

# ON RAMANUJAN AND DIRICHLET SERIES WITH EULER PRODUCTS

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In his unpublished manuscripts (referred to by Birch [1] as Fragment V, pp. 247–249), Ramanujan [3] gave a whole list of assertions about various (transforms of) modular forms possessing naturally associated Euler products, in more or less the spirit of his extremely beautiful paper entitled “On certain arithmetical functions” (in *Trans. Camb. Phil. Soc.* 22 (1916)). It is simply amazing how Ramanujan could write down (with an ostensibly profound insight) a basis of eigenfunctions (of Hecke operators) whose associated Dirichlet series have Euler products, anticipating by two decades the famous work of Hecke and Petersson. That he had further realized, in the event of a modular form  $f$  not corresponding to an Euler product, the possibility of restoring the Euler product property to a suitable linear combination of modular forms of the same type as  $f$ , is evidently fantastic.

Ramanujan’s assertions referred to at the beginning were recently upheld and elucidated in the light of the work of Hecke, Rankin and Serre by Rangachari [4]. Page 5 of [4] refers to “an additional list of Euler product developments found in [3] which is incomplete except for the first one” (already covered by one of Ramanujan’s above-mentioned assertions): namely,

$$(1) \text{ If } \sum a(n)x^n = x^2 \prod (1-x^{3n})^{16} \text{ and } \sum A(n)x^n = x \prod (1-x^{3n})^8 \left(1 + 240 \sum_1^{\infty} \frac{n^3 x^{3n}}{1-x^{3n}}\right)$$

then

$$\begin{aligned} \sum (A(n) + 6a(n)\sqrt{10}) \cdot n^{-s} &= (1 - 6\sqrt{10} \cdot 2^{-s} + 2^{7-2s})^{-1} (1 + 96\sqrt{10} \cdot 5^{-s} + 5^{7-2s})^{-1} \\ &\times (1 - 260 \cdot 7^{-s} + 7^{7-2s})^{-1} (1 + 1920\sqrt{10} \cdot 11^{-s} + 11^{7-2s})^{-1} \dots \end{aligned} \quad (1)$$

$$(2) \text{ If } \sum a(n)x^n = x^3 \prod (1-x^{4n})^{18}, \text{ then}$$

$$\begin{aligned} 156 \sum a(n)n^{-s} &= (1 - 78 \cdot 3^{-s} + 3^{8-2s})^{-1} (1 + 510 \cdot 5^{-s} + 5^{8-2s})^{-1} (1 + 1404 \cdot 7^{-s} + 7^{8-2s})^{-1} \dots \\ &- (1 + 78 \cdot 3^{-s} + 3^{8-2s})^{-1} (1 + 510 \cdot 5^{-s} + 5^{8-2s})^{-1} (1 - 1404 \cdot 7^{-s} + 7^{8-2s})^{-1} \dots \end{aligned} \quad (2)$$

Ramanujan writes further:

Presumably there are analogous results for  $\sum a_n \cdot n^{-s}$  where  $\sum a_n \cdot x^n$  is

$$x^5 \prod (1-x^{12n})^{10}, \quad x^7 \prod (1-x^{12n})^{14}, \quad x^5 \prod (1-x^{6n})^{20} \text{ and } x^{11} \prod (1-x^{12n})^{22} \quad (3)$$

In this brief note—an addendum to [4], our object is merely to point out that Ramanujan’s relation (2) above does not seem to be correct and the “analogous

results for (3) appear to be covered by Ramanujan's assertions referred to in the first paragraph. Of course, as remarked in [4], Ramanujan's formula (1) above is taken care of already by the Euler product written down by him for  $\eta^8 Q + 6\sqrt{10} \eta^{16}$  and verified in [4]; here, as usual, we have

$$\eta(x) = x^{1/24} \prod_1^{\infty} (1 - x^n), \quad E_4(x) = Q(x) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 x^n}{1 - x^n}$$

and

$$E_6(x) = R(x) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 x^n}{1 - x^n}.$$

Let us first consider Ramanujan's assertion (2). Writing  $\prod_{m=1}^{\infty} (1 - x^m)^r = \sum_{n=0}^{\infty} p_r(n) x^n$  for integral  $r \geq 1$ , after Newman [2], we have

$$\eta^{18}(x) = x^{3/4} \sum_{m=0}^{\infty} p_{18}(m) x^m = \sum_{n=3}^{\infty} a(n) x^{n/4}$$

where, as we can check from the tables in [2],

$$\begin{aligned} a(3) &= 1 & a(7) &= -18, & a(11) &= 135, & a(15) &= -510 \\ a(27) &= p_{18}(6) & &= -7038, & a(39) &= -27710 & & \\ a(147) &= p_{18}(36) &= \sum_{j=0}^{36} p_{15}(j) p_3(36-j) &= -767039 \end{aligned} \quad (4)$$

(as also confirmed recently by Prof. M. Newman).

