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Spontaneous Compactification of Generalised Kaluza-Klein theories on Twisted Tori

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ABSTRACT

We explore the possibility of compactification of Kaluza-Klein models containing elementary gauge fields into tori with the internal components of the gauge fields satisfying twisted boundary conditions. We illustrate the idea in a toy model.

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Kaluza-Klein theories provide attractive scenarios for unification of the fundamental interactions. In "pure" Kaluza-Klein models four dimensional gravity and gauge fields are all obtained from pure gravity in higher dimensions. The extra dimensions are supposed to wrap up into a compact manifold; isometries of the latter become gauge symmetries of the effective four dimensional world [1]. In such models, however, it is impossible to obtain chirally coupled fermions [2].

A much less ambitious program starts with gravity and elementary gauge fields in higher dimensions. The motivation behind such models lies in the fact that vector or antisymmetric tensor gauge fields occur naturally in supergravity theories. Otherwise the gauge fields may be thought of as fields obtained by compactification from still higher dimensions. In this case one might have a vacuum in which topologically non-trivial gauge fields provide the energy-momentum tensor necessary for compactification and at the same time ensure the existence of chiral fermions in four dimensions [3,4]. Several examples of this mechanism have been considered in the literature : with two extra dimensions compactifying on S^2, the gauge field is that of a monopole [3]; while with four extra dimensions compactifying on S^4 one has an instanton [4].

In this note we shall explore the possibility of spontaneous compactification induced by gauge field topologies of a different nature. This involves internal manifolds which are tori. Usually a field theory defined on a torus means that all fields satisfy periodic boundary conditions. However, in gauge theory one has the wider possibility of having boundary conditions which are periodic upto gauge transformations. For SU(N) theories such gauge transformations fall into several topologically distinct classes : the topology being characterised by a certain Z_N valued twist tensor [5]. A non-trivial twist (in a sense to be defined shortly) corresponds to some electric and magnetic fluxes running through the torus. With a suitably chosen twist, the field configuration also carries a nonzero Pontryagin number [5]. In the following we shall look for solutions of the equations of motion of Kaluza-Klein type theories which involve nonzero fluxes in the internal dimensions. We shall illustrate the idea by considering a toy model. However, we believe that the idea might be useful in more realistic theories.

Our toy model consists of Einstein gravity, SU(N) Yang-Mills fields and adjoint representation fermions in eight dimensions. With the fermionic fields set to zero (as expected in the ground state),the equations of motion read :

$$\mathcal{R}_{AB} = -k^2\left[T_{AB} - \frac{1}{6} g_{AB} T + \frac{1}{3}\lambda g_{AB}\right]$$

$$\nabla_A F^A{}_B = 0 \qquad \ldots\ldots (1)$$

where R_{AB} is the usual Ricci tensor and T_{AB} is the energy momentum tensor for the gauge fields :

$$T_{AB} = \mathrm{Tr}\left(F_A{}^C F_{CB} - \frac{1}{4} g_{AB} F_{CD} F^{CD} \right) \quad \ldots (2)$$

λ is a cosmological constant and F_{AB} is the gauge field strength of the SU(N) gauge potential A_C. The covariant derivative ∇_A contains both the usual Riemann connection as well as the gauge connection.

In a suitable orthonormal frame the ground state solution has the form :

$$\begin{aligned} g_{AB}(x_C) &= \left(\begin{array}{c|c} g_{\mu\nu}(x_\alpha) & 0 \\ \hline 0 & g_{ij}(x_k) \end{array} \right) \\ A_\mu &= 0 \\ A_i(x_A) &= A_i(x_k) \end{aligned} \quad \ldots . (3)$$

where $g_{\mu\nu}$ and g_{ij}, the metrics in the four-dimensional space-time and the internal space are diagonal. (Upper case latin indices are eight-dimensional, lower case latin indices run over the four internal dimensions and lower case greek indices refer to ordinary four dimensional space-time.) The field equations now separate :

$$\begin{aligned} F_\mu{}^\nu = F_i{}^\mu &= 0 \\ \nabla_i F_j{}^i &= 0 \end{aligned} \quad \ldots (4)$$

$$R_{ij} = -k^2 \left[T_{ij} - \frac{1}{6} g_{ij} T + \frac{1}{3} \lambda g_{ij} \right]$$

$$R_{\mu\nu} = -k^2 \left[T_{\mu\nu} - \frac{1}{6} g_{\mu\nu} T + \frac{1}{3} \lambda g_{\mu\nu} \right]$$

where R_{ij} is the Ricci tensor obtained solely from g_{ij}, and similarly for $R_{\mu\nu}$.

