in the case of $AdS_5 \times S^5$ considered by BMN is that the unbroken symmetry is $SO(4) \otimes SO(4)$. In other words, if we want to use the Penrose contraction as a pattern of the symmetry breaking for the dual $\mathcal{N} = 4$ gauge theory, we should assume that the vacuum is invariant not under the Lorentz group but under a rotation group in a four-dimensional Euclidean space. Insight to this possibility can be gained by recalling aspects of spontaneous conformal symmetry breaking, studied thoroughly some time ago [24]. The idea was to assume an underlying theory invariant under the conformal group and, after spontaneous conformal symmetry breaking, to study the low-energy physics of the corresponding Goldstone bosons. The first peculiar aspect of the spontaneous conformal symmetry breaking, $SO(4,2)$

⁴ In [6] the generators P_i, K_i, D_i correspond respectively to the Killing vectors E_i, E_i^*, ϵ_i .

to $SO(1,3)$, is that the generators of translations are part of the broken symmetries. Being so, only the generators of special conformal transformations and dilatations were considered [24] as real Goldstone bosons. A consequence of this is that these Goldstone bosons, contrary to the standard case, are not massless as the broken symmetries do not commute with the Hamiltonian, viz. with translations in time. In the Penrose contraction, we are facing a similar problem. If we consider $\mathcal{N} = 4$ gauge theory and the standard spontaneous breaking pattern to $SO(3,1) \otimes SO(4)$, we are including among the broken symmetry generators, the translation generators in physical time as well as the spatial translation generators. If we try to understand this breaking in the old-fashioned approach [24], we need to organize the eighteen broken symmetries into a set of nine massless Goldstone bosons, corresponding to the spontaneous breakdown of the internal symmetry $SO(6)$, five massive Goldstone bosons corresponding to the special conformal transformations and dilatations that do not commute with the Hamiltonian and four translations. This is certainly not the picture we get if we use Penrose contraction. In the Penrose contraction, we organize the eighteen broken symmetries into a Heisenberg algebra $\mathfrak{h}(8)$ and an outer-automorphism. What now remains is a concrete interpretation of the Heisenberg algebra $\mathfrak{h}(8)$ and the outer-automorphism entirely within the dual gauge theory formulation.

4.2 Dual Gauge Theory is Euclidean

Let Φ_i $i = 5 \dots 10$ be the Higgs fields of $\mathcal{N} = 4$ super Yang Mills theory. Following BMN, define the field $Z = (\Phi_5 + i\Phi_6)$, and denote by J the $SO(2)$ R-charge corresponding to rotations in the internal (5,6)-plane. Consider decomposing the gauge theory Hilbert space into infinite towers of Hilbert subspaces of definite J quantum number. Evidently, on each subspace, Fock-space “ground state” breaks the internal $SO(6)$ spontaneously to $SO(4)$. We denote the Fock-space vacuum with R-charge equal to J as $|0\rangle_J$. We will be interested in the Hilbert space of quantum fluctuations around this vacua. The first thing to be done is characterizing the state $|0\rangle_J$. The simplest way to define this state is, in radial quantization,

$$\text{Tr} \left(Z^J \right) (\mathbf{x} = 0) |0\rangle_{\text{YM}},$$

where $|0\rangle_{\text{YM}}$ refers to the perturbative vacuum of the dual $\mathcal{N} = 4$ gauge theory. The operators should be considered as functions of the coordinates of a four dimensional Euclidean space, \mathbf{R}^4 , and is *not* related a priori to Euclideanized $\mathcal{N} = 4$ super Yang-Mills theory defined on $\mathbf{R}^3 \times \mathbf{R}_\tau$ after the Wick rotation. On \mathbf{R}^4 , a *local* operator $\text{Tr} \left(Z^J \right) (\mathbf{x})$ is expandable in a complete basis of the Hermite polynomials

$$\text{Tr} \left(Z^J \right) (\mathbf{x}) = \sum_{\{\mathbf{n}\}} c_{\mathbf{n}} \prod_{i=1}^4 \left(e^{-\Lambda^2 x_i^2} H_{n_i}(\Lambda x_i) \right), \quad (20)$$

where Λ is a scale defined within the dual gauge theory, which will be determined later. Thus, we can write

$$|0\rangle_J = c_{\mathbf{n}=0} |0\rangle_{\text{YM}}. \quad (21)$$

The Hilbert space of quantum fluctuations is generated by states $|\mathbf{n}\rangle_J = c_{\mathbf{n}} |0\rangle_{\text{YM}}$. For instance, we get:

$$\sum_{\ell} \text{Tr} \left(Z^{\ell} (D_i Z) Z^{J-\ell} \right) (0) |0\rangle_{\text{YM}} = |\mathbf{n}_i = 1\rangle_J.$$

