the set E_0 described above (following Theorem 5) is a null $\mathfrak{U}$-set and *a fortiori* (Theorem 7) a null $\mathfrak{S}$-set. A simple example of a null $\mathfrak{S}$-set which is not a $\mathfrak{U}$-set is the closed set C of all numbers in [0, 1] which have decimal expansions (scale 10) omitting the integer 5. C is null (see *e.g.* Hardy and Wright, *Theory of Numbers* (1938), 122); but if $x = 0.a_1 a_2 \ldots$ is any number in (0, 1), we may write

$$x = \sum_{r=1}^{\infty} \frac{b_r}{10^r} - \sum_{r=1}^{\infty} \frac{c_r}{10^r},$$

where $b_r = 6$ and $c_r = 1$ if $a_r = 5$, and $b_r = a_r$, $c_r = 0$ if $a_r \neq 5$. This implies $C \in \mathfrak{S}$, but by Theorem 6 Cor. (iii) $C \bar{\in} \mathfrak{U}$.

On the other hand a perfect null set is not always an $\mathfrak{S}$-set. For example, the set of all numbers in [0, 1] which have decimal expansions (scale 10) using only the digits 0, 1 is perfect, but the difference between two of its members can never be 5×10^{-n} if n is an integer. Nor is it true that a set with positive exterior measure must be an $\mathfrak{S}$-set. The well known example of a (non-measurable) set E of real numbers such that every real number differs by a rational number from exactly *one* number in E is a case in point. No rational number is the difference of two members of E, although $|E| > 0$.

University College,
London.

ON A THEOREM OF WALFISZ

S. CHOWLA*.

1. Let $\tau(n)$ denote Ramanujan's function defined by

$$\sum_{1}^{\infty} \tau(n) x^n = x \prod_{1}^{\infty} (1 - x^n)^{24} \quad (|x| < 1).$$

Walfisz† proved that the congruence

$$\tau(n) \equiv 0 \pmod{2^5 . 5^2 . 3^2 . 7 . 691}$$

* Received 28 November, 1946; read 19 December, 1946.

† *Travaux de l'Institut Mathématique de Tbilissi*, 5 (1938), 145–152. In a paper in *Quart. J. of Math.* (Oxford), 18 (1947), 122–123, Bambah and Chowla give a new proof of Walfisz's result. They also observe that the result $\tau(n) \equiv 0 \pmod{23}$ for almost all n is implicitly contained in known results but has never been explicitly stated in the literature before.

is true for almost all n. This was an improvement on earlier results*. The object of this note is to prove the sharper

THEOREM 1. *Let $\theta_1, \theta_2, \ldots, \theta_6$ denote arbitrary non-negative integers. Then the congruence*

$$\tau(n) \equiv 0 \pmod{2^{\theta_1} . 3^{\theta_2} . 5^{\theta_3} . 7^{\theta_4} . 23^{\theta_5} . 691^{\theta_6}}$$

is true for almost all n.

Walfisz's result is the special case

$$\theta_1 = 5, \quad \theta_2 = \theta_3 = 2, \quad \theta_4 = 1, \quad \theta_5 = 0, \quad \theta_6 = 1.$$

2. When p is a prime we say that "n contains the prime p to the power t" if p^t is a factor of n, while p^{t+1} is not. We first prove:

LEMMA 1. *Let r and k denote arbitrary positive integers. Then almost all positive integers n contain at least r different primes congruent to* $-1 \pmod k$ *each to an odd power.*

This result is well-known for $r = 1$. Denote by $N_k(x)$ the number of positive integers n, not exceeding x, such that n contains to an even power every prime p congruent to $-1 \pmod k$. When $k = 4$ it is well-known that $N_k(x)$ represents the number of positive integers, not exceeding x, which can be expressed as a sum of two squares of integers. Further, we have (H, p. 168)

$$N_k(x) = O\left(\frac{x}{\log^c x}\right), \tag{1}$$

where $0 < c < 1$, $c = c(k)$. Now let m_t run through all square-free positive integers such that m_t is a product of exactly t different primes, all congruent to $-1 \pmod k$. Write

$$S_t(x) = \sum_{m_t \leqslant x} N_k\left(\frac{x}{m_t}\right).$$

Clearly $S_t(x)$ represents the number of positive integers n, not exceeding x, such that n contains to odd powers exactly t different primes congruent to $-1 \pmod k$. Further, $S_1(x) + S_2(x) + \ldots + S_r(x)$ represents the number of

* See *Ramanujan* by G. H. Hardy (Cambridge, 1940), 168. We refer to this book as H.

positive integers n, not exceeding x, such that n contains to odd powers at most r different primes congruent to $-1 \pmod{k}$.

3. We write

$$y = x/e^{\sqrt{(\log x)}}$$

and estimate $S_t(x)$ as follows:

$$S_t(x) = \sum_{m_t \leqslant y} N_k\left(\frac{x}{m_t}\right) + \sum_{y < m_t \leqslant x} N_k\left(\frac{x}{m_t}\right)$$

$$= A_t(x) + B_t(x), \tag{2}$$

say.

