

Compact $z = 2$ Electrodynamics in $2 + 1$ dimensions: Confinement with gapless modes

Sumit R. Das* and Ganpathy Murthy †
Department of Physics and Astronomy,
University of Kentucky, Lexington, KY 40513, USA.

We consider $2 + 1$ dimensional compact $U(1)$ gauge theory at the Lifshitz point with dynamical critical exponent $z = 2$. As in the usual $z = 1$ theory, monopoles proliferate the vacuum for any value of the coupling, generating a mass scale. The theory of the dilute monopole gas is written in terms a non-relativistic Sine-Gordon model with two real fields. While monopoles remove some of the massless poles of the perturbative field strength propagator, a gapless mode representing the incomplete screening of monopoles remains, and is protected by a shift invariance of the original theory. Timelike Wilson loops still obey area laws, implying that minimal charges are confined, but the action of spacelike Wilson loops of linear size L goes instead as L^3 .

PACS numbers: 11.15Pg, 11.10Kk

Quantum field theories around Lifshitz fixed points with a dynamical critical exponent $z \neq 1$ have been of interest to a variety of problems in classical and quantum critical phenomena [1]-[7], and have been explored as possible ultraviolet completions of low energy effective actions for applications to particle physics and gravity [8]-[13]. In this paper we study non-perturbative aspects of compact $U(1)$ gauge theory with $z = 2$ in $2 + 1$ dimensions. The action in euclidean signature is

$$S = \frac{1}{2g^2} \int dt d^2x \left[F_{0i} F^{0i} + \frac{1}{2} (\partial_k F_{ij}) (\partial^k F^{ij}) \right] \quad (1)$$

Throughout this paper $i, j = 1, 2$ are spatial indices, while $\mu, \nu = 0, 1, 2$ are space-time indices. Such theories, including their non-abelian generalizations, have been considered in [8]. The action (1) appears as the effective action of $2 + 1$ dimensional CP^{N-1} models at a special multicritical point [14]. For $N = 2$ this in turn can be obtained from an $O(3)$ nonlinear sigma model with $z = 2$ by the usual relation to the CP^1 model $\zeta^\dagger \sigma \zeta = \mathbf{n}$, where \mathbf{n} is the unit length field of the $O(3)$ sigma model, and ζ is a two-component spinor CP^1 field satisfying $\zeta^\dagger \zeta = 1$. Related versions of $z = 2$ gauge theories appear in the description of algebraic spin liquids in $3 + 1$ dimensions [15, 16] and of topological critical phases in $2 + 1$ dimensions [17].

Note that the theory of Eq. (1) has a continuous symmetry with respect to global shifts in $F_{12} = B$. The full action which follows from the spin model [14] also contains essentially singular terms of the form $\frac{B^{\frac{3}{2}} m^{\frac{1}{2}}}{4\pi^2 \sqrt{2}} e^{-\frac{\pi m}{B}}$ (where m is the dynamically generated mass of the spinon fields). These terms are irrelevant by power counting, but violate the shift invariance.

We will consider this theory with an ultraviolet cut-off. If the theory is viewed as the low-energy description of a $SU(2)$ gauge theory broken to $U(1)$ by an adjoint Higgs field, the mass of the off-diagonal components is the cutoff.

The physics of standard compact electrodynamics ($z = 1$) in $2 + 1$ dimensions is well-known [18, 19]. Compactness implies that there are magnetic monopoles (instantons), which disorder the vacuum, resulting in the confinement of minimal charges and the Debye screening of monopoles. All gauge invariant correlators are massive. The suppression of monopoles results in a theory with a gapless photon with potential implications for quantum antiferromagnets [23]-[26].

In this paper we consider the effect of monopoles on the $z = 2$ action defined by Eq. (1). We find that the monopoles are relevant, and minimal charges are still confined. However, a gapless mode with a low energy relativistic dispersion remains. This mode is the remnant of the B shift symmetry mentioned above, and represents the long-range residual interaction between monopoles due to incomplete screening. Finally, the action of a space-like Wilson loop of linear size L behaves as L^3 .