One can define creation and annihilation operators b_0^i and $b_0^{i\dagger}$ obeying the canonical commutation relation $[b_0^i, b_0^{j\dagger}] = \delta^{i,j}$ such that

$$b_0^{i\dagger} |0\rangle_J = |\mathbf{n}_i = 1\rangle_J. \quad (22)$$

These operators generate the Heisenberg algebra $\mathfrak{h}(4)$. As we are working in $\mathcal{N} = 4$ gauge theory, we can also consider fluctuations with respect to the internal directions, namely,

$$\sum_{\ell} \text{Tr} \left(Z^{\ell} \Phi_a Z Z^{J-\ell} \right) (0) |0\rangle_{\text{YM}} \quad (23)$$

with $a = 7, \dots, 10$. In the large- J limit, one can represent these states in terms of the same type of creation and annihilation operators as before, viz. $|\mathbf{n}_a = 1\rangle_J = b_0^{a\dagger} |0\rangle_J$. Both b_0^i with $i = 1, \dots, 4$ and b_0^a with $a = 7, \dots, 10$ transform as vectors under the two $\text{SO}(4)$'s respectively. From now on, we will denote them collectively as $b_0^i, b_0^{i\dagger}$ with $i = 1, \dots, 4, 7, \dots, 10$. These operators generate the Heisenberg algebra $\mathfrak{h}(8)$. Note that this is true only in the large- J limit and for Euclidean gauge theory. In view of the BMN proposal, it is quite natural to identify this Heisenberg algebra $\mathfrak{h}(8)$ with the similar Heisenberg algebra encountered in the Penrose contraction of $\text{SO}(4,2) \otimes \text{SO}(6)$.

The next step would be to identify, within the dual gauge theory, the physical meaning of both the central extension and the outer-automorphism. In the original theory invariant under $\text{SO}(4,2) \otimes \text{SO}(6)$, there are two generators of the symmetry algebra that are of special importance, viz. the generator of dilatations of the space-time coordinates and the generator J of the $\text{SO}(2)$ R-symmetry. Inferring from the discussion on the Penrose contraction in section 2, we should expect that both the central extension and the outer-automorphism are associated with these two generators. In the dual gauge theory, these generators have a clear physical meaning: generator of the space-time dilatations defines *scaling dimension* Δ of the operators, and the generator J defines the corresponding R-charge. Note that in the AdS_5 realization of $\text{SO}(4,2)$ the embedding coordinates $X^A (A = 1, \dots, 6)$ with $(X^1)^2 + (X^2)^2 - (X^3)^2 - \dots - (X^6)^2 = R^2$ are given in terms of the global coordinates $(\bar{t}, \bar{r}, \phi_i)$ as

$$X^1 = R \sqrt{1 + \bar{r}^2} \cos \bar{t}$$

$$\begin{aligned}
X^2 &= R\sqrt{1+\bar{r}^2} \sin \bar{t} \\
X^\alpha &= R\bar{r}\omega^\alpha. \quad (\alpha = 3, \dots, 6)
\end{aligned}
\tag{24}$$

Here, ω^α denote the embedding coordinates of a unit S^3 . The standard "dilatation" generator of the $SO(4,2)$ group, which is J_{12} , generates translation in global time \bar{t} . In the Penrose limit, $\bar{r} \rightarrow 0$, while $r = R\bar{r}$ held fixed. Thus, X^α ($\alpha = 3, \dots, 6$) are $\mathcal{O}(1)$ and become *unconstrained*, while X^1, X^2 are of $\mathcal{O}(R)$. The dual gauge theory is now defined on the Euclidean plane X^3, \dots, X^6 and J_{12} generates the scale transformations on this \mathbf{R}^4 -subspace as in the standard realization of the $SO(4,2)$ group. Let us denote the eigenvalues of these generators, for a given operator, as Δ and J , respectively. For the state $|0\rangle_J$, we have

$$\Delta|0\rangle_J = J|0\rangle_J \quad \text{and} \quad J|0\rangle_J = J|0\rangle_J,$$

while, for states of type $b_0^{i\dagger} \dots b_0^{i\dagger}|0\rangle_J$, we have $\Delta = (J + n)$, where n refers to the number of b^\dagger -oscillators and $J = J$. Thus, on these states, we have $(\Delta + J) = 2J + n$ and $(\Delta - J) = n$. If we work in the large- J , large- N , and small- n limits with

$$4\pi g_{\text{YM}}^2 N := R^2 \rightarrow \infty, \quad J^2 \rightarrow \infty \quad \text{and} \quad \frac{R^2}{J^2} := g_{\text{eff}}^2 \rightarrow \text{finite}, \tag{25}$$

we observe that, in appropriate units, $(\Delta + J)/R^2$ becomes the true central extension commuting with the b_0^i and $b_0^{i\dagger}$ operators⁵, and that $(\Delta - J)$ is simply the number operator for the $b_0^i, b_0^{i\dagger}$ oscillators and therefore is a true outer-automorphism of the Heisenberg algebra.

In summary, built only upon Euclidean gauge theory residing on \mathbf{R}^4 subspace, we succeeded in finding a $SO(4) \otimes SO(4)$ invariant vacuum and a representation of the Heisenberg group $H(8)$ in terms of creation and annihilation operators acting on the Hilbert space of small quantum fluctuations. The corresponding outer-automorphism is just the number operator. Note that the vacuum state $|0\rangle_J$ is not only invariant under $SO(4) \otimes SO(4)$ but also with respect to the one-parameter group generated by the outer-automorphism.

