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Open Wilson Lines in Noncommutative Gauge Theory and Tomography of Holographic Dual Supergravity [1]

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abstract

We study the issue of gauge-invariant observables in $d = 4, \mathcal{N} = 4$ noncommutative gauge theory and UV-IR relation therein. We show that open Wilson lines form a complete set of gauge invariant operators, which are local in *momentum* space and, depending on their size, exhibit two distinct behaviors of the UV-IR relation. We next study these properties in a proposed dual description in terms of supergravity and find agreement.

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1 Introduction

Field theories defined on noncommutative spaces have attracted a lot of attention recently [1], particularly, because of the appearance of noncommutative Yang-Mills (NCYM) theories as low energy limits of open string theories in the background of NS-NS two-form potential $B^{\rm NS}$ [2]. A crucial consequence of such noncommutativity is the growth of transverse size of objects with increasing momentum [3]. On the other hand, one expects that the large-N NCYM theory has a description in terms of a supergravity dual when the 't Hooft coupling $\lambda_{\rm eff} \equiv 4\pi g^2_{\rm YM} N$ is large. Such dual backgrounds with nonzero 2-form $B^{\rm NS}$ have been proposed [4, 5] and several aspects of the holographic map have been studied in [6].

In the absence of $B^{\rm NS}$ background, e.g. for the duality between $N = 4$ SYM theory in 3+1 dimensions and supergravity in $AdS_5 \times S^5$, the scale in the YM theory is related to the radial coordinate in AdS_5 [7, 8]. For example, a source for some supergravity mode located in the bulk of AdS_5 induces an expectation value of the dual operator of the boundary Yang-Mills theory [9]. If the background metric is chosen to be

$$ds^2 = u^2[-(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2] + \frac{(du)^2}{u^2} + (d\Omega_5)^2, \tag{1}$$

then the latter expectation value has a support over a region of size (in the $x_1 \cdots x_3$ directions) $\sim \ \mathcal{O}(1/\bar{u})$, where $\bar{u}$ refers to the location of the supergravity source. As the source moves further away from (closer to) the boundary, the size of the corresponding disturbance in the boundary theory increases (decreases).

In NCYM theory, there are no local, gauge-invariant operators. However, the theory has translation invariance. States in the theory are still labelled by the energy and the momenta and hence there ought to be operators in *momentum* space which create these states. Consequently, if we place a source in the dual supergravity background which has definite momentum along the $x^1 \cdots x^3$ directions and a definite location in the remaining "radial" direction, this should induce an expectation value of some operator in the gauge theory with the same value of the momentum.

In this paper, we argue that the most natural basis of gauge-invariant operators of NCYM theory is provided by open Wilson line operators with definite momentum, originally constructed in [10] and identified with macroscopic fundamental strings in [11]. We find that gauge invariance requires 'size' of the Wilson line proportional to 'momentum' along the noncommutative directions in a way consistent with the expected behavior in noncommutative theories. We then consider the dual supergravity and study the hologram of a source for a given supergravity mode by computing a one-point correlation function. Computation of correlation functions in such supergravity backgrounds via evlauation of the supergravity action is generally ambiguous because of the necessity of momentum-dependent wave function renormalizations. Our computation follows the unambiguous prescription proposed in [12]. From the momentum dependence of the one-point correlators, we 'tomograph' the profile of the hologram. When the source is located deep inside the bulk, we show that the relationship between the location of the source and the size of the hologram approaches the standard relation found in the absence of the $B^{\rm NS}$-field : the size of the hologram increases as the source moves further into the bulk. This is expected, since the region deep in the bulk corresponds to the infrared regime of NCYM theory where the effects of noncommutativity are invisible. However, when the source is located near the boundary we find an opposite relationship : as the source moves closer to the boundary, the size of its hologram *increases*. The relationship between the hologram size and location of the source is found to be consistent with the relationship between the 'size' and the 'momentum' of the open Wilson line operator in NCYM theory. Thus, the proposed supergravity duals indeed encode the UV/IR relationship of NCYM theory.

2 Open Wilson Lines in Noncommutative Gauge Theory

2.1 Noncommutatitve Gauge Theory

We will begin with a brief recapitulation of noncommutative gauge theory and set notations. In this section, for simplicity, we will be considering gauge group U(1). Extension to U(N) group is straightforward and only involves introduction of matrix-valued gauge fields. We start with definition of the generalized Moyal product:

$$\phi_1(x) \star \phi_2(y) \equiv \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu \partial^y_\nu\right) \phi_1(x)\phi_2(y) \tag{2}$$

and Moyal commutator

$$\{\phi_1(x), \phi_2(y)\}_\star \equiv \phi_1(x) \star \phi_2(y) - \phi_2(y) \star \phi_1(x). \tag{3}$$

Throughout this paper, we will be studying 'magnetic' noncommutativity: $\theta^{23} := \theta$ is the only nonvanishing component. In the context of D3-brane in Type IIB string theory, the noncommutativity parameter is determined by nonzero NS 2-form potential $B^{\rm NS}_{23}$:

$$\theta^{\mu\nu} = \left(\frac{1}{B_{\rm NS}}\right)^{\mu\nu} \qquad \text{or,} \quad \text{equivalently,} \qquad \theta^{\mu\nu} B^{\rm NS}_{\nu\lambda} = \delta^\mu_\lambda. \tag{4}$$

Turning on θ breaks the underlying SO(3,1) Lorentz invariance to SO(1,1) Lorentz times SO(2) rotational invariance on the commutative and noncommutative subspaces, respectively [2] and the commutative subspace coordinates as $\mathbf{x}_\perp, \mathbf{y}_\perp, \mathbf{z}_\perp, \cdots$. For both commutative and noncommutative directions, translational symmetry remains intact. As such, energy and momentum are conserved quantities and can be employed in labelling states and operators.

[2] In what follows, we will denote the four-dimensional coordinates as $x, y, z \cdots$ and the noncommutative subspace coordinates as $\mathbf{x}, \mathbf{y}, \mathbf{z} \cdots$.