Let us now briefly recall the idea of a twisted torus [5]. Consider SU(N) Yang-Mills fields on a torus of circumference a in each direction. The boundary conditions are given by :

$$A_i(x_1=a) = \Omega_1 \{ A_i(x_1=0) - (\partial_i \Omega_1^{-1})\Omega_1 \} \Omega_1^{-1}$$
$$\Omega_1 \equiv \Omega_1(x_2, x_3, x_4) \quad \text{etc.} \qquad \ldots (5)$$

The gauge functions Ω_i must satisfy the following consistency conditions :

$$\Omega_i(0)\,\Omega_j(a_i) = \exp\left[\frac{2\pi i}{N} n_{ji}\right] \Omega_j(a)\,\Omega_i(0) \quad \cdots (6)$$

The twist tensor n_{ij} must be a constant to ensure continuity. The twist is said to be orthogonal (or trivial) if

$$k \equiv \frac{1}{4}\epsilon_{ijkl}\, n^{ij} n^{kl} \qquad \ldots .(7)$$

is zero modulo N. For a non-trivial twist, the action is bounded from below :

$$S = \frac{1}{4}\int \mathrm{Tr}\, F_{ij} F^{ij} d^4x \geqslant \min_{\nu}\left(\frac{8\pi^2}{g^2} \Big| \nu - \frac{k}{N} \Big| \right) \quad \ldots (8)$$

and the Pontryagin index is fractional :

$$\mathcal{P} = \frac{g^2}{16\pi^2}\int \mathrm{Tr}\, F_{ij} \tilde{F}^{ij} d^4x = \nu - \frac{k}{N} \qquad \ldots (9)$$

where ν is the integral Pontryagin number in the absence of a non-trivial twist. Evidently , this means that the field strength must be non-vanishing.

Solutions to the gauge field equations in the presence of non-orthogonal twists have been obatined by 't Hooft [5]. For example, in the case :

$$-n_{12} = n_{34} = 1 \; ; \quad n_{13} = n_{14} = n_{23} = n_{24} = 0$$

and N = 2l (l is an integer), one has :

$$A_i(x) = -\frac{1}{2}\omega \sum_j \beta_{ij} x_j$$

$$F_{ij} = -\omega \beta_{ij} \qquad \ldots (10)$$

where $$\beta_{12} = \beta_{34} = \frac{2}{N^2 a^2}$$

and $$\omega = \mathrm{diag}\,(\underbrace{1, 1, \ldots}_{l} ; \underbrace{-1, -1. \ldots -1}_{l})$$

The above configuration, called a "toron" has P = -1/N and a vanishing energy-momentum tensor.

The toron actually describes electric and magnetic fluxes running in the 3 direction. One could, in fact, construct similar configurations in an abelian theory [f1]. Consider, for example, a four dimensional torus containing a U(1) gauge field $A_\mu(x)$. The field strength is given by the self-dual configuration :

$$F_{12} = F_{34} = B \qquad \ldots (11)$$

In a suitable gauge, A_μ is given by :

$$A_\mu = (0, Bx_1, 0, Bx_3) \quad \cdots\cdots (12)$$

The above gauge potential is ,like (10) above, periodic upto a gauge transformation. If a matter field of charge e is coupled to the gauge fields, consistency of the gauge functions relating the fields at the ends of the boundaries lead to quantisation of the magnetic-electric field :

$$B = \frac{2\pi}{ea^2} n$$

where n is an integer. The analog of the Pontryagin index (i.e the integral of $\vec{E}.\vec{B}$ with suitable coefficients) is given by $P = (1/2)n^2$.

Antisymmetric tensor gauge fields ,in suitable number of dimensions, can also form toron configurations. A particular example is a second rank abelian gauge potential A_{mn} in a six dimensional torus.