So far, we have considered only the modes which are chiral primaries. The scaling dimension Δ of the corresponding operator is

$$\Delta = \left(J + \sum_{i=1}^4 n_i + \sum_{a=1}^4 m_a \right), \tag{26}$$

where there are n_i insertions of $D_i Z$, and m_j insertions of Φ^j . Supersymmetry descendants of these would contain factors involving the gauge field, as discussed in section 3. Consider, for example, the dilaton. The operator dual to this should be the integral of

$$\text{Tr}[F_{mn} F^{mn} \Phi^{(i_1} \dots \Phi^{i_\ell)}](t, \phi_i \dots \phi_{d-1}).$$

⁵ Note that once we identify this term with the central extension of the Heisenberg algebra, we need to normalize the b_0^i and $b_0^{i\dagger}$ so that they obey the canonical commutation relations.

For such operators, the scaling dimension Δ is given by

$$\Delta = J + \sum_{i=1}^4 n_i + \sum_{m=1}^4 m_a + 4. \quad (27)$$

These relations are consistent with our interpretation of the holographic coordinate.

The scaling dimension Δ of the dual operator is, however, the eigenvalue of the operator $i\partial_{\bar{t}}$ in the bulk theory. The R-charge is of course the eigenvalue of $-i\partial_{\bar{\theta}}$. These relations are in accord with the solutions of the bulk wave equations. Take the dilaton as an example. From Eq.(15) with $d = 4$ and $\bar{d} = 4$, we have

$$2p^- = \mu (\Delta - J) = \mu \left(\sum_{i=1}^4 n_i + \sum_{a=1}^4 m_a + 4 \right) \quad (28)$$

and find precise agreement with Eq.(27). From the bulk point of view, the additive factor 4 appears as a zero point energy. From the gauge theory viewpoint, this reflects the presence of $F_{mn}F^{mn}$ in the operator. For the bulk mode which is a fluctuation of the four-form RR potential, this zero point energy is absent, which is consistent with the absence of any factor of the gauge field strength in the dual operator. In the $J \rightarrow \infty$ limit, $\Delta \rightarrow \infty$ as well. However, $p_+ = \mu(\Delta - J)$ remains finite. This is the reason why, though it appears natural to consider t as a holographic direction from the bulk point of view, it is actually more natural to consider x^+ as the holographic direction from the gauge theory point of view.

Finally, it should be clear from the definition of the operators that one needs to introduce a cutoff mass scale Λ . Inferring from the expressions for the normal modes of the bulk fields, it is natural to choose the scale to be given, up to $\mathcal{O}(1)$ numerical factor, by

$$\Lambda = \mu p_- . \quad (29)$$

At first sight, this identification appears strange. It would mean that one needs *a priori* a different scale for each operator as p_- is defined by $(\Delta + J)/2\mu R^2$. However, in the approximation adopted in Eq.(25), $n_i, m_a \ll J$. As such, $p_- \sim J/R^2 \sim \mathcal{O}(1)$. This implies that all the operators involved are governed universally by a common renormalization scale.

4.3 Where does the dual gauge theory “live” ?

For the ten-dimensional pp-wave, the states created by the c_n defined in Eq.(20) are to be identified with the states in the light-cone gauge quantized Type IIB string in the bulk created by *transverse* oscillators. It is thus natural to identify one of the transverse \mathbf{R}^4 's of the pp-wave with the space on which the Euclidean gauge theory “resides”. Furthermore, as the gauge theory operator $\text{Tr}(Z^J)$ does not contain any of the Higgs fields $\Phi^7 \cdots \Phi^{10}$, the transverse \mathbf{R}^4 in question would be identifiable with the \mathbf{R}^4 formed by the $X^\alpha, \alpha = 3 \cdots 6$ in Eq.(24).

To understand the meaning of this interpretation, let us trace back the process of taking the Penrose limit. The starting bulk theory, the $AdS_5 \times S^5$ supergravity theory, is dual to $\mathcal{N} = 4$ supersymmetric gauge theory, which is defined on Lorentzian manifold, $S^3 \times \mathbf{R}_{\bar{t}}$, where the S^3 and $\mathbf{R}_{\bar{t}}$ denotes a foliation leaf at $\bar{r} = \bar{r}_0$, with large \bar{r}_0 , and the global time \bar{t} in AdS_5 . One possible way to obtain a gauge theory on a \mathbf{R}^4 is by Wick-rotating the global time $\bar{t} \rightarrow i\tau$, and then performing the standard conformal transformation of the resulting Euclidean $S^3 \times \mathbf{R}_\tau$. We will refer the euclidean space obtained in this way by $\hat{\mathbf{R}}^4$ and its coordinates by \mathbf{z}^i . One may then try to interpret the derivatives D_i appearing in the operator in Eq.(9) as derivatives with respect to the \mathbf{z}^i . However, this cannot be right if the insertion of a $D_i Z$ corresponds in the bulk to a state obtained (in a first quantized description of particles) by applying an operator $\mathbf{c}^{i\dagger}$ defined in Eq.(14). First, the description of string or supergravity states in the light-cone gauge does not, of course, involve any analytic continuation. We should not perform any analytic continuation in the gauge theory as well. Secondly, the index i in $\mathbf{c}^{i\dagger}$ clearly refers to a direction transverse to the pp-wave : in this case the transverse \mathbf{R}^4 is formed by $(\bar{r}, \phi_1, \dots, \phi_3)$. In other words, the symmetry of the set of states created by $\mathbf{c}^{i\dagger}$ includes a $SO(4)$ and this $SO(4)$ does not involve the global time, Wick rotated or not.