It is convenient to define the dual field strength and its fourier transform $H_\mu(t, \vec{x}) = \frac{1}{2} \epsilon_{\mu\nu\lambda} F^{\nu\lambda}(t, \vec{x}) = \int \frac{d\omega d^2\vec{k}}{(2\pi)^3} H_\mu(\omega, \vec{k}) e^{-i(\omega t + \vec{k} \cdot \vec{x})}$. Ignoring the compactness of the gauge field, the correlators of $H_\mu(\omega, \vec{k})$ may be easily computed from the action (1). In terms of redefined fields \tilde{H}_μ with $\tilde{H}_0(\omega, \vec{k}) = \frac{H_0(\omega, \vec{k})}{|\vec{k}|}$, $\tilde{H}_i = H_i$ and with $k_0 \equiv \frac{\omega}{|\vec{k}|}$ we get

$$\langle \tilde{H}_\mu(\omega, \vec{k}) \tilde{H}_\nu(-\omega, -\vec{k}) \rangle_{pert} = \delta_{\mu\nu} - \frac{k_\mu k_\nu \vec{k}^2}{\omega^2 + \vec{k}^4} \quad (2)$$

The poles at $\omega = \pm i \vec{k}^2$ are characteristic of a non-relativistic Lifshitz point.

The equations of motion which follow from (1) are

$$\partial_i F^{0i} = 0, \quad -\partial^0 F_{0i} + \nabla^2 \partial^j F_{ji} = 0 \quad (3)$$

*E-mail: das@pa.uky.edu

†E-mail: murthy@pa.uky.edu

where $\nabla^2 \equiv \partial_i \partial^i$ is the spatial laplacian. The first equation in (3) may be easily solved by $F_{0i} = \epsilon_{ij} \partial^j \chi$. Using the freedom to shift χ by an arbitrary function of time, the second equation in (3) may be written as $\partial_0 \chi + \nabla^2 H_0 = 0 \Rightarrow H_0 = -\frac{\partial_0}{\nabla^2} \chi$. Monopoles are violations of the Bianchi identity for $F_{\mu\nu}$. In terms of χ the monopole charge density $\rho(t, \vec{x})$ is given by

$$\rho(t, \vec{x}) = \partial_\mu H^\mu = \partial_0 H_0 + \nabla^2 \chi \quad (4)$$

The solution to these equations is given by $\chi(t, \vec{x}) = \int dt' d^2 x' G_0(t-t', \vec{x}-\vec{x}') \rho(t', \vec{x}')$ where

$$\left(-\frac{\partial_0^2}{\nabla^2} + \nabla^2 \right) G(t-t', \vec{x}-\vec{x}') = \delta(t-t') \delta^2(\vec{x}-\vec{x}') \quad (5)$$

In momentum space, the Green's function is $G_0(\omega, \vec{k}) = \frac{\vec{k}^2}{\omega^2 + \vec{k}^4}$. A point monopole at the origin has $\rho(t, \vec{x}) = q \delta(t) \delta^2(\vec{x})$. In our conventions Dirac quantization requires $q = 2\pi n$ with $n = 0 \pm 1, \pm 2 \dots$. The Green's function is the magnetic potential for a monopole of charge $q = 1$ at the origin.

The classical action for a monopole charge distribution is

$$S_\rho = \frac{1}{2g^2} \int \frac{d\omega d^2 \vec{k}}{(2\pi)^3} \frac{\vec{k}^2}{\omega^2 + \vec{k}^4} \rho(\omega, \vec{k}) \rho(-\omega, -\vec{k}) \quad (6)$$

To see if monopoles are relevant we need to calculate the action for a single monopole of charge q . From Eq. (6) this is easily seen to be $S_1 = \frac{1}{2g^2} \int \frac{d\omega d^2 \vec{k}}{(2\pi)^3} \frac{\vec{k}^2}{\omega^2 + \vec{k}^4}$. This is of course divergent in the ultraviolet because of self energy [27], but has no infrared divergence as would be present for vortices in two space-time dimensions. Consequently, the entropy factor for a monopole always dominates in the large volume limit. This means that monopoles proliferate in the vacuum for any value of the coupling.