Introduce U(1) gauge connection $\mathbf{A}_\mu(x)$ and define gauge-*covariant* field strength $\mathbf{F}_{\mu\nu}(x)$ in terms of the generalized Moyal product:

$$\mathbf{F}_{\mu\nu}(x) \equiv (\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)(x) + \{\mathbf{A}_\mu, \mathbf{A}_\nu\}_\star . \tag{5}$$

Gauge transformations of the noncommutative U(1) group is defined as

$$\begin{aligned} \delta_\epsilon \mathbf{A}_\mu(x) &= \partial_\mu \epsilon(x) + i\{\epsilon, \mathbf{A}_\mu\}_\star(x) \\ \delta_\epsilon \mathbf{F}_{\mu\nu}(x) &= i\{\epsilon, \mathbf{F}_{\mu\nu}\}_\star(x) \end{aligned} \tag{6}$$

Neutral scalar fields $\mathbf{\Phi}^a(x)$ $(a = 1, \cdots, 6)$ in 'adjoint' representation transform similarly:

$$\delta_\epsilon \mathbf{\Phi}^a(x) = i\{\epsilon, \mathbf{\Phi}^a\}_\star(x). \tag{7}$$

Four-dimensional, noncommutative U(1) gauge theory arising from the low-energy world-volume dynamics of a D3-brane in nonzero $B^{\rm NS}_{\mu\nu}$ background is then defined by the following action:

$$S = \frac{1}{4g^2}\int d^4x \, \left(\mathbf{F}_{\mu\nu}\star\mathbf{F}^{\mu\nu} + D_\mu\mathbf{\Phi}^a \star D^\mu\mathbf{\Phi}^a + \left\{\mathbf{\Phi}^a, \mathbf{\Phi}^b\right\}^2_\star\right). \tag{8}$$

2.2 Parisi's Composite Operators

One distinguishing characteristic of noncommutative gauge theories is that degrees of freedom in spacetime and in color space are all intertwined. This is rather obvious from the following simple observation. In rewriting a conventional gauge theory defined on noncommutative spacetime as a noncommutative gauge theory on commutative spacetime, one transmutes the *color* degrees of freedom into the *spacetime* degrees of freedom along the noncommutative directions. Thus, local observables of the form $\mathrm{Tr}\,\hat{\mathcal{O}}$ in the former theory are now mapped into highly non-local ones of the form $\int d\mathbf{z}\,\hat{\mathcal{O}}(x)$ in the latter, where $\mathbf{z} \subset x$ refers to coordinates along the noncommutative directions. From this observation, it follows that, in general, it is impossible to define local operators in noncommutative gauge theories [3] .

If there are no gauge-invariant operators except the ones integrated over the entire spacetime carrying zero energy and momentum, how can one even probe low-momentum, low-energy excitations? Actually, there is a class of gauge-invariant operators, which are sort of *semi-localized* and hence may be used for probing the noncommutative gauge theory excitations. These so-called Parisi operators [13] carry definite energy and momentum, which are good quantum numbers in noncommutative spacetime, and are defined by Fourier modulation of a string of an elementary fields, $\phi_k(x)$, $(k = 1, 2, \cdots)$:

$$\mathcal{O}_n(x_1, x_2, \cdots, x_n; \mathbf{k}) \equiv \int d^2\mathbf{z}\ \phi_1(\mathbf{z}+x_1)\star\phi_2(\mathbf{z}+x_2)\star\cdots\star\phi_n(\mathbf{z}+x_n)\star e^{i\mathbf{k}\cdot\mathbf{z}}. \tag{9}$$

[3]By the same argument, it also follows that the conventional notion of the operator product expansion or of the multipole expansion does not make sense in noncommutative field theories.

Being integrated over noncommutative coordinates of all elementary fields, the Parisi operators are non-local. As viewed from momentum space, however, they are *local* operators. Thus, we take a viewpoint that physically relevant operators in noncommutative field theory are the ones which are local in configuration space for commutative directions but in momentum space for noncommutative directions. Note also that, at this stage, the multi-locations $(x_1, \cdots, x_n)$ are not directly related to the momentum vector $\mathbf{k}$ along the noncommutative directions. Thus, in a noncommutative field theory, a class of physically relevant m-point correlation functions for probing the theory is provided by:

$$G_m(\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_m) = \Big\langle \mathcal{O}_1(\mathbf{k}_1)\mathcal{O}_2(\mathbf{k}_2)\cdots\mathcal{O}_m(\mathbf{k}_m)\Big\rangle, \tag{10}$$

where, for clarity of the definition, dependence on coordinates $x_1, x_2, \cdots$ in each operators and in the resulting correlation functions are omitted.

Let us illustrate the utility and meaning of the Parisi operators in the simplest context: noncommutative scalar field theory. Take the one-point correlator:

$$G_1(x, \mathbf{k}) = \Big\langle \int d^2\mathbf{z}\ \phi(\mathbf{z}) \star \phi(\mathbf{z} + x) \star e^{i\mathbf{k}\cdot\mathbf{z}} \Big\rangle. \tag{11}$$

In order to visualize the spacetime picture, consider first the limit $x \to 0$. One finds that the Parisi operator reduces to a sort of $[\phi^2]$ composite operator, partially Fourier-transformed in the noncommutative directions, except that now all the products (including Fourier transform) are defined in terms of the Moyal product. Indeed, the above one-point correlator may be understood as follows. Introduce, in the noncommutative scalar field theory Lagrangian, a bilocal mass 'spurion' term:

$$\mathcal{S}_{m^2} = \frac{1}{2}\int d^4x \int d^4y\ \ m^2(x, y) \star [\phi(x) \star \phi(y) + \phi(y) \star \phi(x)]\,, \tag{12}$$

where all the products are the generalized Moyal product, Eq.(2). Consider the situation that $x = x, y = x + \mathbf{z}$, viz. non-local split only within the noncommutative subspace. Take the suprion mass-squared a slowly varying function in x and Fourier expand:

$$m^2(x, \mathbf{z})\ =\ \int \frac{d^2\mathbf{k}}{(2\pi)^2}\ e^{i\mathbf{k}\cdot\mathbf{z}}\,\widetilde{m^2}(x, \mathbf{k}). \tag{13}$$

One then immediately find that the one-point correlators can be derived from the partition function of the noncommutative scalar field theory:

$$G_1(x, \mathbf{k})\ \ =\ \ \frac{\delta \ln Z_{\rm NC}[\widetilde{m^2}]}{\delta \widetilde{m^2}(x, \mathbf{k})}. \tag{14}$$

Extending the result, it should be fairly obvious that generic multi-point correlators involving the non-local operators of the type Eq.(9) are derivable from variation of the partition function with respect to a set of suitable spurion couplings.

2.3 Gauge Invariance = Spacetime Translation Invariance

As mentioned above, in noncommutative field theories, degrees of freedom in spacetime and in internal space are all intertwined. In particular, this has implied, in noncommutative gauge theory, there is *no* gauge invariant, local observables. In view of its profound implication, in this subsection, we would like to understand better the meaning of the noncommutative gauge invariance. Indeed, we will be showing explicitly that noncommutative gauge invariance is identical to spacetime translational invariance and hence vast reduction of the degrees of freedom – intimately related to the Eguchi and Kawai reduction [14]. In fact, noncommutative gauge theories can be derived from twisted version of the Eguchi-Kawai models [15].