In concrete terms, our statement that the dual theory can be thought of as living on the transverse $\mathbf{R}^4 = (\bar{r}, \phi_1, \dots, \phi_3)$ means that the derivatives which appear in Eq.(9) should be derivatives with respect to this transverse \mathbf{R}^4 . This provides a manifest linear realization of the symmetries $SO(4) \times SO(4)$ and the associated Heisenberg algebra $\mathfrak{h}(8)$.

However, the Penrose limit is supposed to be a sector of the original Yang-Mills theory defined on the $S^3 \times \mathbf{R} = (\bar{t}, \phi_1 \dots \phi_3)$. How can we then regard the operators of this theory to depend on $(\bar{r}, \phi_1, \dots, \phi_3)$? The answer is provided by the IR-UV connection in AdS holography. In the gauge theory composite operators are functions on the space on which the theory is defined, $\mathcal{O}(\mathbf{z}) = \mathcal{O}(\bar{t}, \phi_1, \dots, \phi_3)$. At the quantum level, however, these local operators also depend on the (position space) ultraviolet cutoff a , or equivalently a renormalization scale. So, these local operators should really be expressed as $\mathcal{O}(\mathbf{z}; a) = \mathcal{O}(\bar{t}, \phi_1 \dots \phi_3; a)$. The AdS/CFT correspondence identifies this operator with a *bulk operator* (e.g. for one of the supergravity fields) $\Phi(\bar{t}, \phi_1, \dots, \phi_3, \bar{r})$ evaluated at large $\bar{r} = \bar{r}_0 = 1/a$. We can thus think of the transverse $\mathbf{R}^4 = \{\vec{\mathbf{x}}\}$ above as being formed by combining the spatial coordinates ϕ_i with the scale a . In terms of the original Yang-Mills theory living on $S^3 \times \mathbf{R}_{\bar{t}}$ derivatives which appear in Eq.(9) therefore involve derivatives with respect to the scale.

Before considering any Penrose limit, it is natural to arrange ‘twist’ expansion of operators as in $\mathcal{O}(\mathbf{z} = 0; a)$ and $D_{\mathbf{z}} \dots D + \mathbf{z} \mathcal{O}(\mathbf{z} = 0; a)$, while, after the Penrose limit, the more convenient ‘twist’ expansion is with respect to \mathbf{x}^i , viz. $D_{\mathbf{x}} \dots D_{\mathbf{x}} \mathcal{O}(\mathbf{x} = 0, \bar{t})$. Upon swapping \bar{r} with the light-cone time, we are also led to do so for the ‘holography’ direction. In this sense, the light-cone time plays the role of the ‘holography’ direction, while \bar{r} is now part of the Euclidean base

space \mathbf{R}^4 .

Evidently, the above two choices of ‘twist’ expansions are intimately related. Schematically, the renormalization group equation of the operator \mathcal{O} takes the form

$$a \frac{d}{da} \mathcal{O}(a) = \gamma \cdot \mathcal{O}(a).$$

By virtue of the AdS/CFT correspondence the renormalization group equations of the gauge theory are related to the (suitably formulated) bulk equations of motion. Replacing $1/a$ with the radial coordinates \bar{r} , the renormalization group can then be recasted as a first-order light-cone time evolution obeyed by a data specified on \mathbf{R}^4 formed out of \bar{r} and (ϕ_1, \dots, ϕ_3) . An implicit assumption in the two alternative ‘twist’ expansions is that the locality, which was automatically built in the Minkowski gauge theory on $S^3 \times \mathbf{R}_{\bar{t}}$, is maintained in the Euclidean gauge theory on \mathbf{R}^4 . This is highly nontrivial, as, in the standard AdS_5 holography, map between the gauge theory scale and the AdS_5 radial position is given in a complicated and nonlocal manner. Apparently, in the Penrose limit, the map becomes simplified dramatically and becomes local, as inferred from the precise matching between the light-cone string dynamics and the Euclidean gauge theory dynamics.

Summarizing, we emphasize that the question regarding where the gauge theory dual to a pp-wave light-cone string lives is intimately connected with the way we identify the string transverse oscillators into the Hilbert space of quantum fluctuations of a fixed R-charge ground-state in the gauge theory. Because we are focusing on each Hilbert subspace of fixed R-charge, the question is also intimately tied with the realization of the pp-wave isometries in the gauge theory. The normal subgroup of the latter is $SO(4) \otimes SO(4)$, where one $SO(4)$ originating from the kinematical conformal group in four dimensions and the other from the internal R-symmetry. These symmetries are manifest once operators and states in the dual gauge theory are formulated on the transverse \mathbf{R}^4 . In doing so, we interchange the role between the light-cone time \bar{t} and the bulk radial coordinate \bar{r} , and now view the former as the ‘holography’ coordinates of the dual, Euclidean gauge theory.

