PERIODIC SIMPLE CONTINUED FRACTIONS

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1. Let $N(R)$ denote the number of elements in the periodic part of the simple continued fraction for $\sqrt{R}$. Vijayaraghavan† has proved that

$$N(R) = O(R^{\frac{1}{2}} \log R) \tag{1}$$

for all positive integers R, and that

$$N(R) > R^{\frac{1}{2}-\delta} \tag{2}$$

for infinitely many values of the positive integer R, where δ is an arbitrarily small positive number.

In this paper we prove the following results.

THEOREM 1. *If R is not a perfect square and if no zero of the function*

$$\sum_{(n,\, 2R)=1} (R \mid n) n^{-s} \quad (s = \sigma + it,\ \sigma > 0)$$

lies on the right of $\sigma = \frac{1}{2}$, then

$$N(R) = O(\sqrt{R} \log \log R). \tag{3}$$

Here $(R \mid n)$ is the generalized Legendre (not Kronecker) symbol.

THEOREM 2A. *There are positive constants C_1 and C_2 such that*

$$C_1 \sqrt{R} < N(R) < C_2 \sqrt{R} \tag{4}$$

is true for infinitely many positive integers R.

This theorem is an improvement of (2).

THEOREM 3.

$$\sum_{x \leqslant R} N(x) = O(R^{\frac{3}{2}}). \tag{5}$$

In words, $N(R)$ is, on the average, of order $\sqrt{R}$.

2. The hypothesis that, for a particular k and χ, $L(s, \chi)$ has no zeros in $\sigma > \frac{1}{2}$ we call $X(k, \chi)$. Littlewood‡ has proved that

if χ is a real non-principal character to modulus k and the hypothesis

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† *Proc. London Math. Soc.* (2), 26 (1927), 403–414.

‡ *Proc. London Math. Soc.* (2), 27 (1928), 367, Theorem 1.

$X(k, \chi)$ is true, then

(6) $$L(1) = O(\log\log k).$$

Since we know that*

(7) $$h(D)\log(T+U\sqrt{D}) = 2\sqrt{D} \sum_{(n,\,2D)=1} (D\,|\,n)\, n^{-1}$$

and†

(8) $$N(D) = O\{\log(T+U\sqrt{D})\},$$

Theorem 1 is an immediate corollary.

3. Before proving the harder Theorem 2A, it is worth while to notice that we have

THEOREM 2B. *There are positive constants C_1 and C_2 such that*

(9) $$\frac{C_1\sqrt{R}}{\log R} < N(R) < C_2\sqrt{R}$$

is true for infinitely many R.

This is an immediate corollary of the well known result (7). For put $R = 5^{2n+1}$ in (7). Observing that‡

(10) $$h(5^{2n+1}) = 1,$$

we obtain (9) from (8), (10), and the simple result that

(11) $$N(D) > b\,\frac{\log(T+U\sqrt{D})}{\log D},$$

where b is positive and independent of D.

4. We require the following lemmas for the proof of Theorem 2A.

LEMMA 1. *If* §

(i) $$1 < m < \sqrt{R},$$

(ii) $$(m,\, 2R) = 1,$$

(iii) $$m\,|\,(x^2 - R) \text{ for some } x,$$

(iv) $$h(R) = 1,$$

* Mathews, *Theory of numbers*, 238.

† Vijayaraghavan, *loc. cit.*

‡ See § 5 of this paper.

§ (x, y) denotes the greatest common divisor of x and y. $a\,|\,b$ means that a is a divisor of b.

then m occurs exactly $2^{\varpi(m)}$ times as a partial quotient in the first period of the simple continued fraction for $\sqrt{R}$. Here $\varpi(m)$ denotes the number of distinct prime factors of m.

Proof. There is a well known theorem of Lagrange that if

$$1 < m < \sqrt{R}, \quad m = u^2 - Rv^2, \quad (u, v) = 1,$$

then m occurs as a partial quotient in the simple continued fraction for $\sqrt{R}$. Now from (ii), (iii), and (iv) it follows that

$$m = u^2 - Rv^2, \quad (u, v) = 1.$$

Hence, by Lagrange's theorem, u/v is a convergent to the simple continued fraction for $\sqrt{R}$. Now, from the theory of binary quadratic forms, there are exactly $2^{\varpi(m)}$ solutions of

$$\text{(v)} \qquad x^2 - Ry^2 = m,$$

$$\text{(vi)} \qquad (x, y) = 1,$$

$$\text{(vii)} \qquad 1 < x + y\sqrt{R} < t + u\sqrt{R},$$

where $x > 0$, $y > 0$, and t and u are the least positive integral values of x, y in the Pellian equation $x^2 - Ry^2 = 1$. Since $h(R) = 1$, $x^2 - Ry^2 = -1$ is solvable, and hence t/u is a convergent just before the end of the *second* period. Hence there are exactly $2^{\varpi(m)}$ *even* convergents within the first two periods which give solutions of $x^2 - Ry^2 = m$; from this and a known property of continued fractions it follows that there are exactly $2^{\varpi(m)}$ *odd* convergents within the first two periods which furnish solutions of $x^2 - Ry^2 = -m$. It follows that m occurs as a partial quotient exactly $2^{\varpi(m)}$ times in the first period.

5. LEMMA 2. $h(5) = h(5^3) = \dots = h(5^{2n+1}) = 1$,
where n is any positive integer.

Proof. It is known that* $h(5) = 1$.

