ON THE NUMBER OF POSITIVE INTEGERS $\leqq x$ ALL OF WHOSE PRIME FACTORS ARE $\leqq y$

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Recently de Bruijn [1; 2] estimated the functions $f(x, y)$ and $g(x, y)$, where $f(x, y)$ denotes the number of positive integers $\leqq x$, all of whose prime factors are $\leqq y$, and where $g(x, y)$ denotes the number of positive integers $\leqq x$, all of whose prime factors are $>y$. A special and interesting case are the estimates

$$(1') \qquad f[x, (\log x)^h] = O(x^{1-1/h+\epsilon}) ,$$

$$(1'') \qquad g[x, (\log x)^h] = x \prod_{p \leqq (\log x)^h} (1 - p^{-1}) + O(x^{1-1/h+2/h^2+\epsilon})$$

where ϵ is an arbitrarily small positive number, p denotes primes, $h>2$, and the constants implied in the O-symbols depend on ϵ and h.

Vinogradoff [4] in his proof of the theorem that every large odd number is the sum of three primes, gives a much cruder estimate than (1′). Recently Ankeny[2] proved that if $p \equiv 3 \pmod 4$, then $n(p) < p^\epsilon$, for $p > p_0(\epsilon)$, where $n(p)$ is the least positive quadratic non-residue of the prime p. This proof uses de Bruijn's estimates for $f(x, y)$ and $g(x, y)$ in a sharper form, however, than those given above.

We follow de Bruijn in proving (1′) and (1″) which are listed as Theorems 1 and 2 below. Although these results are only special cases of de Bruijn's work, they seem important enough to merit simple proofs. Although our proof follows de Bruijn's very closely, we believe it is considerably simpler.

Writing

$$(2) \qquad \prod_{p \leqq y} (1 - p^{-s})^{-1} = \sum_1^\infty c_n n^{-s},$$

$$(3) \qquad f(x, y) = \sum_{n \leqq x} c_n,$$

we have

$$(4) \qquad \zeta(s) \prod_{p \leqq y} (1 - p^{-s}) = \sum_1^\infty b_n n^{-s},$$

$$(5) \qquad 1 + g(x, y) = \sum_{n \leqq x} b_n$$

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[2] N. C. Ankeny, *Quadratic residues*, Duke Math. J. vol. 21 (1954) pp. 107–112.

where $\zeta(s)$ is the zeta function of Riemann and c_n and b_n equal 1 or 0.

Let $a=1+1/\log x$, $b=1-1/h$, $T=x^{2/h}$, and $h>2$. For convenience let x be half an odd integer. This choice of x does not mean a restriction, as can easily be seen after the proof.

LEMMA 1. *For* $c>1$, $0<y\neq 1$, $R>0$,

$$\left|\frac{1}{2\pi i}\int_{c-Ri}^{c+Ri}\frac{y^s}{s}\,ds - E(y)\right| \leqq \frac{y^c}{\pi R|\log y|}$$

where

$$E(y) = \begin{cases} 1 \textit{ if } y > 1, \\ 0 \textit{ if } y < 1. \end{cases}$$

The proof follows Landau [**3**, p. 112].

Let $y>1$ and apply Cauchy's Theorem to the rectangle with corners $c\pm Ri$, $d\pm Ri$, $d<0$.

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{c-Ri}^{c+Ri}\frac{y^s}{s}\,ds - 1\right| &= \left|\frac{1}{2\pi i}\int_{c-Ri}^{d-Ri}\frac{y^s}{s}\,ds + \frac{1}{2\pi i}\int_{d-Ri}^{d+Ri}\frac{y^s}{s}\,ds + \frac{1}{2\pi i}\int_{d+Ri}^{c+Ri}\frac{y^s}{s}\,ds\right| \\ &\leqq \frac{1}{2\pi R}\int_{-\infty}^{c} y^r dr + \frac{Ry^d}{-d\pi} + \frac{1}{2\pi R}\int_{-\infty}^{c} y^r dr = \frac{y^c}{\pi R\log y} - \frac{Ry^d}{\pi d},\end{aligned}$$

and the result follows by letting $d\to-\infty$.

Let $0<y<1$ and apply Cauchy's Theorem to the rectangle with corners $c\pm Ri$, $d\pm Ri$, $d>c$.

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{c-Ri}^{c+Ri}\frac{y^s}{s}\,ds\right| &= \left|\frac{1}{2\pi i}\int_{c-Ri}^{d-Ri}\frac{y^s}{s}\,ds + \frac{1}{2\pi i}\int_{d-Ri}^{d+Ri}\frac{y^s}{s}\,ds + \frac{1}{2\pi i}\int_{d+Ri}^{c+Ri}\frac{y^s}{s}\,ds\right| \\ &\leqq \frac{1}{2\pi R}\int_{c}^{\infty} y^r dr + \frac{Ry^d}{\pi d} + \frac{1}{2\pi R}\int_{c}^{\infty} y^r dr = \frac{y^c}{\pi R|\log y|} + \frac{Ry^d}{\pi d},\end{aligned}$$

and the result followed by letting $d\to\infty$.