4.4 Light-Cone Holography

Once we have defined the outer-automorphism, call it H , we can trivially use it to define a one parameter family of operators. In fact, we can introduce a formal parameter x^+ as a conjugate variable to H and define

$$\sum_{\ell} \text{Tr} \left(Z^{\ell} (D_i Z) Z^{J-\ell} \right) (0, x^+) |0\rangle = e^{-ix^+ H} b_0^{i\dagger} |0\rangle_J.$$

This is the prescription for introducing an extra *holographic* light-cone time coordinate in the Euclidean gauge theory. In fact, what we get is simply

$$\sum_{\ell} \text{Tr} \left(Z^{\ell} (D_i Z) Z^{J-\ell} \right) (0, x^+) |0\rangle_{\text{YM}} = e^{-ix^+(\Delta-J)} \sum_{\ell} \text{Tr} \left(Z^{\ell} (D_i Z) Z^{J-\ell} \right) (0) |0\rangle_{\text{YM}} \quad (30)$$

and the outer-automorphism $H = i\partial_{x^+}$. If we wish, we can also introduce an another holographic coordinate, say, x^- conjugate to the central extension $C = (\Delta + J)/\lambda$ by

$$\sum_{\ell} \text{Tr} \left(Z^{\ell} (D_i Z) Z^{J-\ell} \right) (0, x^+, x^-) |0\rangle_{\text{YM}} = e^{ix^- C} e^{-ix^+ H} b_0^{i\dagger} |0\rangle_J.$$

This dependence on x^- is trivial, as it is the same for all operators. It depends only on the particular finite value we choose for R^2/J^2 in the double large- J and large- N limit. Once we introduce the coordinate x^- , the central extension C can be represented as $-i\partial_{x^-}$.

We can readily make a contact with the spacetime Killing vectors in the Penrose limit. To illustrate this, consider the case of $\text{AdS}_3 \times S^3$. In the Penrose limit, the isometry group $\text{SO}(2,2) \otimes \text{SO}(4)$ is contracted to $[\text{H}(2) \otimes \text{H}(2)] \otimes \text{SO}(2) \otimes \text{SO}(2)$. We are interested in the generators of the Heisenberg groups $\text{H}(2)$'s:

$$\begin{aligned} P_i(x^+) &= \cos(\mu x^+) \frac{\partial}{\partial x^i} + \mu x^i \sin(\mu x^+) \frac{\partial}{\partial x^-}, \\ K_i(x^+) &= \sin(\mu x^+) \frac{\partial}{\partial x^i} - \mu x^i \cos(\mu x^+) \frac{\partial}{\partial x^-}. \end{aligned}$$

These are the generators that we should put in correspondence with the operators b_0^i and $b_0^{i\dagger}$ in the dual gauge theory. As discussed above, the operator $-i\partial_{x^-}$ plays the role of the central extension C of the Heisenberg algebra. In the standard coordinate-momentum notation, we can write

$$\begin{aligned} P_i(x^+) &= p_i \cos(\mu x^+) + \mu C x^i \sin(\mu x^+) \\ K_i(x^+) &= p_i \sin(\mu x^+) - \mu C x^i \cos(\mu x^+), \end{aligned}$$

where the parameter C represents the central extension term, and

$$[p_i, x^j] = -i\delta_i^j.$$

Evidently, we can write

$$P_i(x^+) = e^{-ix^+ H} P_i(0) e^{+ix^+ H} \quad (31)$$

for H the light-cone Hamiltonian defined by $H = i\partial_{x^+}$. This shows that the Hamiltonian H is an outer-automorphism of the Heisenberg algebra.

We close this section with comments. Quantum mechanically, the value Δ of the scaling dimension is corrected by the contribution of anomalous dimensions, whereas the value J of

the R-symmetry charge remains unchanged. As such, one ought to expect quantum corrections to the outer-automorphism $(\Delta - J)$. How is it corrected? The interaction part of $\mathcal{N} = 4$ gauge theory contains a term of the type $\frac{1}{2\pi g_{\text{YM}}^2} \text{Tr}[Z, \Phi_a][Z, \Phi_a]$. This term induces a non-vanishing transitions of the type:

$$\langle \text{Tr} \left(Z^\ell \Phi_a Z^{J-\ell} \right) (x) \text{Tr} \left(Z^{\ell+1} \Phi_b Z^{J-(\ell+1)} \right) (0) \rangle = \delta_{ab} 2\pi g_{\text{YM}}^2 N \mathcal{I}(x) \frac{1}{4\pi^2 x^{2J-2}} \quad (32)$$

and

$$\langle \text{Tr} \left(Z^\ell \Phi_a Z^{J-\ell} \right) (x) \text{Tr} \left(Z^\ell \Phi_b Z^{J-\ell} \right) (0) \rangle = \delta_{ab} 2\pi g_{\text{YM}}^2 N \mathcal{I}(x) \frac{1}{4\pi^2 x^{2J-2}}.$$