For  $z$  in the complex upper half-plane and  $q = e^{2\pi iz}$ , let  $\eta^6(q) E_6(q) = \sum_{n=1}^{\infty} b(n) e^{2\pi inz/4}$  so that  $b(1) = 1$ ,  $b(5) = -510$ ,  $b(13) = -27710$ ,  $b(21) = 504 \times 720$ . Then (as functions of  $z$ ),  $\eta^{18}$ ,  $\eta^6 E_6$  are two cusp forms generating a 2-dimensional subspace  $S$  of the space of cusp forms in  $M_9(12)$  of weight 9, divisor 3 and character  $\varepsilon$  in the sense of Hecke, where for  $(n, 12) = 1$ ,  $\varepsilon(n) = \sin^9(\pi n/2)$ ; actually in the notation of Rankin,  $S = M_9^{(0)} \oplus M_9^{(6)}$  and further  $S$  is invariant under the Hecke operators  $T_n$  for  $(n, 12) = 1$  (See [5]). In fact, for primes  $p \equiv 1, 5 \pmod{12}$ ,

$$T_p(\eta^{18}) = a(3p)\eta^{18}, \quad T_p(\eta^6 E_6) = b(p)\eta^6 E_6.$$

For primes  $r \equiv 7, 11 \pmod{12}$ ,  $T_r$  permutes  $\eta^{18}$  and  $\eta^6 E_6$  up to scalar multiples and indeed

$$T_7(\eta^6 E_6) = b(21)\eta^{18} = (504 \times 720)\eta^{18}, \quad T_7(\eta^{18}) = a(7)\eta^6 E_6 = -18\eta^6 E_6.$$

The eigenvalues of the matrix  $A = \begin{pmatrix} 0 & -18 \\ 504 \times 720 & 0 \end{pmatrix}$  are  $\pm 432\sqrt{-35}$ , and if  $\lambda = 432 i\sqrt{35}$  and  $P = \begin{pmatrix} -\lambda/18 & 1 \\ \lambda/18 & 1 \end{pmatrix}$ , we have  $PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ . Thus  $\eta^6 E_6 \pm (\lambda/18)\eta^{18}$  are normalized

eigenfunctions of the Hecke algebra and as a consequence, the associated Dirichlet series have Euler product expansions. (Incidentally, we have, from above, a justification, although indirect, for the application—untenable on the face of it—of formula (1) in page 6 of [4] even for the prime  $p = 3$  with  $t = 3$ , in order to find out the value of  $\lambda_3$  in line 3 from the bottom of page 9 of [4].) One now concludes that for  $48 i\sqrt{35} \eta^{18}$ , the associated Dirichlet series is the difference of two Euler products. To this extent, therefore, Ramanujan’s assertion (2) is true.

However, if assertion (2) were true exactly as stated, we would have

$$\begin{aligned} 156 a(147) \cdot (147)^{-s} &= 78 \cdot 3^{-s} \cdot 7^{-2s} ((1404)^2 - 7^8) - (-78 \cdot 3^{-s} \cdot 7^{-2s} ((1404)^2 - 7^8)) \\ &= -156 \cdot (147)^{-s} \times 3793585 \end{aligned}$$

which gives a contradiction to  $a(147) = -767039$  from (4).