THEOREM 1. *For any fixed* $h>2$

$$f[x, (\log x)^h] = O(x^{1-1/h+\epsilon}),$$

where ϵ *is an arbitrary positive number.*

From Lemma 1, (2), and (3) we obtain, since $c_n = 0$ or 1,

$$f(x, y) = \frac{1}{2\pi i}\int_{a-Ti}^{a+Ti} x^s \prod_{p\leqq y} (1 - p^{-s})^{-1}\frac{ds}{s} + O\left\{\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a \frac{1}{T\,|\log x/n\,|}\right\}. \tag{6}$$

To estimate

$$\sum_{n=1}^{\infty}\frac{1}{n^a\,|\log x/n\,|}$$

break the summation into $\sum_1 + \sum_2 + \sum_3$, where $n < x/2$ in $\sum_1$, $n \geqq 2x$ in $\sum_2$, and $x/2 < n < 2x$ in $\sum_3$.

$$\sum_1 = O\left(\sum_{n\leqq x}\frac{1}{n}\right) = O(\log x),$$

$$\sum_2 = O\left(\int_x^{\infty} y^{-a}dy\right) = O\left(\frac{x^{1-a}}{a-1}\right) = O(\log x).$$

The estimate for $\sum_3$ is obtained by using

$$|\log x/n\,| = \left|\log\left(1 + \frac{x-n}{n}\right)\right| > \begin{cases} \dfrac{1}{2}\left(\dfrac{x-n}{n}\right) & \text{if } n < x, \\ \dfrac{n-x}{n} & \text{if } n > x. \end{cases}$$

Hence, remembering that x is half an odd integer,

$$\sum_3 = O\left(\sum_{x/2<n<2x}\frac{n}{n^a\,|\,n-x\,|}\right) = O\left(\sum_{x/2<n<2x}\frac{1}{|\,n-x\,|}\right) = O(\log x).$$

Therefore the O-term in (6) is

$$O\left(\frac{x^a \log x}{T}\right) = O(x^{1-2/h+\epsilon}).$$

The integral term in (6) is evaluated by applying Cauchy's Theorem to the rectangle with corners $a \pm Ti$, $b \pm Ti$, and is seen to have the value $(1/2\pi i)(J_1 + J_2 + J_3)$, where

$$J_1 = \int_{a-Ti}^{b-Ti}, \qquad J_2 = \int_{b-Ti}^{b+Ti} \qquad J_3 = \int_{b+Ti}^{a+Ti}.$$

On this rectangle

$$\left|\prod_{p\leq y}(1-p^{-s})^{-1}\right| = O\left[\prod_{p\leq y}(1+p^{-b})\right] = O\left[\exp \sum_{p\leq y} p^{-b}\right].$$

$$\sum_{p\leq y} p^{-b} = \sum_{2\leq t\leq y}\frac{\pi(t)-\pi(t-1)}{t^b} = \sum_{2\leq t\leq y}\pi(t)\left[t^{-b}-(t+1)^{-b}\right]$$

$$+O\left[\frac{\pi(y)}{y^b}\right]$$

$$= O\left(\sum_{2\leq t\leq y}\frac{t}{\log t}\frac{1}{t^{1+b}}\right)+O\left(\frac{y^{1-b}}{\log y}\right)$$

$$= O\left(\int_2^{y^{1/2h}}\frac{dt}{t^b\log t}+\int_{y^{1/2h}}^{y}\frac{dt}{t^b\log t}\right)+O\left(\frac{y^{1-b}}{\log y}\right)$$

$$= O(y^{1/2h})+O\left(\frac{1}{\log y}\int_1^y\frac{dt}{t^b}\right)+O\left(\frac{y^{1-b}}{\log y}\right)$$

$$= O(y^{1/2h})+O\left(\frac{y^{1-b}}{\log y}\right)=O\left(\frac{y^{1-b}}{\log y}\right)=O\left(\frac{y^{1/h}}{\log y}\right).$$

Hence, for some constant k,

$$\left|\prod_{p\leq y}(1-p^{-s})^{-1}\right| = O\left(\exp\frac{k\log x}{\log\log x}\right).$$

Therefore the order of J_1 and J_3 is

$$O\left(\frac{x^{1+\epsilon'}}{T}\right) = O(x^{1-2/h+\epsilon'}),$$

and that of J_2 is

$$O\left(x^{b+\epsilon'}\int_{-T}^{T}\frac{dt}{(b^2+t^2)^{1/2}}\right) = O(x^{1-1/h+\epsilon}),$$

which gives the theorem.

THEOREM 2. *For any fixed* $h>2$,

$$g[x,(\log x)^h] = x\prod_{p\leq(\log x)^h}(1-p^{-1})+O(x^{1-1/h+2/h^2+\epsilon}).$$

Following the proof of Theorem 1, we have

$$(7)\qquad g(x,y)=\frac{1}{2\pi i}\int_{a-Ti}^{a+Ti}x^s\zeta(s)\prod_{p\leq y}(1-p^{-s})\frac{ds}{s}+O(x^{1-2/h+\epsilon}).$$

The integral term in (7) can be written as

$$x \prod_{p \leqq y} (1 - p^{-1}) + \frac{1}{2\pi i} (K_1 + K_2 + K_3),$$

where the K_i are integrals with the same limits as the J_i.

Using the completely elementary result [3, Satz 407]

$$\zeta(\sigma + it) = O(t^{1-\sigma}) \qquad [1 > \sigma \geqq 1/2],$$

the rest of the proof proceeds exactly as before.

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