The function $\mathcal{I}(x)$ is given by

$$\mathcal{I}(x) = \frac{1}{4\pi^2} \log|x| \Lambda + \text{finite},$$

where Λ defines the ultraviolet cutoff. Both contributions cancel each other, meaning that, for operators of the type $\sum_\ell \text{Tr} \left(Z^\ell (D_i Z) Z^{J-\ell} \right)$ or $\sum_\ell \text{Tr} \left(Z^\ell \Phi_a Z^{J-\ell} \right)$, the value of $(\Delta - J)$ is not corrected at least at first-order in the weak coupling perturbation theory. As pointed out by BMN already, the situation is different for operators of the type

$$\sum_\ell e^{i\frac{2\pi n\ell}{J}} \text{Tr} \left(Z^\ell (D_i Z) Z^{J-\ell} \right)$$

or

$$\sum_\ell e^{i\frac{2\pi n\ell}{J}} \text{Tr} \left(Z^\ell \Phi_i Z^{J-\ell} \right), \quad (33)$$

modulated by the ‘separation-dependent’ phase-factors. We will discuss aspects of these corrections in the next subsection.

4.5 Strings out of Dual Gauge Theory

Let us first recapitulate what we have done so far. We have considered Euclidean $\mathcal{N} = 4$ gauge theory around a vacuum state invariant under $\text{SO}(4) \otimes \text{SO}(4)$ and under the outer-automorphism H . The Hilbert space of small fluctuations around this vacua define a representation of the Heisenberg algebra $\mathfrak{h}(8)$. The outer-automorphism H is simply the number operator associated with the creation and annihilation operators generating $\mathfrak{h}(8)$. In addition, using the gravity dual of the Minkowskian $\mathcal{N} = 4$ gauge theory, we can define a precise map between the creation annihilation operators and the outer-automorphism H on the dual gauge theory side and the Killing vectors in the Penrose limit of $AdS_5 \times S^5$ on the gravity side. The formal conjugate variables of the Hamiltonian H and the central extension C become the coordinates x^+ and x^- of the Penrose limit of the bulk spacetime.

Probably the most surprising result of all this is the connection between the Penrose limit of Minkowskian $\mathcal{N} = 4$ gauge theory and the Euclidean $\mathcal{N} = 4$ gauge theory around a particular vacua. The main reason for this strange connection has to do with the peculiarities of spontaneous breakdown of conformal invariance. In fact, one can in principle think this as a spontaneous breakdown of $SO(4,2)$ to the Euclidean subgroup $SO(4)$. In this case, the Lorentz invariance of the original Minkowskian theory should be hidden somehow in the dynamics of the Goldstone bosons around the vacua used to break the conformal symmetry spontaneously.

How then is the Lorentz invariance realized in the Hilbert space of small fluctuations around the chosen vacua $|0\rangle_J$? The answer descending from the BMN proposal is quite surprising and in fact extremely interesting. It asserts that the Hilbert space of small fluctuations of the Euclidean $\mathcal{N} = 4$ gauge theory around the vacua $|0\rangle_J$ in the large- N and large- J limit is the Hilbert space of a ten-dimensional string theory in a suitable Minkowskian background.

To understand this, we begin with recalling some salient features of string dynamics in the light-cone gauge. In flat ten-dimensional space-time and for the bosonic sector, the light-cone gauge-fixed string is defined by:

i) **string oscillators:** infinite tower of Heisenberg algebras

$$[a_n^i, a_m^{j\dagger}] = \delta_{n,m} \delta_{i,j}$$

with $i, j = 1 \dots 8$ the transversal coordinates,

ii) **light-cone Hamiltonian:**

$$H_{LC} = \sum_n \frac{n}{2\alpha' p^+} a_n^\dagger a_n + (\text{h.c.}),$$

iii) **string parameter space:** total length of the light-cone string is given by p^+ ,

iv) **Virasoro constraint:** infinite tower of constraints satisfying the Virasoro algebra.

The way string dynamics emerges out of the Euclidean $\mathcal{N} = 4$ gauge theory around the vacuum $|0\rangle_J$ relies crucially on the existence of the Heisenberg algebra $\mathfrak{h}(8)$ and of the outer-automorphism H . In order to establish a connection between the two structures, the first thing we should do is to extend the Heisenberg algebra $\mathfrak{h}(8)$ to an infinite family of Heisenberg algebras of the type displayed in (i). Remarkably, this is achieved by the phase-factor-modulated operators, Eq.(33). Introduce creation and annihilation operators b_n, b_n^\dagger via:

$$\sum_\ell e^{i\frac{2\pi n\ell}{J}} \text{Tr} \left(Z^\ell \Phi_a Z^{J-\ell} \right) (0)|0\rangle_{\text{YM}} := b_n^\dagger |0\rangle_J.$$

One can show readily that the newly introduced creation and annihilation operators obey the requisite infinite towers of Heisenberg algebras:

$$[b_n^i, b_m^{j\dagger}] = \delta_{n,m} \delta_{i,j} \quad (m, n = 0, 1, 2, \dots).$$