To grasp the physical meaning of the noncommutative gauge invariance, begin with Fourier decomposition of the gauge parameter along the noncommutative directions:

$$\epsilon(\mathbf{x}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\; e^{i\mathbf{k}\cdot\mathbf{x}}\; \widetilde{\epsilon}(\mathbf{k}). \tag{15}$$

Dependence on the commutative coordinates are omitted for notational simplicity. One can then re-express the noncommutative gauge transformations Eqs.(6, 7) as:

$$\begin{aligned} \delta_\epsilon \mathbf{A}_\mu(\mathbf{x}) &= i\int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}}\, \widetilde{\epsilon}(\mathbf{k}) \Big[(\mathbf{A}_\mu(\mathbf{x}+\theta\cdot\mathbf{k}) - \mathbf{A}_\mu(\mathbf{x}-\theta\cdot\mathbf{k})) + i\mathbf{k}_\mu \Big], \\ \delta_\epsilon \mathbf{\Phi}^a(\mathbf{x}) &= i\int \frac{d^2\mathbf{k}}{(2\pi)^2}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \widetilde{\epsilon}(\mathbf{k}) \Big[\mathbf{\Phi}^a(\mathbf{x}+\theta\cdot\mathbf{k}) - \mathbf{\Phi}^a(\mathbf{x}-\theta\cdot\mathbf{k}) \Big]. \end{aligned} \tag{16}$$

We recognize that the noncommutative gauge transformation is identified with the spacetime translation (plus an additional constant shift in case of the gauge fields). Take $\widetilde{\epsilon}(\mathbf{k})$ to be a Gaussian distribution with dispersion $\Delta\mathbf{k}$. Then, Eq.(16) implies that dispersion of the spacetime translation $\Delta\mathbf{x}$ is given by

$$\Delta\mathbf{x}^\mu \quad \sim \quad \theta^{\mu\nu}\Delta\mathbf{k}_\nu, \tag{17}$$

viz. size of the spacetime translation is proportional to the Fourier wave vector of the gauge transformation.

To construct gauge invariant observables, one needs to average over the gauge orbits. According to the above interpretation, such observables are the ones integrated over the entire spacetime, as the gauge orbits are identified with orbits of the spacetime translation. While this is quite true and is intimately related to the Eguchi-Kawai reduction, we show below that there exists a class of gauge-invariant operators in noncommutative gauge theory, open Wilson line operators. In fact, they turn out precisely the gauge theory counterpart of the Parisi's composite operators discussed in the last subsection.

2.4 Open Wilson Line Operators

In noncommutative gauge theory, there is a distinguished class of such *semi-local* operators, which are labelled by momentum along the noncommutative directions – *open* Wilson lines:

$$W_{\mathbf{k}}[C] = \int d^2\mathbf{x}\,\mathcal{P}\,\exp_\star\left[i\int_C \left(\dot{y}(t)\cdot\mathbf{A}(x+y(t)) + \sqrt{\dot{y}^2(t)}\hat{\Omega}(t)\cdot\mathbf{\Phi}(x+y(t))\right)\right]\star e^{i\mathbf{k}\cdot\mathbf{x}}. \qquad (18)$$

This is a generalization of the operator constructed in [10]. Here, $t = [0, 1]$ denotes an affine parameter along a contour C specifying the open Wilson line and $\mathcal{P}$ denotes the path ordering along C. The spacetime position of the base point ($t = 0$) is denoted as x^μ, which may also be viewed as encoding center-of-mass position of the Wilson line. Similarly, along the Wilson line, the spacetime image of the point t as measured relative to x^μ is denoted as $y^\mu(t)$. Distance between the two endpoints of the open Wilson line is given by:

$$\Delta y^\mu \equiv y^\mu(1) - y^\mu(0). \qquad (19)$$

$\hat{\Omega}^a(t)$ ($a = 1, 2, \cdots, 6$) refers to the angular coordinates on S_5 in Eq.(1) having unit modulus, $\hat{\Omega}(t)\cdot\hat{\Omega}(t) = 1$. For simplicity, in this paper, we will be studying open Wilson lines consisting only of the gauge fields. A cartoon view of the open Wilson line is depicted in Fig.1

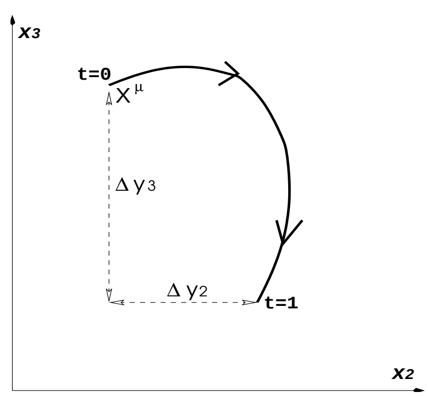

Figure 1: *Projection of the open Wilson line on the noncommutative plane. The base-point at* $t = 0$ *is denoted as* x^μ*, splitting between the two endpoints* ($t = 0, 1$) *as* $\Delta y^{2,3}$.

Let us now examine explicitly under what conditions the open Wilson lines would be gauge invariant. Interestingly enough, we will be discovering that the momentum $\mathbf{k}_\mu$ is not arbitrary but is directly related to the splitting Δy^μ. Denote finite gauge transformation parameter as $\mathrm{U}(x) = \exp_\star(i\epsilon(x))$. First, under a gauge transformation that depends only on the commutative coordinates, the open Wilson line is trivially invariant, as the two ends of the Wilson line are splitted only along the noncommutative directions. Thus, let us focus on the case where the gauge transformation parameter depends on the noncommutative coordinates. We then find that

$$W_{\mathbf{k}}[C] \quad \rightarrow \quad W_{\mathrm{k}}^{\mathrm{U}}[C] = \int d^2\mathbf{x}\, \mathcal{P}\, \mathrm{U}(\mathbf{x} + \Delta y) \star e^{i\int \dot{y}\cdot \mathbf{A}(x+y)} \star \mathrm{U}^{-1}(\mathbf{x}) \star e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{20}$$

From the definition of the Moyal product, it follows that

$$e^{+i\mathbf{k}\cdot\mathbf{x}} \star \mathrm{F}(\mathbf{x} + \theta\cdot\mathbf{k}) \star e^{-i\mathbf{k}\cdot\mathbf{x}} = \mathrm{F}(\mathbf{x}) \qquad \text{and} \qquad e^{-i\mathbf{k}\cdot\mathbf{x}} \star \mathrm{F}(\mathbf{x}) \star e^{+i\mathbf{k}\cdot\mathbf{x}} = \mathrm{F}(\mathbf{x} + \theta\cdot\mathbf{k}) \tag{21}$$

for any function $\mathrm{F}(\mathbf{x})$ and hence

$$e^{i\mathbf{k}\cdot\mathbf{x}} \star \mathrm{U}^{-1}(\mathbf{x} + \theta\cdot\mathbf{k}) = \mathrm{U}^{-1}(\mathbf{x}) \star e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{22}$$

This is nothing but the Moyal product manifestation of the result shown in the last subsection that noncommutative gauge transformation is equivalent to translation along noncommutative directions.

