AVERAGES OF CHARACTER SUMS

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Suppose that χ is a primitive residue character[1] modulo k, $k>1$, and that for y non-negative, $S(y)=\sum_{0\leq l\leq y}\chi(l)$. It is important [see, for example, **11**] in the analytic theory of numbers to have as much information as possible about the sums $S(y)$, in particular about their maximum order of magnitude; it is known (cf. [**13**; **14**; **8**]), for example, that $S(y)<k^{1/2}\log k$, but unknown whether or not $M(\chi)=o(k^{1/2}\log k)$ as k tends to infinity, where $M(\chi)$ is the maximum of $|S(1)|, \cdots, |S(k-1)|$. Hua [**4**; **5**; **6**] has shown that it is often helpful to consider the averages $n^{-1}\sum_{m=0}^{n} S(m)$. In this paper we consider some further developments of this idea.

1. **Preliminaries.** We recall [**7**, pp. 483–486, 492–494] that if χ is a primitive residue character mod k and if $\tau(\chi)=\sum_{n=1}^{k}\chi(n)e^{2\pi in/k}$, then $|\tau(\chi)|=k^{1/2}$ and

$$\sum_{n=1}^{k}\chi(n)e^{2\pi imn/k}=\bar{\chi}(m)\tau(\chi) \tag{1}$$

for any integer m, $\bar{\chi}$ being the complex conjugate of χ.

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[1] For the basic facts about residue characters see [**7**, pp. 401–414, 478–494]. Numbers in brackets refer to the bibliography.

The function $S(kx)$ is of bounded variation and period unity, and as such has an everywhere convergent Fourier series. It may be deduced from (1) (cf. [**13**, pp. 23–24] and [**8**, pp. 81–82]) that this Fourier series is

$$A(\chi) - \frac{\tau(\chi)}{2\pi i}\sum_{m\neq 0}\frac{\bar{\chi}(m)}{m}e^{-2\pi i m x}, \tag{2}$$

where m runs over the positive and negative integers and

$$A(\chi) = \int_0^1 S(kx)dx = \frac{1}{k}\sum_{m=0}^{k-1} S(m). \tag{3}$$

Thus if we define a function $S^*(y)$ as $S(y)$ if y is not an integer and as $S(y)-\chi(y)/2$ if y is an integer, we have

$$S^*(kx) = \begin{cases} A(\chi) + \dfrac{\tau(\chi)}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)}{n}\sin 2\pi n x & \text{if } \chi(-1) = 1 \\ A(\chi) - \dfrac{\tau(\chi)}{\pi i}\displaystyle\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)}{n}\cos 2\pi n x & \text{if } \chi(-1) = -1 \end{cases} \tag{4}$$

for all non-negative x.

2. **Another proof of Paley's Ω-result for** $S(n)$. If we put $x=0$ in (4) we get a formula for the arithmetic mean $A(\chi)$ of $S(0), \cdots, S(k-1)$, namely

$$A(\chi) = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ \dfrac{\tau(\chi)}{\pi i}L(1, \bar{\chi}) & \text{if } \chi(-1) = -1, \end{cases} \tag{5}$$

where $L(s, \bar{\chi} = \sum_{n=1}^{\infty}\bar{\chi}(n)n^{-s}$ for $R(s)>0$. The second half of formula (5) can of course be proved by the method used in [**9**] to prove the second part of Satz 217 (of which the second half of (5) is a generalization), but some use of Fourier series is essential.[2]

The second part of (5) enables us, when $\chi(-1)=-1$, to use information about the order of magnitude of $L(1, \bar{\chi})$ to get information about the order of magnitude of $A(\chi)$. For example, a slight change in the argument of [**2**] shows that for real primitive χ with $\chi(-1)=-1$ we have

$$\limsup_{k\to\infty}\frac{L(1, \chi)}{\log\log k} \geqq e^{\gamma}, \tag{6}$$

[2] Naturally (5) can be proved in a roundabout way by showing that both sides are equal to $L(0, \chi)$. The argument indicated here essentially stems from Minkowski (cf. [**13**, p. 26]).

γ being Euler's constant. (The lim sup is unambiguous, since for each k there is at most one real primitive χ with $\chi(-1)=-1$.) Now $\tau(\chi)=ik^{1/2}$ for real primitive χ with $\chi(-1)=-1$ [9, Satz 215] and thus (6) gives the following result.

