The following papers were taken as read:—

A theorem in arithmetic: S. Chowla.

A universal Waring theorem for eleventh powers: L. E. Dickson.

The asymptotic paths of integral functions of finite orders: A. J. Macintyre.

A note on Fourier transforms: G. W. Morgan.

## A THEOREM ON IRRATIONAL INDEFINITE QUADRATIC FORMS

S. CHOWLA*.

THEOREM†. *If the c's are not all of one sign and if all the ratios* $c_s/c_t$ $(s \neq t)$ *are irrational, we can find integers* $n_1, \ldots, n_r$ *(not all zero) such that*

$$\left| \sum_{s=1}^{r} c_s n_s^2 \right| < \epsilon,$$

*where* $\epsilon$ *is an arbitrary positive number and* $r \geqslant 9$.

*Proof.* Of nine positive or negative numbers, at least five must have the same sign. Hence we may suppose $c_1, \ldots, c_5$ positive. It is well known from the modern theory of lattice points that‡ (since $c_s/c_t$ is irrational for $s \neq t$)

$$(1) \qquad \sum_{c_1 n_1^2 + \ldots + c_5 n_5^2 \leqslant x} 1 = A x^{\frac{5}{2}} + o(x^{\frac{3}{2}}).$$

Here $n_1, n_2, \ldots$ are integers, positive, negative, or zero, $x$ is not necessarily an integer, and $A$ is independent of $x$. Hence

$$(2) \qquad \sum_{x < c_1 n_1^2 + \ldots + c_5 n_5^2 \leqslant x + \epsilon} 1 > 0$$

for every positive $\epsilon$ and all $x \geqslant x_0(\epsilon)$. Since at least one of the numbers $c_6, \ldots, c_r$ is negative, we can find $y \geqslant x_0(\epsilon)$ such that

$$(3) \qquad \sum_{s=6}^{r} c_s n_s^2 = -y.$$

* Received 8 January, 1934; read 18 January, 1934.

† It has been conjectured that the theorem is true for $r \geqslant 5$. See A. Oppenheim, "The minima of indefinite quaternary quadratic forms", *Proc. Nat. Acad. Sc.*, 15 (1929), No. 9, 724–727 (7).

‡ V. Jarník and A. Walfisz, "Über Gitterpunkte in mehrdimensionalen Ellipsoiden", *Math. Zeitschrift*, 32 (1930), 152–160 (154). The force of my argument rests on the "little" $o$ in (1).

Having fixed $y$, we can find from (2) integers $n_1, \ldots, n_5$ (not all zero) such that

(4) $$y < c_1 n_1^2 + \ldots + c_5 n_5^2 \leqslant y + \epsilon.$$

From (3) and (4),

$$\left| \sum_{s=1}^{r} c_s n_s^2 \right| \leqslant \epsilon,$$

where the $n$'s are not all zero; this proves the theorem.

# A THEOREM IN ARITHMETIC

S. Chowla*.

Hypothesis. *Let $\theta_1, \ldots, \theta_5$ be positive numbers and such that at least one of the ratios $\theta_s/\theta_1$ ($s = 2, 3, 4, 5$) is irrational. Let $[y]$ denote the greatest integer contained in $y$.*

Theorem. *Every $n \geqslant n_0(\theta_1, \ldots, \theta_5)$ satisfies*

$$n = [\theta_1 n_1^2] + \ldots + [\theta_5 n_5^2] + c,$$

*where $c$ may be 0, 1, 2, 3, or 4, and the $n$'s are integers.*

*Remarks.* Two points about this theorem are:

(i) It is not a consequence of Schnirelmann's recent generalization† of Waring's problem.

(ii) It is not capable, as proved here, of generalization to higher powers.

*Proof.* It follows from (1) of the preceding paper that the number of solutions of

$$x < \theta_1 n_1^2 + \ldots + \theta_5 n_5^2 \leqslant x + \tfrac{1}{2}$$

is asymptotically $Bx^{\frac{3}{2}}$ for all $x \geqslant x_0(\theta_1, \ldots, \theta_5)$, where $B > 0$. Hence

$$[\theta_1 n_1^2] + \ldots + [\theta_5 n_5^2]$$

is equal to one of $x$, $x-1$, $x-2$, $x-3$, $x-4$, where $x$ is a sufficiently large integer. This proves the theorem.

* Received 27 January, 1934; read 15 March, 1934.

† "Über additive Eigenschaften von Zahlen", *Math. Annalen*, 107 (1933), 649–691 (682, § 3).