Applying Eq.(22) to the last expression in Eq.(20) and using the cyclic property of the Moyal product, we finally find that

$$\begin{aligned} W_{\mathbf{k}}^{\mathrm{U}}[C] &= \int d^2\mathbf{x}\; \mathcal{P}\, \mathrm{U}(\mathbf{x} + \Delta y) \star e^{i\int \dot{y}\cdot\mathbf{A}(x+y)} \star e^{i\mathbf{k}\cdot\mathbf{x}} \star \mathrm{U}^{-1}(\mathbf{x} + \theta\cdot\mathbf{k}) \\ &= \int d^2\mathbf{x}\; \mathcal{P}\, \mathrm{U}^{-1}(\mathbf{x} + \theta\cdot\mathbf{k}) \star \mathrm{U}(\mathbf{x} + \Delta y) \star e^{i\int \dot{y}\cdot\mathbf{A}(x+y)} \star e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= W_{\mathbf{k}}[C] \end{aligned} \tag{23}$$

provided the splitting Δy is nonzero only along the noncommutativity directions and is related to the momentum $\mathbf{k}$ of the open Wilson line as:

$$\mathbf{k}_\mu \;=\; \left(\frac{1}{\theta}\right)_{\mu\nu} \Delta y^\nu \,. \tag{24}$$

Thus, we have proven that the open Wilson lines Eq.(18) are indeed gauge invariant but only under the condition that the 'momentum' $\mathbf{k}$ is related to the 'size' Δy precisely as in Eq.(24). Along with Eq.(17), the relation Eq.(24) stems from the underlying noncommutative gauge invariance. As such, we take Eq.(24) as the fundamental defining relation characterizing all gauge invariant open Wilson lines.

One may wonder how possibly the above Wilson lines can be encoded into the noncommutative gauge theory. The Parisi's prescription alluded above tells us what ought to be done. In the gauge theory partition function, consider a 'naive' open Wilson line along a contour C and couple it to a non-local 'spurion' field:

$$\mathcal{S}_J = \int d^4y \int d^4x\; \mathcal{P}\; \exp_\star\left(i\int_C \dot{y}(t)\cdot\mathbf{A}(x + y(t))\right) \star J[x, C(y)]. \tag{25}$$

Note that the naive Wilson line is *not* gauge invariant. Therefore, under the noncommutative U(1) gauge transformation, the spurion coupling $J[x, C(y)]$ ought to transform appropriately so that S_J is rendered gauge invariant. Fortuitously, due to the equivalence of the gauge transformation and the spacetime translation, it is possible to extract the gauge variant piece out of the 'spurion' field and adjoin it to the naive Wilson line. To do so, consider the 'spurion' field slowly varying along the commutative directions. One can partially Fourier-expand it along the noncommutative direction as:

$$J[x, C(y)] = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \widetilde{J}_{\mathbf{k}}[\mathbf{x}_\perp, C(y)]. \tag{26}$$

Inserting this to Eq.(25), one finds

$$\mathcal{S}_J = \int d^4y \int d^2\mathbf{x}_\perp \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, W_{\mathbf{k}}[\mathbf{x}_\perp, C(y)]\, \widetilde{J}_{\mathbf{k}}[\mathbf{x}_\perp, C(y)]. \tag{27}$$

Here, $W_{\mathbf{k}}[C]$ is indeed the gauge invariant open Wilson line. Thus, even though $J[x, C]$ is not gauge invariant, Fourier-transformed 'spurion' coupling $\widetilde{J}_{\mathbf{k}}[\mathbf{x}_\perp, C]$ has become gauge invariant. Moreover, in the integrand, the open Wilson line operator and the source $\widetilde{J}_{\mathbf{k}}[\mathbf{x}_\perp, C]$ are multiplied as an ordinary product, as they are functions of the commutative coordinates only. As such, one-point correlator of the gauge invariant open Wilson line can be obtained from functional response of the gauge theory with respect to the *Fourier-transformed* 'spurion' couplng:

$$\Big\langle W_{\mathbf{k}}[\mathbf{x}_\perp, C(y)]\Big\rangle_{\rm NCYM} \quad = \quad \frac{\delta \ln Z_{\rm NCYM}[\widetilde{J}]}{\delta \widetilde{J}_{\mathbf{k}}[\mathbf{x}_\perp, C(y)]}. \tag{28}$$

Incidentally, the spurion coupling to the Wilson lines also indicates an interesting reshuffling between the noncommutative coordinates. Namely, even though the coupling Eq.(25) involves sum over size and base point location of the open Wilson lines, the more natural, gauge-invariant coupling Eq.(27) indicates that the sum ought to be interpreted in terms of the commutative coordinates $\mathbf{x}_\perp$ and momentum $\mathbf{k}$ along noncommutative directions.

Lastly, much as the closed Wilson loops form a complete set of gauge invariant observables in conventional gauge theory, we can take the open Wilson lines Eq.(18) as a complete set of gauge invariant observables in the noncommutative gauge theory.

2.5 UV-IR Relation and Spacetime Uncertainly Principle

The relation Eq.(24) is the most important, salient feature of the noncommutative Wilson lines in that it relates the 'momentum' and the 'size' of the Wilson line. It should be emphasized that, by assigning the Fourier mode $\mathbf{k}$, two spacetime aspects of the Wilson line are specified — the momentum of the base point x and the size of the open Wilson line Δy. Typical open Wilson lines have a shape whose endpoints are splitted within the noncommutative subspace but not in commutative subspace. While the standard notion of the operator product expansion does not make sense along the noncommutative directions, it still holds along the commutative directions. Recalling that the Wilson line does not have any endpoint splitting in the commutative subspace, it would make a sense to expand a generic open Wilson line in the ambient spacetime into multipoles of a subset of open Wilson lines which lie entirely within the noncommutative subspace. This is illustrated in Fig.(2).

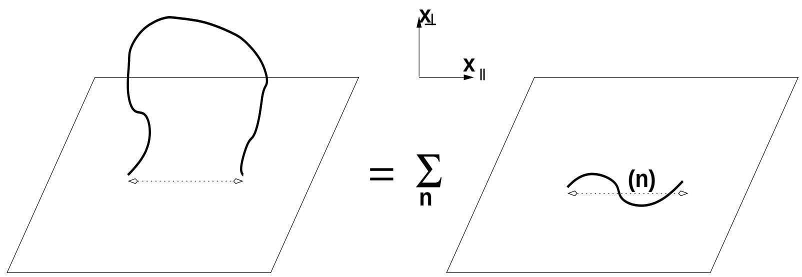

Figure 2: *Multipole expansion of open Wilson lines.*

To grasp the physical meaning, we shall be exploring the Wilson line operators at two different regimes of the spatial momentum, $\mathbf{k}$.