This means that, if all the prime factors of m are of the forms $20k+1$, 9, 11, 19, then m is of the form $x^2 - 5y^2$. The lemma will be proved if we can show that when a number m, prime to 10, is of the form

$$x^2 - 5^{2n+1} y^2,$$

then it is also expressible in the form

$$x^2 - 5^{2n+3} y^2.$$

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* Mathews, *loc. cit.*, 248.

Let $$m = a^2 - 5^{2n+1}\beta^2, \quad 1 = \gamma^2 - 5^{2n+1}\delta^2. \qquad \text{(i, ii)}$$

It follows, on multiplication, that

$$m = (a\gamma + 5^{2n+1}\beta\delta)^2 - 5^{2n+1}(a\delta + \beta\gamma)^2. \qquad \text{(iii)}$$

If $\beta \equiv 0 \pmod 5$, then m is of the form $x^2 - 5^{2n+3}y^2$, and we have proved what we wanted. Hence we may suppose that $\beta \not\equiv 0 \pmod 5$. Now a and β are given while γ is either $\equiv 1 \pmod 5$ or $\equiv -1 \pmod 5$. It is easy to prove that we can find δ in (ii) so as to be of any of the forms $5k+1$, $5k+2$, $5k+3$, $5k+4$ that we choose. Hence, since $\beta \not\equiv 0 \pmod 5$, in (iii), we can make $a\delta + \beta\gamma \equiv 0 \pmod 5$. Hence m is of the form $x^2 - 5^{2n+3}y^2$ in all cases. The lemma now follows by induction.

6. LEMMA 3. *If* $R = 5^{2n+1}$, *then*

$$N(R) \geqslant \sum_{1}^{\sqrt{R}} 2^{\varpi(m)} f(m),$$

where $f(m) = 1$ *if all the prime factors of* m *are of the forms* $20k+1$, 9, 11, 19, *and* $f(m) = 0$ *otherwise.*

This follows at once from the preceding lemmas.

LEMMA 4. *If* $s > 1$, *then*

$$\sum_{m=1}^{\infty} \frac{2^{\varpi(m)} f(m)}{m^s} = \frac{\zeta(s)\, L(s)}{(1+2^{-s})(1+5^{-s})\,\zeta(2s)},$$

where $\zeta(s)$ *is Riemann's zeta function and*

$$L(s) = 1^{-s} - 3^{-s} - 7^{-s} + 9^{-s} + 11^{-s} - 13^{-s} - 17^{-s} + 19^{-s} + - - + \ldots.$$

This is proved by expressing the right-hand side as an infinite product.

LEMMA 5. $$\sum_{m \leqslant x} 2^{\varpi(m)} f(m) = \frac{10L(1)}{3\pi^2} x + O(x^{\frac{1}{2}} \log x).$$

Proof. Let, for $s > 1$,

$$\zeta(s)\, L(s) = \sum_{1}^{\infty} a_n n^{-s}.$$

Then we know that*

$$\sum_{n \leqslant x} a_n = L(1)x + O(x^{\frac{1}{2}}). \qquad (12)$$

Lemma 5 follows easily from (12) and Lemma 4.

* Landau, *Handbuch*, 472–4.

7. Since $L(1)$ is positive, Lemmas 3 and 5 give

$$N(R) > c\sqrt{R} \quad (R = 5^{2n+1}), \tag{13}$$

where c is an absolute positive constant. Theorem 2A follows from (13) and the second inequality in (9).

8. We now prove Theorem 3. Since*

$$N(R) \leqslant \sum_{y \leqslant \sqrt{R}} g(R-y^2, R, y), \tag{14}$$

where $g(R-y^2, R, y)$ is the number of divisors of $R-y^2$ between $\sqrt{R}-y$ and $\sqrt{R}+y$, it follows that

$$\begin{aligned} \sum_{R \leqslant x} N(R) &\leqslant \sum_{R \leqslant x} \sum_{y \leqslant \sqrt{R}} g(R-y^2, R, y) \\ &= \sum_{y \leqslant \sqrt{x}} \sum_{R=y^2+1}^{x} g(R-y^2, R, y) \\ &= \sum_{y \leqslant \sqrt{x}} \sum_{m=1}^{y+\sqrt{x}} \theta(m), \end{aligned} \tag{15}$$

where $\theta(m)$ is the number of solutions of values of R such that

$$R-y^2 \equiv 0 \pmod{m} \tag{16}$$

and

$$\sqrt{R}-y < m < \sqrt{R}+y. \tag{17}$$

From the latter condition

$$m^2+y^2-2my < R < m^2+y^2+2my.$$

Hence R takes $4my-1$ consecutive values, of which only† $[(4my-1)/m] = 4y-1$ satisfy (16). Hence $\theta(m) = 4y-1$, and it follows from (15) that

$$\begin{aligned} \sum_{R \leqslant x} N(R) &\leqslant \sum_{y \leqslant \sqrt{x}} (4y-1)(y+\sqrt{x}) \\ &= \tfrac{4}{3}x^{\frac{3}{2}}+2x^{\frac{3}{2}}+O(x) = \tfrac{10}{3}x^{\frac{3}{2}}+O(x) = O(n^{\frac{3}{2}}). \end{aligned} \tag{18}$$

This is Theorem 3.

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* Vijayaraghavan, *loc. cit.*, 407.

† $[x]$ denotes the greatest integer contained in x.