Let us now investigate whether the equations of motion of our toy model, equations (4) admit solutions in which the internal space is a twisted torus. This means we must have $g_{ij} = \delta_{ij}$, but a constant nonzero F_{ij} given by (9) or its abelian analog. Since a toron has $T_{ij} = 0$, $R_{ij} = 0$ requires :

$$\lambda = \frac{1}{2} T = -\frac{1}{2} \mathrm{Tr}\, F^2 \quad \cdots (14)$$

The Einstein equations for ordinary space-time now become :

$$R_{\mu\nu} = -k^2 T_{\mu\nu}$$

where

$$T_{\mu\nu} = -\mathrm{Tr}\ F_{\mu}{}^{A} F_{\nu A} - \frac{1}{4} g_{\mu\nu} \mathrm{Tr}\ F^2 = -\frac{1}{4} g_{\mu\nu} \mathrm{Tr}\ F^2$$

Thus one has

$$R_{\mu\nu} = g_{\mu\nu} \frac{k^2}{4} \mathrm{Tr}\ F^2 = \frac{k^2}{4\pi^2 a^4} g_{\mu\nu} \qquad \ldots(15)$$

Ordinary space-time thus has a constant negative curvature, e.g. an anti de Sitter space. The scale of the curvature is set by the size of the internal dimensions. This might not be as disastrous as it looks : adS space-times occur frequently in compactifications of higher dimensional supergravity [7] and it has been, in fact, suggested that quantum effects tend to decrease the curvature [8].

The fact that T_{ij} is zero is crucial in ensuring that the twisted torus is a solution of the Einstein equations. This, in turn, is ensured by the self-duality of the toron. It may be, in fact, argued that our model compactifies into a twisted four torus and a four dimensional space-time simply because the field F_{ij} can be self-dual only in four dimensions.

Adjoint representation fermions may be coupled to the above background field in a straightforward way. Let us consider such fermions having an eight dimensional chirality $\bar{\Gamma}_5 = +1$. (A similar discussion holds for $\bar{\Gamma}_5 = -1$ fermions). The low energy fermions in the effective four dimensional space-time must be zero modes of the internal Dirac operator [2] :

$$\not{D}^{(int)} \psi = 0$$
$$\not{D}^{(int)} \equiv \Gamma^i D_i$$

where Γ_i are the gamma matrices and D_i the covariant derivatives in the internal space. One can choose a representation of the gamma matrices such that

$$\bar{\Gamma}_5 = \gamma_5 \, \Gamma_5^{(int)}$$

γ_5 and Γ_5^{int} denote the chiralities in ordinary space-time and internal space respectively. This implies that solutions of (16) which automatically has a definite Γ_5^{int} also has a definite chirality in the effective four dimensional world. One would have low energy chiral fermions if each $\Gamma_5^{int} = +1$ solution of equation (16) is not accompanied by a $\Gamma_5^{int} = -1$ solution.

In our background field D denotes the Dirac operator in a flat torus in the presence of a gauge field $A_i(x)$ given by eqn.(10). Since the Pontryagin index of the background field is $P = -1/N$, the Atiyah-Singer index theorem implies

:

$$n_+ - n_- = 2NP = -2$$

where $n_+(n_-)$ denotes the number of $\Gamma_5^{int} = +1(-1)$ zero modes of D. (See Ref.[9] for a discussion of these zero modes). Since the background is self-dual eqn.(17) is satisfied minimally, i.e. with $n_+ = 0$, $n_- = 2$. (The opposite is true of an anti-self-dual background). Compactification on twisted tori thus leads to chiral fermions. These are coupled to the $(U(1))^4$ gauge fields arising from gravity by compactification.

In our model the gauge fields obtained from gravity are abelian. However, it is believed that in higher dimensional supergravity theories non-Abelian gauge fields are dynamically generated in a fashion analogous to two dimensional sigma models [10]. If such models undergo compactification in twisted tori, it is conceivable that fermions shall form complex representations of the non-Abelian symmetry group.

One possibility is that the twisted torus exists as a solution to the N=2 Chiral Supergravity theory in 10 dimensions [6]. The bosonic sector of this theory consists of a complex scalar, a complex second rank antisymmetric tensor gauge field, a real rank four antisymmetric tensor and a graviton. As mentioned above a rank two tensor gauge field can form a toron in six dimensions. One might wonder whether there are solutions of the equations of motion in which six of the dimensions wrap up in a twisted torus. We do not know whether such a solution exists.(There is such a solution if the supersymmetry is broken by adding a cosmological term).If it does, it would necessarily involve several of the abovementioned scalar fields acquiring vacuum expectation values.

Admittedly, the model described in this note has little resemblence to reality. The twisted torus has,however, some interesting properties. The toron is a fairly general object ; it occurs in abelian and non-abelian theories and it is present in theories with antisymmetric tensor gauge fields as well as in usual vector gauge theories. It is certainly worthwhile to try to implement the idea in more realistic theories.

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FOOTNOTE

f1. This was suggested to me by E.Witten.

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