The partition function of this monopole gas may be represented as a functional integral over two scalar fields ϕ_1 and ϕ_2 , $e^{-S_\rho} = \int D\phi_1 D\phi_2 e^{-S[\phi_1, \phi_2]}$, where

$$S[\phi_i] = \frac{1}{2} \int d^3 x \left[2i\phi_1 \partial_0 \phi_2 + (\nabla \phi_1)^2 + (\nabla \phi_2)^2 - \frac{2i}{g} \rho \phi_1 \right] \quad (7)$$

Assuming a dilute gas of monopoles with charges $0, \pm 1$ [18], we get $Z_{gas} = \int D\phi_1 D\phi_2 e^{-S_{SG}[\phi_1, \phi_2]}$, where $S_{SG}[\phi_1, \phi_2]$ is a non-relativistic Sine-Gordon model

$$S_{SG}[\phi_1, \phi_2] = \frac{g^2}{8\pi^2} \int d^3 x \left[2i\phi_1 \partial_0 \phi_2 + (\nabla \phi_1)^2 + (\nabla \phi_2)^2 - 2M^2 \cos \phi_1 \right] \quad (8)$$

where we have rescaled the fields ϕ_1 and ϕ_2 . The mass scale is $M^2 = \frac{8\pi^2 \zeta}{g^2}$ where ζ is the fugacity determined by the monopole self-action (which includes the one loop contribution).

The theory of Eq. (8) has gapless modes, unlike its relativistic counterpart. From the Lorentzian signature action corresponding to (8) we see that the momentum conjugate of ϕ_1 is $\Pi_1 = -\phi_2$. The corresponding hamiltonian is

$$H = \int d^2 x \frac{1}{2} \left[\frac{4\pi^2}{g^2} (\nabla \Pi_1)^2 + \frac{g^2}{4\pi^2} \{ (\nabla \phi_1)^2 - 2M^2 \cos \phi_1 \} \right] \quad (9)$$

The original shift symmetry of the field $B = F_{12}$ now manifests itself as a shift symmetry of Π_1 . It is easy to check that the energy of a single particle state of the linearized hamiltonian is

$$E(\vec{k}) = |\vec{k}| \sqrt{\vec{k}^2 + M^2} \quad (10)$$

Thus the presence of a gapless mode results from the shift invariance of F_{12} , and is protected by it to all orders in perturbation theory. The gapless mode is fact a goldstone mode for a spontaneously broken shift symmetry.

The propagator matrix for the fields (ϕ_1, ϕ_2) is given by $G_{ab} = \frac{1}{\omega^2 + M^2 \vec{k}^2 + \vec{k}^4} \begin{pmatrix} \vec{k}^2 & -\omega \\ \omega & \vec{k}^2 + M^2 \end{pmatrix}$ with poles at $\omega = \pm i|\vec{k}| \sqrt{\vec{k}^2 + M^2}$. It is significant that the monopole density ρ couples only to ϕ_1 , because ϕ_2 remains massless to all orders in perturbation theory. In fact, the saddle point equation for ϕ_2 is $i\partial_0 \phi_1 = -\nabla^2 \phi_2 \Rightarrow \phi_2 = -i\frac{\partial_0}{\nabla^2} \phi_1$. Noting that up to a factor of i , ϕ_1 is none other than the field χ of Eq. (5), we realize that $i\phi_2 = H_0 = F_{12}$.