First, let us consider probing physics at 'infrared' as compared to the noncommutativity scale, $|\mathbf{k}| \ll 1/\sqrt{\theta}$. The Moyal phase factors in Eq.(2) are completely suppressed that all the Moyal products are reduced to ordinary products. This implies that, in the infrared regime, the noncommutative gauge theory converges to the conventional commutative gauge theory [4]. In this regime, $|\mathbf{k}| \ll 1/\sqrt{\theta}$ and Eq.(24) indicates that $|\Delta y| \ll \sqrt{\theta}$, viz. 'size' of the open Wilson line is smaller than the noncommutativity scale, a minimum distance scale one can probe. Altogether with vanishing Moyal phase factor, the Wilson line operator is also reduced effectively to a Fourier-transform of the standard, commutative, *closed* Wilson loop operator. Therefore, in the infrared regime, the conventional $\mathcal{N} = 4$ supersymmetric gauge theory would describe dynamics well and the well-known UV-IR *duality* [7] would follow immediately at large-N and strong 't Hooft coupling limit.

Incidentally, the closed Wilson loop operators are known to form a complete set of gauge-invariant physical observables in the conventional, commutative gauge theories. Uniqueness of the Moyal deformation (up to gauge equivalence) as a noncommutative but associative deformation then implies that the open Wilson line operators in noncommutative gauge theory also form a complete set of gauge invariant observables. Hence, together with the fact that they can carry nonzero momentum along the noncommutative directions, we conclude that the open Wilson line operators are the most physically suitable probes for dynamical aspects of the

[4] This reduction remains the same even at quantum level, as the underlying $\mathcal{N} = 4$ supersymmetry ensures absence of ultraviolet divergences in the noncommutative gauge theory and hence no possibility of UV/IR mixing.

noncommutative gauge theories.

Next, consider probing physics at 'ultraviolet' as compared to the noncommutativity scale, $|\mathbf{k}| \gg 1/\sqrt{\theta}$. From Eq.(24), one finds that the 'size' of the open Wilson line would be larger than the noncommutativity scale, $|\Delta y| \gg \sqrt{\theta}$. In fact, these excitations look like macroscopically large fundamental open strings stretched along the noncommutativity subspace. As such, in the 'ultraviolet' regime, we would find UV-IR *proportionality* instead of *duality*. Note that we have reached such a conclusion purely based on noncommutative gauge theory — open Wilson lines are the only known operators with gauge invariance and nonzero momentum along the noncommutative directions.

A remarkable consequence of Eq.(24) is that the open Wilson lines satisfy a version of (Euclidean) spacetime uncertainty relation [16]. If the macroscopic open Wilson lines are treated quantum mechanically, $\mathbf{k}_{1,2} \sim \Delta \mathbf{k}_{1,2} \geq \hbar/\Delta x^{1,2}$, where x^μ denotes the coordinates of the open Wilson line base point at $t = 0$. Thus, utilizing the UV-IR relation Eq.(24), one immediately finds

$$\Delta x^1 \Delta y^2 \gtrsim \frac{1}{2}\hbar\theta \qquad \text{and} \qquad \Delta y^1 \Delta x^2 \gtrsim \frac{1}{2}\hbar\theta. \tag{29}$$

Here, recall that Δy refers to the 'classical' distance between the two endpoints of the open Wilson line. As such, the above inequalities define a version of (Euclidean) spacetime uncertainty relation satisfied between the classical size and the quantum mechanical uncertainty of the base point position of the open Wilson lines. In other words, base-point coordinates and the open Wilson line splitting distances form a set of conjugate variables each other.

3 Tomography of Five-Dimensional Supergravity Dual

We now turn to supergravity description of the noncommutative gauge theories, as initiated in [4] and [5]. As in the previous sections, we shall be considering scaling limit of D3 branes (oriented along $x^1 \cdots x^3$ directions) in the presence of $B^{\rm NS}_{23} \neq 0$. The resulting background is described by a classical solution of the Type IIB supergravity, whose string frame metric is given by

$$ds^2_{\rm IIB} = \alpha' R^2 \left[u^2(-(dx^0)^2 + (dx^1)^2) + \frac{u^2}{1 + a^4 u^4}((dx^2)^2 + (dx^3)^2) + \frac{du^2}{u^2} + d\Omega_5^2 \right], \tag{30}$$

while the dilaton ϕ, NS-NS 2-form potential $B_2^{\rm NS}$ and the R-R 2-form potential $C_2^{\rm RR}$ and the R-R self-dual, 5-form field strength $F_5^{\rm RR}$ are given by

$$\begin{aligned} e^{2\phi} &= g^2 \frac{1}{1 + a^4 u^4} \qquad & B^{\rm NS}_{23} &= \frac{\alpha' R^2}{a^2} \frac{a^4 u^4}{1 + a^4 u^4} \\ C^{\rm RR}_{01} &= \frac{1}{g} \frac{\alpha' R^2}{a^2} a^4 u^4 \qquad & F^{\rm RR}_{0123u} &= \frac{\alpha'^2}{g} \frac{1}{1 + a^4 u^4} \partial_u (u^4 R^4). \end{aligned} \tag{31}$$

Here, $R^4 = 4\pi g N$ and g refers to the *open* string coupling. In the infrared, $ua \ll 1$, $B_2^{\rm NS}, C_2^{\rm RR}$ tend to vanish and the spacetime asymptotes to $AdS_5 \times S^5$, the supergravity dual to the large-N limit of the standard $d = 4, \mathcal{N} = 4$ supersymmetric gauge theory.

In this background, the graviton fluctuation $h_{01}(x, u)$ with zero momenta along x^0, x^1 and zero angular momenta along the S^5 satisfy a simple decoupled equation. Denoting $\phi(x, u) = g^{00} h_{01} = g^{11} h_{01}$ and expanding the Type IIB supergravity action, one easily finds that the ϕ-field equation in string frame is given by

$$\partial_\mu (\sqrt{g} e^{-2\phi} g^{\mu\nu} \partial_\nu \ \phi) = 0. \tag{32}$$

In terms of Fourier modes along the noncommutative directions

$$\phi(u, \mathbf{x}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \, e^{-i\mathbf{k}\cdot\mathbf{x}} \, \widetilde{\phi}(\mathbf{k}, u), \tag{33}$$

the field equation becomes

$$\partial_u \Big(u^5 \partial_u \widetilde{\phi}(\mathbf{k}, u) \Big) - \mathbf{k}^2 u \Big(1 + a^4 u^4 \Big) \, \widetilde{\phi}(\mathbf{k}, u) = 0. \tag{34}$$

Here, $\mathbf{k}^2 = k_2^2 + k_3^2$. Eq.(34) represents a perturbation of the background with zero-energy. As such, it does not make sense in Lorentzian signature. Hence, in what follows, we will work always in the Euclidean signature.