THEOREM 1. *If χ runs through all real primitive characters with $\chi(-1)=-1$ in order of increasing size of the modulus k of χ and if $A(\chi)$ is as defined in* (3), *then*

$$\limsup_{k\to\infty} \frac{A(\chi)}{k^{1/2}\log\log k} \geqq \frac{e^{\gamma}}{\pi}.$$

This theorem is a slightly stronger form of Paley's result (see [12; 10; 1]) that if χ is a primitive character modulo k, then $M(\chi)=\Omega(k^{1/2}\log\log k)$, where $M(\chi)$ is the maximum of $|S(0)|, \cdots, |S(k-1)|$; for our theorem shows that this Ω-result is true even for the arithmetic mean of $S(0), \cdots, S(k-1)$.

We remark in passing that Parseval's formula applied to (2) or (4) gives

$$\begin{aligned}\frac{1}{k}\sum_{m=0}^{k-1}|S(m)-A(\chi)|^2 &= \int_0^1 |S(kx)-A(\chi)|^2dx\\ &= \frac{k}{2\pi^2}\sum_{n=1}^{\infty}{}' \frac{1}{n^2} = \frac{k}{12}\prod_{p|k}\left(1-\frac{1}{p^2}\right),\end{aligned}$$

where the $'$ indicates summation over the positive integers relatively prime to k. Alternatively we have, in view of (5),

$$\frac{1}{k}\sum_{m=0}^{k-1}|S(m)|^2 = \begin{cases} \dfrac{k}{2\pi^2}\displaystyle\sum_{n=1}^{\infty}{}'\frac{1}{n^2} & \text{if } \chi(-1)=1\\ \dfrac{k}{\pi^2}\left(\dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty}{}'\frac{1}{n^2}+|L(1,\chi)|^2\right) & \text{if } \chi(-1)=-1.\end{cases}$$

3. **Estimation of certain partial averages of $S(n)$.** We saw above that if χ is a primitive character with $\chi(-1)=1$, then the arithmetic mean of the numbers $S(0), \cdots, S(k-1)$ is zero. Hence in this case we should expect a fairly good estimate of the arithmetic mean of $S(0), \cdots, S(n-1)$ for $n<k$. And in fact Hua [4; 5; 6] has proved that we have

$$(7) \qquad \left|\frac{1}{n}\sum_{m=0}^{n-1}S(m)\right| \leqq \frac{1}{2}\left(k^{1/2}-\frac{n}{k^{1/2}}\right) \quad \text{for} \quad 0\leqq n\leqq k,\ \chi(-1)=1.$$

If χ is a primitive character with $\chi(-1)=-1$, such a neat result cannot be expected, since the arithmetic mean of $S(0), \cdots, S(k-1)$ is not zero. However we prove the following result, which is valuable for n not too small relative to k, especially if $|L(1,\chi)|$ is large.

THEOREM 2. *If χ is a primitive character modulo k with $\chi(-1)=-1$ and if $\alpha k \leq n \leq k$, where α is a number between* 0 *and* 1, *then*

$$\left|\frac{1}{n}\sum_{m=0}^{n-1} S(m) - \frac{\tau(\chi)}{\pi i} L(1,\bar{\chi})\right| < \left(2.1 + \frac{\pi}{4}\log\frac{1}{\alpha}\right) k^{1/2}.$$