Now let us get back to monopoles. Following the steps in [18], introducing a source J for the monopoles and shifting the field ϕ_1 , the generating functional for correlation functions of the monopole density is seen to be $Z[J] = \int D\phi_1 D\phi_2 e^{-S_{SGJ}[\phi_1, \phi_2, J]}$, where

$$S_{SGJ}[\phi_1, \phi_2] = \frac{g^2}{8\pi^2} \int d^3 x \left[2i(\phi_1 - J) \partial_0 \phi_2 + (\nabla(\phi_1 - J))^2 + (\nabla \phi_2)^2 - M^2 \cos \phi_1 \right] \quad (11)$$

In the quadratic approximation, ($\cos \phi_1 \sim 1 - \frac{1}{2} \phi_1^2$) we can now easily obtain the two point function of the monopole density in momentum space, $\langle \rho(\omega, \vec{k}) \rho(-\omega, -\vec{k}) \rangle = \frac{M^2(\omega^2 + \vec{k}^4)}{\omega^2 + \vec{k}^2(\vec{k}^2 + M^2)}$, which shows that the monopoles have a residual long-range interaction, and are incompletely screened. The full two point function of the gauge invariant field strength is a sum of the classical contribution from the monopole gas and the one loop contribution from fluctuations around the monopole gas. Since the theory (1) is quadratic, the latter is the same as that in the absence of the monopole gas background, i.e. equation (2). The contribution from the monopole gas is obtained by using (5) to obtain $H_\mu^{monopole}(\omega, \vec{k})$ in terms of $\rho(\omega, \vec{k})$ and then using the correlator of $\rho(\omega, \vec{k})$.

The result is

$$\langle \tilde{H}_\mu(k^\alpha) \tilde{H}_\nu(-k^\alpha) \rangle_{mon} = \frac{M^2 k_\mu k_\nu \vec{k}^4}{(\omega^2 + \vec{k}^4)(\omega^2 + M^2 \vec{k}^2 + \vec{k}^4)} \quad (12)$$

Adding the contributions from (2) and (12) we finally get the following total correlators

$$\langle \tilde{H}_\mu(k^\alpha) \tilde{H}_\nu(k^\alpha) \rangle_{total} = \delta_{\mu\nu} - \frac{k_\mu k_\nu \vec{k}^2}{\omega^2 + M^2 \vec{k}^2 + \vec{k}^4} \quad (13)$$

The perturbative poles at $\omega = \pm i\vec{k}^2$ have been removed by the monopole gas. The poles of the full propagator are again at $\omega = \pm i|\vec{k}|\sqrt{M^2 + \vec{k}^2}$, as in the parent sine-Gordon theory. For $\vec{k}^2 \ll M^2$ this is a relativistic dispersion relation with the speed of light given by M . However, we do not regain $z = 1$ electrodynamics in this limit since the redefined field strengths are related to the original field strengths nonlocally.

As mentioned previously, the remaining gapless mode in our theory is a result of the invariance of the original action to shifts in F_{12} . However, recall that (1) is obtained from a CP^{N-1} model in the large- N limit by integrating out the spinon fields[14]. The action has an additional irrelevant, but essentially singular term of the form $\frac{B^{\frac{3}{2}} m^{\frac{1}{2}}}{4\pi^2 \sqrt{2}} e^{-\frac{\pi m}{B}}$, where m is the dynamically generated mass of the spinon fields [14]. In our present analysis m has been taken to be at the cutoff scale. It is possible that this violation of the shift invariance, though irrelevant, could lead to a nonperturbative gapping of the gapless mode. Note that $1/N$ corrections will merely shift the multicritical point where Eq. (1) applies[14], and are not capable of generating a full gap for the initially gapless mode.