The background Eqs.(30, 31) has been proposed as the holographic dual of the noncommutative gauge theory residing at the boundary, $u = \infty$. Noting that the value of $B_{23}^{\rm NS}$ at the boundary is $B_{23}^{\rm NS} = \alpha' R^2 / a^2$, noncommutativity scale in the gauge theory is identified as $\sqrt{\theta} \equiv \sqrt{\alpha' / B_{23}^{\rm NS}} = a/R$. In the supergravity background, however, the spacetime metric Eq.(30) indicates that the scale of departure from $AdS_5 \times S^5$ is really a rather than a/R. This difference may be attributed to strong coupling dynamics of the noncommutative gauge theory, much as in the standard AdS/CFT correspondence, where similar non-analytic enhancement has been discovered [7].

Our goal in this section would be to understand the UV/IR relation exhibited by the open Wilson line operators from the dual supergravity side. In parallel to the gauge theory analysis in the previous section, we will be focusing on supergravity counterpart of the one-point correlators of gauge-invariant operators. For doing so, we need to begin with a general prescription for the one-point correlator.

3.1 One-Point Correlator from Dual Supergravity

From the supergravity side, to study one-point correlator, we add a suitable nondynamical source to the graviton fluctuation, ϕ, in Eq.(32). As in the previous section, we assume the source is prescribed by a definite momentum $\mathbf{k}$ in the noncommutative directions, (x^2, x^3), and by a definite location $\bar{u}$ in the radial direction. In the dual gauge theory, turning on the source simply refers to a situation that we have excited the system from its ground state. Thus, in the presence of the source, the operator $\mathcal{O}(\mathbf{k})$ dual to the ϕ-field would be acquiring a nonvanishing vacuum expectation value. We are interested in 'tomography' of the one-point correlator — profile of $\langle \mathcal{O}(\mathbf{x}) \rangle_{\bar{u}}$ as a function of $\mathbf{x}$ and $\bar{u}$.

In the standard AdS/CFT correspondence, where the NS-NS 2-form potential $B^{\rm NS}$ is turned off, prescription for the one-point correlator is well understood — perturb the supergravity field around the normalizable solution $\phi_{\rm norm}$ of Eq.(34) in the presence of a source:

$$\phi(\mathbf{k}, u) \;=\; \phi_{\rm norm}(\mathbf{k}, u) + \delta\phi_{\rm non-norm}(\mathbf{k}, \bar{u}). \tag{35}$$

Here, $\delta\phi_{\rm non-norm}(\mathbf{k}, \bar{u})$ denotes a suitable non-normalizable mode. Evaluating the supergravity action for $\phi(\mathbf{k}, u)$ and taking functional variation of the action with respect to $\delta\phi_{\rm non-norm}(\mathbf{k}, \bar{u})$ would then yield the one-point correlator, $\langle \mathcal{O}(\mathbf{k}) \rangle_{\bar{u}}$.

In the presence of the $B^{\rm NS}$ background, a similar procedure was carried out to calculate one-point correlator in the presence of a D-instanton in [17]. There, it was found that there is an ambiguity in extracting the one-point correlators due to necessity of momentum-dependent wave-function renormalizations. A similar ambiguity was found for two-point correlators in [5].

For two-point correlators, a possible resolution [5] of the ambiguity has been suggested in [12]. The idea of [12] was to postulate an operator-field correspondence between the gauge theory and the dual supergravity along the lines of [9]. After Wick-rotation to Euclidean signature, it then followed that the two-point correlator of a gauge theory operator $\mathcal{O}$ ought to be given by

$$\Big\langle \mathcal{O}(\mathbf{k}) \mathcal{O}(-\mathbf{k}) \Big\rangle_{\rm E} = {\rm Lim}_{u, u' \to \infty} \left(\frac{\mathbf{k}^4}{\Psi_{\mathbf{k}}(u) \Psi_{-\mathbf{k}}(u')} \right) \Big[\mathcal{G}_{\rm E}(u, u'; \mathbf{k}) - \mathcal{G}_0(u, u'; \mathbf{k}) \Big]. \tag{36}$$

Here, $\mathcal{G}_{\rm E}(u, u'; \mathbf{k})$ denotes the Euclidean *bulk* Green function in the supergravity background Eqs.(30, 31), $\mathcal{G}_0(u, u'; \mathbf{k})$ is the bulk Green function in flat space, and $\Psi_{\mathbf{k}}(u)$ is the (Wick-rotated) normalized wave function pertinent to the perturbation. Clearly, the normalizations in all the quantities in Eq.(36) are well defined such that we have an unambiguous supergravity prediction for the gauge theory correlator. It was shown in [12] that the correlator Eq. (36) has the correct low-(ka) behavior.

Here, we utilize the same prescription for the one-point correlator in the presence of a source located at $\bar{u}$ in the radial direction. Then, the classical solution of ϕ at a radial position u is simply given by the bulk Green function $\mathcal{G}_{\rm E}(u, \bar{u}; \mathbf{k})$. Following the same steps as in [12], we finally get

$$\Big\langle \mathcal{O}(\mathbf{k}) \Big\rangle_{\bar{u}} \;=\; {\rm Lim}_{u \to \infty} \left(\frac{\mathbf{k}^2}{\Psi_{\mathbf{k}}(u)} \right) \Big[\mathcal{G}_{\rm E}(u, \bar{u}; \mathbf{k}) - \mathcal{G}_0(u, \bar{u}; \mathbf{k}) \Big]. \tag{37}$$

[5] A closely related prescription was proposed in [18] in the context of holography in the full D3-brane geometry.

Provided the limit specified does exist and yields a u-independent result, Eq.(37) may be taken as *the* defintion of the one-point correlators from the dual supergravity side.

3.2 Consistency Check – AdS/CFT Correspondence

One can easily confirm that the prescription Eq.(37) yields the correct result for $AdS_5 \times S^5$ background. In this case, the Euclidean bulk Green function for a massless scalar field is given by

$$\mathcal{G}_{\rm E}(u, \bar{u}; k)\Big|_{\rm AdS} = \left(\frac{1}{u\bar{u}}\right)^2 K_2\left(\frac{k}{\bar{u}}\right) I_2\left(\frac{k}{u}\right) \qquad \text{for} \qquad u > \bar{u}. \tag{38}$$

Here, k refers to the magnitude of the Euclidean 4-momentum along the D3-brane worldvolume directions. After Wick rotation to the Euclidean signature, the orthonormal wave function is given by $\Psi_{\mathbf{k}}(u) u^{-2} I_2(k/u)$. Moreover, for a fixed $\bar{u}$, as $u \to \infty$, the bulk Green function in *flat* space

$$\mathcal{G}_0(u, \bar{u}; k) = \frac{1}{2ka^2}\frac{1}{(u\bar{u})^{5/2}} \exp\Big(-ka^2\,(u - \bar{u})\Big) \tag{39}$$

goes to zero exponentially, while the wave function $\Psi_{\mathbf{k}}(u)$ goes to zero only in powers of $1/u$. Thus, as $u \to \infty$ limit is taken, the subtraction of $\mathcal{G}_0(u, \bar{u}; k)$ in Eq.(37) is irrelevant for $AdS_5 \times S^5$ background. One obtains

$$\Big\langle \mathcal{O}(k) \Big\rangle_{\bar{u}}\Big|_{\rm AdS} = \left(\frac{k}{\bar{u}}\right)^2 K_2\left(\frac{k}{\bar{u}}\right). \tag{40}$$