Denote by $\pi_t(x)$ the number of positive square-free integers containing exactly t different primes congruent to $-1 \pmod{k}$, so that

$$\pi_t(x) = \sum_{m_t \leqslant x} 1.$$

By a well-known result*

$$\pi_t(x) = O\left(\frac{x}{\log x}(\log\log x)^{t-1}\right).$$

Using the last result we deduce from (1) that

$$A_t(x) = O\left(\sum_{m_t \leqslant y} \frac{x}{m_t \log^c (x/y)}\right)$$

$$= O\left(\frac{x}{\log^{\frac{1}{2}c}(x)} \sum_{m_t \leqslant x} \frac{1}{m_t}\right)$$

$$= O\left(\frac{x}{\log^{\frac{1}{2}c} x}\right) \sum_{n=1}^{x} \frac{\pi_t(n) - \pi_t(n-1)}{n}$$

$$= O\left(\frac{x}{\log^{\frac{1}{2}c} x}\right) \left\{\sum_{2}^{x-1} \frac{\pi_t(n)}{n(n+1)} + \frac{\pi_t(x)}{x}\right\}$$

$$= O\left(\frac{x(\log\log x)^t}{\log^{\frac{1}{2}c} x}\right). \tag{3}$$

* See e.g. Landau's *Handbuch der Lehre von der Verteilung der Primzahlen* (Leipzig, 1909).

To estimate $B_t(x)$ we use the crude inequality $N_k(x) \leqslant x$. Hence

$$B_t(x) = O\Big(\sum_{y<m_t\leqslant x} \frac{x}{m_t}\Big)$$

$$= O(x) \sum_{y<n\leqslant x} \frac{\pi_t(n) - \pi_t(n-1)}{n}$$

$$= O(x)\Big\{-\frac{\pi_t(y)}{y+1} + \frac{\pi_t(x)}{x} + \sum_{y<n<x} \frac{\pi_t(n)}{n(n+1)}\Big\}$$

$$= O\Big\{\frac{x(\log\log y)^{t-1}}{\log y} + \frac{x(\log\log x)^{t-1}}{\log x} + \frac{x(\log\log y)^{t-1}}{\log y} \sum_{y<n<x} \frac{1}{n}\Big\}$$

$$= O\Big(\frac{x(\log\log x)^{t-1}}{\sqrt{(\log x)}}\Big), \tag{4}$$

since $\log y \sim \log x$, $\log\log y \sim \log\log x$.

By (2), (3) and (4) we have, for $1 \leqslant m \leqslant r$,

$$S_m(x) = o(x), \tag{5}$$

whence

$$\sum_{t=1}^{r} S_t(x) = o(x). \tag{6}$$

From (6) and the meaning given to the left side of (6) at the end of §2, Lemma 1 follows at once.

4. It is known (*cf.* H, p. 161) that

$$\tau(mn) = \tau(m)\tau(n) \quad \text{if} \quad (m, n) = 1, \tag{7}$$

a result conjectured by Ramanujan and proved by Mordell. Denote by $\sigma_k(n)$ the sum of the k-th powers of the divisors of n. Then we have the following known* results:

$$\tau(n) \equiv \sigma_1(n) \pmod 8 \quad \text{if} \quad n \equiv 1 \pmod 2, \tag{8}$$

$$\tau(n) \equiv \sigma_1(n) \pmod 3 \quad \text{if} \quad (n, 3) = 1, \tag{9}$$

$$\tau(n) \equiv n\sigma_1(n) \pmod 5, \tag{10}$$

$$\tau(n) \equiv n\sigma_3(n) \pmod 7, \tag{11}$$

$$\tau(n) \equiv 0 \pmod{23} \quad \text{if} \quad n \equiv 22 \pmod{23}, \tag{12}$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}. \tag{13}$$

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* For (8) see Gupta, *Journal Indian Math. Soc.* 9 (1945), 59–60, where (9) is also proved; for (11) see Ramanathan, *Journal Indian Math. Soc.* 9 (1945), 55–59; for (10), (12) and (13) see H, p. 166, H, p. 166 and H, p. 169 respectively. Another proof of (8) has recently been given by Bambah, Chowla, Gupta and Lahiri in *Quart J. of Math.* (Oxford), 18 (1947), 143–146,

From these results it is easy to deduce

LEMMA 2. *The congruence*

$$\tau(n) \equiv 0 \pmod{k^r}$$

holds if n contains at least r different primes $p \equiv -1 \pmod{k}$, *each to an odd power, where k is any of the numbers* 2, 3, 5, 7, 23, 691.

From Lemmas 1 and 2 we get at once

THEOREM 2. *The congruence*

$$\tau(n) \equiv 0 \pmod{k^r}$$

is true for almost all n, where r is an arbitrary positive integer and k is any of the numbers 2, 3, 5, 7, 23, 691.

Theorem 1 follows at once from Theorem 2.

I should like to express my thanks to a referee for several useful suggestions, and in particular for the neat formulation of Lemma 2.

Government College,
Lahore, India.

THE RESIDUE OF RAMANUJAN'S FUNCTION $\tau(n)$ TO THE MODULUS 2^8

R. P. BAMBAH *and* S. CHOWLA*.

1. Ramanujan's function $\tau(n)$ is defined by

$$\sum_{n=1}^{\infty} \tau(n)\, x^n = x \prod_{1}^{\infty} (1-x^n)^{24}.$$

Bambah proved† that

(1) $\tau(n) \equiv \sigma_3(n) \pmod{2^5}$ *when* $n \equiv 1 \pmod{2}$,

where $\sigma_k(n) = \sum_{n|d} d^k; \quad \sigma(n) = \sigma_1(n).$

We‡ prove here the following theorem:

(2) $\tau(n) \equiv \sigma_{11}(n) \pmod{2^8}$ *when* $n \equiv 1 \pmod{2}$.

* Received 8 October, 1946; read 14 November, 1946.

† *Journal London Math. Soc.*, 21 (1946), 91–93.

‡ The result of this paper was arrived at independently and almost simultaneously by the two authors. We should like to record our thanks to Prof. Hardy and the referees for several helpful suggestions.