Let us now turn to another aspect of $2 + 1$ compact electrodynamics, namely, the confinement of infinitely heavy quarks. To understand this we need to calculate the behavior of Wilson loops as they grow large. Consider a Wilson loop along a contour \mathcal{C} , $W_{\mathcal{C}} = \exp\left(i e \int_{\mathcal{C}} A_\mu dx^\mu\right) = \exp\left(i e \int_{\mathcal{S}} H_\mu d\sigma^\mu\right)$ where \mathcal{S} is the surface which is bounded by \mathcal{C} . $W_{\mathcal{C}}$ can be factored into a product of a ‘‘classical’’ monopole contribution, which we will evaluate via saddle point, and a ‘‘quantum’’ contribution due to fluctuations around the saddle point. The classical contribution may be rewritten via $\int_{\mathcal{S}} H_\mu d\sigma^\mu = \int d^3x \rho(x) \eta_{\mathcal{C}}(x)$, and subsequently in terms of the generating function of monopole density correlations as $[W_{\mathcal{C}}]_{classical} = Z[J = e\eta_{\mathcal{C}}]$, where the source $\eta_{\mathcal{C}}$ may be written down explicitly for simple loops. Let us first focus on the canonical ‘‘timelike’’ Wilson loop in the $x_2 = 0$ plane. For this loop we have $\eta_{\mathcal{C}} = \frac{\partial}{\partial x_2} \int dt' d^2x' G_0(t - t', \vec{x} - \vec{x}') \delta(x'_2) \Theta_{\mathcal{S}}(t'x'_1)$. Here G_0 is the Green’s function of Eqs. (5), and the Θ -function is unity on the surface \mathcal{S} and zero outside it. To evaluate

$Z[J = e\eta_{\mathcal{C}}]$, we shift the field ϕ_1 by $e\eta_{\mathcal{C}}$, integrate out ϕ_2 (possible because it appears purely quadratically), and look at the saddle point equation for ϕ_1 . Using (5) the saddle point equation becomes

$$\left(-\frac{\partial_0^2}{\nabla^2} + \nabla^2\right) \phi_1 = 2\pi \frac{e}{g} \delta'(x_2) \Theta_{\mathcal{S}} + M^2 \sin \phi_1 \quad (14)$$

Consider a timelike loop with linear dimensions T, L . For $TM^2 \gg 1$ we can ignore the dependence of ϕ_1 on t . In fact, at any point far enough away from the boundary of the loop, ϕ_1 is independent of both t and x_1 . In this case the differential equation reduces to the corresponding equation in the $z = 1$ case, and has the solution

$$\phi_1(x_2) = 4 \operatorname{sgn}(x_2) \tan^{-1} \left(e^{-M|x_2|} \tan\left(\frac{\pi e}{4g}\right) \right) \quad (15)$$

For a quantized charge $e = g \times \text{integer}$, one has a non-trivial solution only for *odd* multiples of g , leading to an area law. Even multiples of g lead, as in the $z = 1$ case, to $\phi_1 = 4 \operatorname{sgn}(x_2)$ which has zero action [20, 21] [28]. However, this solution has to be modified in the region close to the loop, as well as far away from the loop in the x_2 direction to satisfy the boundary condition for large values of x_2 [22].

We now proceed to an explicit calculation in the case of e being an odd multiple of g , in a linearized approximation, ($\sin \phi_1 \sim \phi_1$), valid far from the surface. Using the standard representation of the step function, the momentum space solution is $\phi_1(\omega, \vec{k}) = 8i\pi \frac{k_2 \sin(\omega T/2) \sin(k_1 L/2)}{(\omega - i\epsilon)(k_1 - i\epsilon)(\frac{\omega^2}{\vec{k}^2} + \vec{k}^2 + M^2)}$ where we will let $\epsilon \rightarrow 0^+$ in the end. Now we evaluate the saddle point action, $S = \int \frac{d\omega d^2k}{(2\pi)^3} (\frac{\omega^2}{\vec{k}^2} + \vec{k}^2 + M^2) |\phi_1(\omega, \vec{k})|^2$. Neglecting unimportant overall factors we obtain