The 'tomograph' of the source distributed on the holographic boundary at $u = \infty$ is then provided by Fourier-transform of Eq.(40):

$$\begin{aligned}\Big\langle \mathcal{O}(\Delta x) \Big\rangle_{\bar{u}} &= \int \frac{d^4 k}{(2\pi)^4} e^{-ik\cdot\Delta x} \left(\frac{k}{\bar{u}}\right)^2 K_2\left(\frac{k}{\bar{u}}\right) \\ &= \frac{\bar{u}^{-4}}{[(\Delta x)^2 + \bar{u}^{-2}]^4}.\end{aligned} \tag{41}$$

Thus, as the bulk source approaches the boundary, $\bar{u} \to \infty$, the tomograph of $\langle \mathcal{O}(\Delta x)\rangle_{\bar{u}}$ asymptotes to Dirac delta function, $\delta^{(4)}(\Delta x)$. As such, characteristic size of the hologram may be extracted from equal-altitude contours of $\langle \mathcal{O}(\Delta x)\rangle_{\bar{u}}$ being a constant multiple of $\bar{u}^4$. It yields

$$\Big(|\Delta x|\Big)_0 \sim \frac{1}{\bar{u}} \qquad \text{for all} \quad \bar{u}, \tag{42}$$

increasing (decreasing) *monotonically* as the source moves into (out of) the bulk. This is a manifestation of the UV/IR-*duality* [7] of the AdS/CFT correspondence reflected in the one-point correlator.

Incidentally, it is also instructive to understand the relationship Eq.(42) qualitatively for small $\bar{u}$ regime. In this regime, $K_2(k/\bar{u}) \sim \sqrt{\bar{u}} \exp(-k/\bar{u})$ and the integral over k in Eq.(41) leads to a vanishing contribution for $\Delta x \gg 1/\bar{u}$.

3.3 Noncommutative One-Point Correlator

We finally study the one-point correlator Eq.(37) in the presence of $B_{23}^{\rm NS} \neq 0$. In [17, 12], the bulk Green function for a nonzero $B^{\rm NS}$ background has been computed. In Euclidean space, the result is:

$$\mathcal{G}_E(u, \bar{u}; \mathbf{k}) = \frac{\pi}{4i} \frac{C(ka)}{A(ka)} \frac{1}{u^2\bar{u}^2} H^{(1)}\left(\nu, \bar{w} + \frac{i\pi}{2}\right) H^{(2)}\left(\nu, -w - \frac{i\pi}{2}\right) \qquad \text{for} \qquad u > \bar{u}, \tag{43}$$

where, as in the previous sections, $\mathbf{k}$ is the momentum vector along the (x^2, x^3) noncommutative directions, k is its magnitude [6], and $A(ka), B(ka), C(ka)$ and $\nu(ka)$ are known functions of ka (for their power series expansions, see e.g. Ref.[12]) satisfying the 'unitarity' relation $B^2(ka) = A^2(ka) + C^2(ka)$. The parameter a is the same as in Eqs.(30, 31). In Eq.(43), the variables $w, \bar{w}$ are related to the coordinates $u, \bar{u}$ by

$$au = e^{-w} \qquad \text{and} \qquad a\bar{u} = e^{-\bar{w}}, \tag{44}$$

and $H^{(i)}\left(\nu, w + i\pi/2\right)$ for $i = 1, 2$ are the associated Mathieu functions of the third and the fourth kinds, respectively.

We also need the normalized wave functions $\Psi_{\mathbf{k}}(u)$. After Wick rotation to the Euclidean signature, they are given by

$$\Psi_{\mathbf{k}}(u) = N(ka)\; e^{-i\frac{\pi}{2}(\nu+1)}\; \frac{1}{u^2} H^{(2)}\left(\nu, -w - \frac{i\pi}{2}\right), \tag{45}$$

where $N(ka)$ is a normalization factor, which is again a power series in (ka) and whose low-ka behavior is known [12]. It is now possible to extract the one-point correlator $\langle \mathcal{O}(\mathbf{k}) \rangle_{\bar{u}}$ using Eq.(37). For our purposes, it suffices to analyze Eq.(37) in two asymptotic regimes – $a\bar{u} \ll 1$ and $a\bar{u} \gg 1$.

For $a\bar{u} \ll 1$, the associated Mathieu functions asymptote to the Hankel functions:

$$H^{(1,2)}\left(\nu, \bar{w} + \frac{i\pi}{2}\right) \quad \longrightarrow \quad H_\nu^{(1,2)}\left(i\frac{k}{\bar{u}}\right), \tag{46}$$

where $H_\nu^{(1,2)}(z)$ are Hankel functions of the first and the second kinds, respectively. As in the case of $AdS_5 \times S^5$, at $u \to \infty$, subtraction of the flat space Green function is irrelevant and the one-point function reduces precisely to the AdS_5 result:

$$\Big\langle \mathcal{O}(\mathbf{k}) \Big\rangle_{\bar{u}} \;=\; \left(\frac{k}{\bar{u}}\right)^2 K_2\left(\frac{k}{\bar{u}}\right) + \cdots \qquad \text{for} \qquad \bar{u} \ll \frac{1}{a}. \tag{47}$$

[6] The bulk Green function Eq.(43) is valid also for nonzero Euclidean momentum $\mathbf{k}_\perp = (k_0, k_1)$ provided k is replaced by $(\mathbf{k}^2\mathbf{k}_\perp^2)^{1/4}$.

To find the 'tomographic' size of the hologram of the bulk source, we perform Fourier transform of Eq.(47) over the momenta $\mathbf{k}$:

$$\begin{aligned}\Big\langle \mathcal{O}(\Delta x)\Big\rangle_{\bar{u}} &= \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{-i\mathbf{k}\cdot\Delta x}\Big\langle \mathcal{O}(\mathbf{k})\Big\rangle_{\bar{u}} \\ &= \frac{4}{\pi}\frac{\bar{u}^{-4}}{[(\Delta x)^2+\bar{u}^{-2}]^3},\end{aligned} \tag{48}$$

yielding a tomograph identical to the AdS_5 result, Eq.(41). Thus, when the source is located deep inside the bulk where the space-time asymptotes to $AdS_5 \times S^5$, one finds 'tomograph' of the one-point correlator is such that the size of the hologram at the boundary exhibits UV/IR duality:

$$\Big(|\Delta x|\Big)_0 \quad \sim \quad \frac{1}{\bar{u}} \qquad \text{for} \qquad \bar{u} \ll \frac{1}{a}. \tag{49}$$