Compared to the light-cone string in flat spacetime, the main difference is the existence of the Heisenberg algebra for the $b_0^i, b_0^{i\dagger}$ harmonic oscillator operators. In fact, in flat spacetime, one only has the Heisenberg algebra for the center-of-mass part

$$[x_0^-, p^+] = i.$$

Let us now see how the outer-automorphism H is modified by quantum effects. As computed by BMN, the scaling dimension Δ for the state $b_n^\dagger|0\rangle_J$ is given in first-order in weak-coupling perturbation theory by

$$\Delta = J + 1 - \frac{1}{\pi} g_{\text{YM}}^2 N \left(\cos\left(\frac{2\pi n}{J}\right) - 1 \right).$$

Thus, the outer-automorphism at the quantum level is given by

$$\begin{aligned} H = (\Delta - J) &= 1 - \frac{1}{\pi} g_{\text{YM}}^2 N \left(\cos\left(\frac{2\pi n}{J}\right) - 1 \right) \\ &= 1 + \frac{2\pi g_{\text{YM}}^2 N}{J^2} n^2 + \dots, \end{aligned}$$

where, in obtaining the second expression, we have taken the large- J and large- N limit.

The light-cone Hamiltonian in the bulk may be therefore written as

$$p_+ = \mu \sum_n \sqrt{1 + \frac{4\pi g_{\text{YM}}^2 N}{J^2} n^2} \quad b_n^\dagger b_n$$

However, from Eq.(10), we have

$$p_- = \frac{1}{2\mu R^2} (\Delta + J) \sim \frac{J}{\mu R^2} = \frac{J}{\mu \sqrt{4\pi g^2 N}}$$

where we have used $\alpha' = 1$ units throughout. Thus,

$$p_+ = \mu \sum_n \sqrt{1 + \frac{n^2}{\mu^2 p_-^2}} \quad b_n^\dagger b_n,$$

yielding exactly the light-cone Hamiltonian of the string theory in this background.

An issue we have not elaborated in detail is the string Virasoro constraints. We end this section with brief remarks on it. By inserting phases into the dual gauge theory operators, the Heisenberg algebra $\mathfrak{h}(8)$ is extended to the family of Heisenberg algebras Eq.(34). It is also natural to extend the outer-automorphism to this collection of Heisenberg algebras by introducing operators H_n , ($n = 0, 1, 2, \dots$) such that $[H_n, b_m] = b_{n+m}$ and $H_0 = H$ for the light-cone Hamiltonian H . These outer-automorphisms then generate the string Virasoro algebra. A very interesting question left for the future would be to uncover meaning of these Virasoro constraints entirely within the dual gauge theory viewpoint.

4.6 Light-Cone Hamiltonian and Renormalization Group Flow

The renormalization group equation for the correlators $\langle \mathcal{O}(x)\mathcal{O}^*(0) \rangle$ is

$$[a\partial_a + 2\gamma(\mathcal{O})]\langle \mathcal{O}(x)\mathcal{O}^*(0) \rangle = 0.$$

where a refers to the renormalization scale. If we consider the phase-modulated operator $\mathcal{O} = \sum_n e^{2\pi n\ell} \text{Tr} \left(Z^\ell \Phi_a Z^{J-\ell} \right)$, in the large- J and large- N limit, we get

$$2\gamma(\mathcal{O}) = (\Delta - J)[\mathcal{O}] - 1, \tag{34}$$

where $(\Delta - J)[\mathcal{O}]$ is the value of $(\Delta - J)$ for the operator \mathcal{O} . This equation is re-expressible in a more suggestive form

$$2\gamma(\mathcal{O})\mathcal{O} = [H, \mathcal{O}] - \mathcal{O},$$

where H is, as usual, the string light-cone Hamiltonian $(L_0 + \bar{L}_0)$.

Note that the anomalous dimension $\gamma(\mathcal{O})$ appearing in Eq.(34) is, for the operator $\mathcal{O} = \sum_n e^{2\pi i n\ell} \text{Tr} \left(Z^\ell \Phi_a Z^{J-\ell} \right)$, given by

$$\gamma(\mathcal{O}) = J\gamma(Z) + \gamma(\Phi_a),$$

where $\gamma(Z)$ represents the anomalous dimension of the operator Z . Generically, the anomalous dimension $\gamma(Z)$ is affected by radiative effects through the self-energy corrections⁶. If not protected by supersymmetry, these contributions would grow as $g_{\text{YM}}^2 N$ and the connection with the string light-cone Hamiltonian would be lost. Moreover, the scaling dimension of the operators \mathcal{O} , in that case, would grow with t'Hooft's coupling constant and eventually disappear out of the physical spectrum. In the supersymmetric case we are considering, supersymmetry renders $\gamma(Z) = 0$. This is the reason behind regarding $\gamma(\mathcal{O})$ as the anomalous dimension of the field Φ_a 's. From the previous discussion, it should be evident that changes in the holographic coordinate x^+ are equivalent to changes of the renormalization scale a if we interpret $(\Delta - J)$ as the anomalous dimension⁷.

5 Symmetry Enhancement as ‘‘Post Mortem’’ Effect

In recent papers [12, 13], the results of BMN have been generalized to gauge theories with less supersymmetry, in particular, to the gravity duals of $AdS_5 \times T^{1,1}$. The fact that the Penrose limit of this space is the same as the one of $AdS_5 \times S^5$ raises the question regarding reason

⁶These are so-called zero-momentum effects in the nomenclature of BMN.