$$S \simeq \int \frac{d\omega d^2k}{(2\pi)^2} \frac{1 - \cos(\omega T)}{\omega^2 + \epsilon^2} \frac{1 - \cos(k_1 L)}{k_1^2 + \epsilon^2} k_2^2 \vec{k}^2 \left(\frac{1}{\omega^2 + \vec{k}^2(\vec{k}^2 + M^2)} - (M \rightarrow \Lambda) \right) \quad (16)$$

where the linear ultraviolet divergence of the k_2 integral has been removed by a Pauli-Villars subtraction with a cutoff Λ . Clearly there is no divergence in the infrared, even when $\epsilon \rightarrow 0$. The ω integral can be carried out by contour integration. The poles at $\pm i\epsilon$ produce terms proportional to T as $T\epsilon \rightarrow 0^+$. It is evident that the term which is not proportional to T is also not divergent as $T \rightarrow \infty$, and can therefore at most lead to a perimeter correction. Ignoring this, we carry out the k_1 integration by exactly the same methods, and obtain a dominant contribution proportional to TL ,

$$E \simeq TL \int \frac{dk_2}{2\pi} k_2^2 \left(\frac{1}{k_2^2 + M^2} - \frac{1}{k_2^2 + \Lambda^2} \right) \quad (17)$$

This explicitly shows the area law, showing the corresponding charges are confined.

Another surprise is obtained when we calculate the action for a spacelike Wilson loop. Here one starts from the exponent $\int dx_1 dx_2 H_0$. Recalling that $H_0 = -\frac{\partial_0}{\nabla^2} \phi_1$ we obtain the saddle-point equation

$$\left(-\frac{\partial_0^2}{\nabla^2} + \nabla^2\right)\phi_1 = 2\pi\frac{\partial_0}{\nabla^2}(\delta(t)\Theta(x_1x_2)) + M^2 \sin \phi_1 \quad (18)$$

In the linearized approximation the spacelike Wilson loop can be calculated using the same procedure used for time-like loops. In this case, the integrals are convergent in both the ultraviolet and the infrared. For the special case $L_1 = L_2 = L$ we can scale out L , and for $LM, L\Lambda \gg 1$, we find that $\langle W(C) \rangle \sim e^{-S}$ where $S \simeq L^3(\Lambda - M)$. In this anisotropic theory, there is no reason to expect the action of the spacelike loop to go with the area law, but it still surprising to find that it goes faster. The reason is the nonlocal right hand side in the saddle point equation (18).

In the $z = 1$ theory, a monopole source is Debye screened by the surrounding gas of monopoles. In our case, because of the anisotropic nature of the bare interaction, there is a residual long-range interaction even after screening. The interaction between two monopoles in the gas behaves as $(|\Delta\vec{x}|^2 - 2M^2(\Delta t)^2)/(|\Delta\vec{x}|^2 + M^2(\Delta t)^2)^{\frac{5}{2}}$ for large $|\Delta\vec{x}|$ and Δt , showing the incomplete screening. However, the potential between static electric charges is sensitive to only the zero frequency part of the Green's function of Eq. (5), which is identical to the corresponding quantity in the $z = 1$ theory, and leads to confinement. Thus, the phenomena of the confinement of minimal charges, and the Debye screening of monopoles, which were coupled in the $z = 1$ model, are now decoupled due to the space vs time anisotropy.

The presence of the gapless mode would be manifested in the behavior of bulk quantities, e.g. the low temperature behavior of the specific heat. It remains to be seen if non-analytic terms in the action which arise in the effective gauge theory which follows from CP^{N-1} model [14] changes this conclusion. Naively these terms are irrelevant since they vanish faster than any power of B , and would lead to a gap for this mode which is much smaller than the scale of the string tension. One must also consider the possibility that the instantons of the theory come with different phases on the different plaquettes of the lattice, as in [26]. Finally, it remains to be seen if a concrete spin model such as the one in Ref. [7] for the one-component Lifshitz theory can be constructed which displays the behavior presented here.