Now the Hecke algebra acting on the 2-dimensional space  $S$  contains the representation matrices  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $A$  and further, being commutative and semisimple over  $\mathbb{Q}$  of rank  $\leq 2$ , is necessarily isomorphic to  $\mathbb{Q}(\sqrt{-35})$ . Since the representation matrix corresponding to the Hecke operator  $T_p$  on  $S$  for primes  $p \equiv 1, 5 \pmod{12}$  has the form  $\begin{pmatrix} a(3p) & 0 \\ 0 & b(p) \end{pmatrix}$ , it follows necessarily that  $a(3p) = b(p)$  for all such  $p$ . But then we have

$$\sum_{n=0}^{\infty} a(3p^n) p^{-ns} = \sum_{n=0}^{\infty} b(p^n) p^{-ns} = (1 - a(3p) p^{-s} + p^{8-2s})^{-1}$$

and these are precisely the Eulerian factors occurring in the above two Euler products corresponding to primes  $p \equiv 1, 5 \pmod{12}$ . The Eulerian factors corresponding to primes  $r \equiv 7, 11 \pmod{12}$  have the form  $(1 - \lambda(r) \cdot r^{-s} + \varepsilon(r) r^{8-2s})^{-1}$ . In the light of the different structure of the Eulerian factors in (2) for primes  $r \equiv 5, 7 \pmod{12}$ , perhaps Ramanujan had in mind some other cusp form  $g$  of weight 9 and level 12 such that  $g + 78 \eta^{18}$  is an eigenfunction of the Hecke algebra.

We next come to the “analogous” Euler products connected with  $\eta^{10}(12z)$ ,  $\eta^{14}(12z)$ ,  $\eta^{20}(6z)$  and  $\eta^{22}(12z)$  in (3) and hinted at by Ramanujan. From [3] and [4], we know, in fact, that for  $\rho, \delta = \pm 1$ ,  $\rho \cdot 48^{10} + \eta^2 E_4$ ,  $\rho \cdot 360 i\sqrt{3} \eta^{14} + \eta^2 E_6$ ,  $\rho \cdot 288\sqrt{70} \eta^{20} + \eta^4 E_4^2$ ,  $\delta \cdot 103680 i\sqrt{7} \eta^{22} + \eta^2 E_4 E_6 + \rho \cdot 96\sqrt{1045} \eta^{10} E_6 + \delta \rho \cdot 216 i\sqrt{7315} \eta^{14} E_4$  are eigenfunctions of the Hecke algebra and the associated Dirichlet series have respectively the Euler product expansions

$$\begin{aligned} \Pi_{\rho}(5) &= (1 - \rho \cdot 48 \cdot 5^{-s} + 5^{4-2s})^{-1} (1 - 238 \cdot 13^{-s} + 13^{4-2s})^{-1} \dots \\ \Pi_{\rho}(7) &= (1 - \rho \cdot 360 i\sqrt{3} \cdot 7^{-s} - 7^{6-2s})^{-1} (1 + 506 \cdot 13^{-s} + 13^{6-2s})^{-1} \dots \\ \Pi_{\rho}(10) &= (1 - \rho \cdot 288\sqrt{70} \cdot 5^{-s} + 5^{9-2s})^{-1} (1 - 476 \cdot 7^{-s} - 7^{9-2s})^{-1} \dots \\ \Pi_{\rho,\delta}(11) &= (1 - \rho \cdot 96\sqrt{1045} \cdot 5^{-s} + 5^{10-2s})^{-1} (1 - \delta \rho \cdot 216 i\sqrt{7315} \cdot 7^{-s} - 7^{10-2s})^{-1} \\ &\quad \times (1 - \delta \cdot 103680 i\sqrt{7} \cdot 11^{-s} - 11^{10-2s})^{-1} \dots \end{aligned} \tag{5}$$

and we have by appropriate elimination, Ramanujan’s “analogous results”:

THEOREM (Ramanujan). If  $\sum_n a(n)x^n$  is the expansion for  $96 \eta^{10}(x^{12})$ ,  $720 i\sqrt{3} \eta^{14}(x^{12})$ ,  $576\sqrt{70} \eta^{20}(x^6)$ ,  $414720 i\sqrt{7} \eta^{22}(x^{12})$ , respectively, then  $\sum_n a(n) \cdot n^{-s}$  is equal respectively to

$$\begin{aligned} & \Pi_1(5) - \Pi_{-1}(5), \quad \Pi_1(7) - \Pi_{-1}(7), \quad \Pi_1(10) - \Pi_{-1}(10), \\ & \Pi_{1,1}(11) - \Pi_{1,-1}(11) + \Pi_{-1,1}(11) - \Pi_{-1,-1}(11) \end{aligned}$$

for the Euler products  $\Pi_*(*)$  given by (5).

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