⁷ Definition of the renormalization group $\gamma(\mathcal{O})$ as $(\Delta - J)$ or as $(\Delta - J) - 1$ depends on whether one adopts the canonical dimensions for the fields and masses or not. For a lucid discussion on this point, see [26]

behind supersymmetry enhancement from $\mathcal{N} = 1$ to $\mathcal{N} = 4$. The point of this seemingly mysterious result goes back to the fact that, in the *strict* Penrose limit, there always appear extra isometries. In [6], these extra isometries are referred to as a post mortem effect. These isometries define always a Heisenberg algebra. In the case of $AdS_5 \times T^{1,1}$, the isometries are $SO(4,2) \otimes (SU(2) \otimes SU(2) \otimes U(1))$, and are in correspondence, respectively, with the conformal invariance and the R-symmetry of the dual gauge theory. Following our approach, we can think in terms of a spontaneous breakdown of this symmetry to $SO(4) \otimes SO(4)$. The difference with the case of $\mathcal{N}=4$ supersymmetric gauge theory is that we now have a smaller number of Goldstone bosons associated with the broken symmetries, viz. ten instead of eighteen in the $\mathcal{N} = 4$ supersymmetric case. The deficit eight Goldstone bosons are precisely the ones associated with the Higgs fields in the $\mathcal{N} = 4$ supersymmetric gauge theory. These are the fields that would render the theory $\mathcal{N} = 4$ supersymmetric. In the Penrose limit of any ten-dimensional background, we have always a Heisenberg algebra $\mathfrak{h}(8)$ of the isometries. In the $AdS_5 \times T^{1,1}$ case, the Heisenberg algebra is composed of two Heisenberg subalgebras $\mathfrak{h}(4)$ with a common central extension. The eight generators of one of the two algebras are the ones we are going to use as the eight missing broken symmetries. These “post mortem” Goldstone bosons are simply states that, in the $\mathcal{N} = 1$ gauge theory, are degenerate in the light-cone mass (the eigenvalue of the outer-automorphism or the light-cone Hamiltonian) with the real Goldstone bosons. As the enhancement of $\mathcal{N} = 4$ supersymmetry is true only in the strict Penrose limit, viz. for the Penrose scaling factor $R \rightarrow \infty$, we expect this enhancement to be violated by $\mathcal{O}(1/N)$ -corrections.

6 Conclusions

Our main conclusion is that the supersymmetric gauge theory dual to the IIB string theory in a ten dimensional pp-wave background may be thought to “live” on a Euclidean four dimensional space. The indications has come from several corners. The most direct reason is that the dual operators are in one to one correspondence with the states of the string theory which are created by *transverse* oscillators in the light-cone gauge. This is in fact apparent in the zero-mode sector of the string relevant for the supergravity modes and follows from field equations in this background. Furthermore, understanding of the low-energy Nambu-Goldstone modes resulting from symmetry breaking of the original symmetry group $SO(4,2) \times SO(6)$ to $SO(4) \times SO(4) \times H(4) \times H(4)$ also indicates that the dual theory lives in a Euclidean space.

One may think of this Euclidean space as the space spanned by four of the transverse directions. In this case, the light-cone time has a natural explanation as the holographic coordinate representing the scale of the dual gauge theory. Details of the proposed holographic correspondence remain to be understood better. The fact that a Euclidean theory can give rise

to strings living in a space-time with Lorentzian signature is intriguing and deserves a better understanding. We expect that this would enhance our understanding of holography in general.

Extending the results of this work, one can abstract the main ingredients needed in order to generalize the BMN proposal to generic situations. The minimal starting point would be a four-dimensional gauge theory invariant under the conformal group and with a nonanomalous global $U(1)$ symmetry. The spontaneous breakdown of $SO(4,2)\otimes U(1)$ to $SO(4)$ provides ten broken generators that we can try to organize, taking Euclidean signature dual theory, into a Heisenberg algebra $\mathfrak{h}(4)$ and an outer-automorphism H . This is the structure that we can try to map into the Hilbert space of a non-critical six-dimensional string theory in the light-cone gauge with the $SO(4)$ rotational invariance acting on the transversal coordinates. Of course, we also need controllable, finite contributions- in the large 't Hooft coupling limit - to the anomalous dimensions of the fields representing the small quantum fluctuations around the selected vacuum or, equivalently, finite light-cone mass (defined by the outer-automorphism) for the Goldstone bosons. Taking the real-world QCD, one now has *ab initio* two related problems: anomaly for the conformal invariance (viz. a non vanishing beta function) and anomaly for the axial $U(1)$ symmetry, which would serve as a natural candidate for the global $U(1)$ symmetry. We can try to solve these problems by introducing two extra scalar fields, namely the dilaton D and the axion A . We then find an indication that the field $(D + iA)$ is a natural candidate to play the role of the Z -field in the BMN proposal.

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Note Added:

After this paper was submitted in the preprint archive, two preprints [27], [28] investigating related issues have appeared. The paper [27] argues that the hologram is eight dimensional, based on the classical Cauchy problem - rather than four dimensional. In [28] the holographic screen is argued to be four-dimensional Minkowski space, but the realization of the $SO(4)$ symmetry from this viewpoint was not addressed.

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