We would like to thank Noah Bray-Ali, Al Shapere and Spenta Wadia for fruitful conversations, and the NSF for partial support under grants PHY-0555444 and PHY-0855614 (SD) and DMR-0703992 (GM).

-
- [1] R. M. Hornreich, M. Luban and S. Shtrikman, Phys. Rev. Lett. **35**, 1678 (1975); G. Grinstein, Phys. Rev. B **23**, 4615 (1981).
 - [2] D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988); E. Fradkin and S. A. Kivelson, Mod. Phys. Lett. B **4**, 225 (1990).
 - [3] E. Fradkin, D. A. Huse, R. Moessner, V. Oganesyan, and S. L. Sondhi, Phys. Rev. B **69**, 224415 (2004).
 - [4] E. Ardonne, P. Fendley and E. Fradkin, Annals Phys. **310**, 493 (2004) [arXiv:cond-mat/0311466]; S. Papanikolaou, E. Luijten and E. Fradkin, Phys. Rev. **134514** (2007) [arXiv:cond-mat/0607316]
 - [5] C. L. Henley, Jour. Stat. Phys. **89**, 483 (1997).
 - [6] L. S. Levitov, Phys. Rev. Lett. **64**, 92 (1990).
 - [7] A. Vishwanath, L. Balents, and T. Senthil, Phys. Rev. B **69**, 224416 (2004).
 - [8] P. Horava, arXiv:0811.2217 [hep-th].
 - [9] P. Horava, Phys. Rev. D **79**, 084008 (2009) [arXiv:0901.3775 [hep-th]].
 - [10] M. Visser, arXiv:0902.0590 [hep-th].
 - [11] D. Anselmi, Annals Phys. **324**, 874 (2009) [arXiv:0808.3470 [hep-th]]; D. Anselmi, Annals Phys. **324**, 1058 (2009) [arXiv:0808.3474 [hep-th]]; D. Anselmi, arXiv:0904.1849 [hep-ph].
 - [12] A. Dhar, G. Mandal and S. R. Wadia, arXiv:0905.2928 [hep-th].
 - [13] R. Iengo, J. G. Russo and M. Serone, arXiv:0906.3477 [hep-th].
 - [14] S. R. Das and G. Murthy, Phys. Rev. D **80**, 065006 (2009), [arXiv:0906.3261 [hep-th]].
 - [15] R. Moessner and S. L. Sondhi, Phys. Rev. B **68**, 184512 (2003).
 - [16] M. Hermele, M. P. A. Fisher, and L. Balents, Phys. Rev. B **69**, 064404 (2004).
 - [17] M. Freedman, C. Nayak, and K. Shtengel, Phys. Rev. Lett. **94**, 147205 (2005).
 - [18] A. M. Polyakov, Phys. Lett. B **59**, 82 (1975); A. M. Polyakov, Nucl. Phys. B **120**, 429 (1977).
 - [19] A. M. Polyakov, *Gauge Fields and Strings*, Chur, Switzerland: Harwood (1987).
 - [20] S. R. Wadia and S. R. Das, Phys. Lett. B **106**, 386 (1981) [Erratum-ibid. B **108**, 435 (1982)].
 - [21] S. R. Das and S. R. Wadia, Phys. Rev. D **53**, 5856 (1996) [arXiv:hep-th/9503184].
 - [22] N. J. Snyderman and P. Q. Hung, LBL Preprint No. 15353 (1982).
 - [23] M. Lau and C. Dasgupta, Phys. Rev. B **39**, 7212 (1989).
 - [24] M. Kamal and G. Murthy, Phys. Rev. Lett. **71**, 1911 (1993).
 - [25] O. Motrunich and A. Vishwanath, Phys. Rev. B **70**, 075104 (2005).
 - [26] T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, *Science* **303**, 1490 (2004); Phys. Rev. B **70**, 14407 (2004).
 - [27] Note that our theory always has a finite cutoff
 - [28] When the $U(1)$ theory is embedded in $SU(2)$ this leads to screening of adjoint representations.