For $a\bar{u} \gg 1$, the 'tomograph' would be quite different as it covers the region close to the boundary, where the supergravity mode in question practically perceives flat space-time. In this region, the Euclidean bulk Green function becomes, for $\bar{u} < u$,

$$\mathcal{G}_{\rm E}(u,\bar{u};\mathbf{k}) \;=\; \mathcal{G}_0(u,\bar{u};\mathbf{k}) - \frac{1}{2ka^2}\frac{1}{(u\bar{u})^{5/2}}\frac{\hat{B}(ka)}{iA(ka)}\; e^{-ka^2(u+\bar{u})} + \cdots, \tag{50}$$

where $\hat{B}(ka)$ refers to the real part of $B(ka)$. Thus, the one-point correlator in this region is given by

$$\Big\langle \mathcal{O}(\mathbf{k})\Big\rangle_{\bar{u}} \;=\; \left(\frac{\pi}{8ka^2\bar{u}^5}\right)^{1/2}\frac{i\hat{B}(ka)}{N(ka)A(ka)}\; e^{-ka^2\bar{u}} + \cdots \qquad \text{for} \qquad \bar{u} \gg \frac{1}{a}. \tag{51}$$

As anticipated, dependence on the cutoff u has cancelled out completely so that the limit in Eq.(37) indeed exists. From the form of power series expansion of the known coefficients $A(ka), B(ka), C(ka)$ and $N(ka)$ [12], one finds that

$$\frac{iB(ka)}{N(ka)A(ka)} \;=\; \left(-\frac{2}{3a^2}\right)\Big[1+\alpha_1(ka)^4+\alpha_2(ka)^4\log ka+\cdots\Big]. \tag{52}$$

Here, $\alpha_1, \alpha_2, \cdots$ are calculable numerical coefficients. To tomograph the hologram of the supergravity source on the boundary, we perform Fourier transform of Eq.(51) over the momentum $\mathbf{k}$:

$$\Big\langle \mathcal{O}(\Delta x)\Big\rangle_{\bar{u}} \;=\; -\frac{\sqrt{\pi}}{3a^2}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\; e^{-i\mathbf{k}\cdot\Delta x}\;\frac{1}{(2ka^2\bar{u}^5)^{1/2}}\; e^{-ka^2\bar{u}}\;\Big[1+\alpha_1(ka)^4+\cdots\Big], \tag{53}$$

The result is

$$\Big\langle \mathcal{O}(\Delta x)\Big\rangle_{\bar{u}} \;=\; -\frac{(a\bar{u})^{5/2}}{12\sqrt{2}a^2}\;\Big[1+\alpha_1 a^{-4}\partial_{\bar{u}}^4+\cdots\Big]\,P_{1/2}(Z(\bar{u}))\;\left(\frac{Z(\bar{u})}{a\bar{u}}\right)^{3/2} \qquad \text{for} \qquad \bar{u} \gg \frac{1}{a}. \tag{54}$$

Here, $P_\nu(Z)$ denotes the Legendre function and $Z(\bar{u}) = (1 + (\Delta x/a^2\bar{u})^2)^{-1/2}$. It is easily seen that the profile drops to zero fast for $|\Delta x| \gg a^2\bar{u}$. The characteristic size of the hologram can be determined from equal-altitude contours of Eq.(54) being a multiple of $\bar{u}^2$ — intersection locus of $P_{1/2}(Z)$ and $Z^{-3/2}$. The result exhibits UV/IR *proportionality*:

$$\Big(|\Delta x|\Big)_0 \quad \sim \quad a^2\bar{u} \qquad \text{for} \qquad \bar{u} \gg \frac{1}{a}. \tag{55}$$

Thus, in this regime, the hologram size actually *increases* as the source moves closer to the boundary. If we impose an infrared cutoff u_0 in the supergravity background, then a source placed at the cutoff would correspond to a state in the gauge theory in which the dual operator is spread over a size a^2u_0 [7].

The result is certainly consistent with the fact that gauge-invariant open Wilson line operators in the noncommutative gauge theory have the property that their size increases with increasing momentum. The following heuristic picture may provide yet another way of understanding the UV-IR proportionality. Consider Fourier-transforming the open Wilson line operator, Eq.(18), over **k**-momenta with an *ultraviolet* cutoff Λ and construct a position space 'tomograph'. This means that, due to Eq.(24), we are fixing the base point x^μ of the Wilson line and integrating over all possible separations Δy^μ — the Fourier integral with some ultraviolet cutoff Λ would involve a sum over such Wilson lines with sizes upto $\theta\Lambda$. Since the infrared cutoff of the bulk is related to the ultraviolet cutoff of the boundary theory by $u_0 \sim \Lambda$, one is summing over open Wilson lines of size up to θu_0. If we take into account of the strong-coupling effect that the true noncommutativity scale following from the dual supergravity differs from naive gauge theory estimate θ by a factor of R, the maximal size of the open Wilson lines is the same as predicted by the dual supergravity, viz a^2u_0.

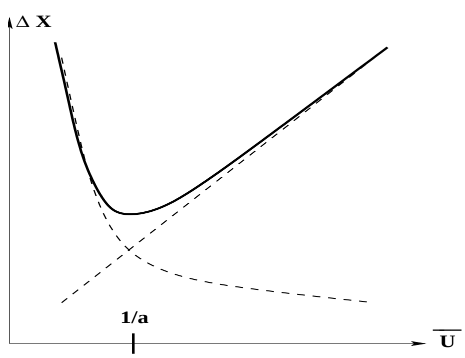

Figure 3: *Schematic view of the UV-IR relation in noncommutative gauge theory.*

Combining the two regimes – Eq.(49) at low-energy regime and Eq.(55) at high-energy

[7]This UV/IR proportionality between the size and the location was also observed in [18] in the context of holography in full D3-brane geometry.

regime, we conclude that the supergravity prediction for the UV/IR relation agrees well with that in noncommutative gauge theory, and takes a form depicted in Fig.(3). The agreement may be taken as a convincing evidence for holographic dual relation between the supergravity in the background Eqs.(30, 31) and the noncommutative gauge theory.

The open Wilson line operator along a given contour and with definite momentum would be, in general, dual to a combination of various supergravity fields with the same momentum but with various quantum numbers. To understand the detailed operator-field relationship, one needs to learn how to decompose the Wilson line operator irreducibly according to the quantum numbers of supergravity modes. We will address this issue in a separate publication. It should be noted, however, that the IR/UV properties demonstrated here are quite independent of the details of such a decomposition.

Finally, it would be also interesting to extend the methods we have developed in this paper to the noncritical open string theories [19] which arises as a decoupling limit in open string theory with 'electric' noncommutativity and to the dynamics of the noncommutative monopoles, fluxons [20] and vortices [21]. We hope to return this question in future publications.

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