PROOF. Put $q=[k/2]$, $\rho = e^{2\pi i/k}$. Then by (1) we have

$$\begin{aligned}
\tau(\bar{\chi})S(m) &= \sum_{l=0}^{m} \chi(l)\tau(\bar{\chi}) = \sum_{l=0}^{m} \sum_{h=-q}^{q} \bar{\chi}(h)\rho^{hl} \\
&= \sum_{h=1}^{q} \bar{\chi}(h) \sum_{l=0}^{m} (\rho^{hl} - \rho^{-hl}) \\
&= \sum_{h=1}^{q} \bar{\chi}(h) \frac{\rho^{h(m+1)} - 1 - \rho^{h} + \rho^{-hm}}{\rho^{h} - 1} \\
&= \sum_{h=1}^{q} \bar{\chi}(h) \frac{\rho^{h(m+1)} + \rho^{-hm}}{\rho^{h} - 1} \\
&\quad + i\sum_{h=1}^{q} \bar{\chi}(h) \cot\frac{\pi h}{k}\cdot
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\tau(\bar{\chi})}{n}\sum_{m=0}^{n-1} S(m) &= \frac{1}{n}\sum_{h=1}^{q} \bar{\chi}(h) \frac{\rho^{h(n+1)} - \rho^{-h(n-1)}}{(\rho^{h}-1)^2} \\
&\quad + i\sum_{h=1}^{q} \bar{\chi}(h)\cot\frac{\pi h}{k} \\
&= \frac{1}{2in}\sum_{h=1}^{q} \bar{\chi}(h) \frac{\sin(2\pi hn/k)}{\sin^2(\pi h/k)} \\
&\quad + i\sum_{h=1}^{q} \bar{\chi}(h)\cot\frac{\pi h}{k}\cdot
\end{aligned}$$

Now $\tau(\chi)\tau(\bar{\chi}) = -\tau(\chi)\overline{\tau(\chi)} = -k$ for $\chi(-1)=-1$ and thus

$$\begin{aligned}
\tau(\bar{\chi})&\left\{\frac{1}{n}\sum_{m=0}^{n-1} S(m) - \frac{\tau(\chi)}{\pi i} L(1,\bar{\chi})\right\} \\
&= \frac{\tau(\bar{\chi})}{n}\sum_{m=0}^{n-1} S(m) - \frac{ik}{\pi} L(1,\bar{\chi})
\end{aligned}$$

$$(8)\qquad \begin{aligned} &= \frac{1}{2in}\sum_{h=1}^{q}\bar{\chi}(h)\frac{\sin(2\pi hn/k)}{\sin^2(\pi h/k)} \\ &\quad + i\sum_{h=1}^{q}\bar{\chi}(h)\left(\cot\frac{\pi h}{k} - \frac{1}{\pi h/k}\right) \\ &\quad - \frac{ik}{\pi}\sum_{h=q+1}^{\infty}\frac{\bar{\chi}(h)}{h} = R_1 + R_2 + R_3, \end{aligned}$$

say. If we put $S_1(h) = \sum_{n=q+1}^{h}\bar{\chi}(n)$ and use the fact that $|S_1(h)| < 2^{-}k$, we have

$$(9)\qquad \begin{aligned} |R_3| &= \left|\frac{k}{\pi}\sum_{h=q+1}^{\infty}\frac{S_1(h)}{h(h+1)}\right| \leqq \frac{2^{-1}k^2}{\pi}\sum_{h=q+1}^{\infty}\frac{1}{h(h+1)} \\ &= \frac{2^{-1}k^2}{\pi(q+1)} \leqq \frac{k}{\pi}. \end{aligned}$$

Using the fact that $0 < x^{-1} - \cot x \leqq 2/\pi$ for $0 < x \leqq \pi/2$, we find

$$(10)\qquad |R_2| \leqq \frac{2}{\pi}q \leqq \frac{1}{\pi}k.$$

We divide the summation in R_1 into two parts, namely $1 \leqq h \leqq [\alpha^{-1}+1]$ and $[\alpha^{-1}+2] \leqq h \leqq q$. In the first part we use $\sin(2\pi hn/k) \leqq 2\pi hn/k$, in the second part we use $\sin(2\pi hn/k) \leqq 1$, and in both parts we use $\sin(\pi h/k) \geqq 2h/k$. Thus

$$(11)\qquad \begin{aligned} |R_1| &\leqq \frac{1}{2n}\sum_{h=1}^{[\alpha^{-1}+1]}\frac{2\pi hn/k}{4h^2/k^2} + \frac{1}{2n}\sum_{h=[\alpha^{-1}+2]}^{q}\frac{1}{4h^2/k^2} \\ &\leqq \frac{\pi}{4}k\sum_{h=1}^{[\alpha^{-1}+1]}\frac{1}{h} + \frac{k^2}{8n}\sum_{h=[\alpha^{-1}+2]}^{\infty}\frac{1}{h^2} \\ &\leqq \frac{\pi}{4}k\left(\frac{3}{2} + \log\frac{1}{\alpha}\right) + \frac{k^2}{8n}\alpha \\ &\leqq \left(\frac{3\pi}{8} + \frac{\pi}{4}\log\frac{1}{\alpha} + \frac{1}{8}\right)k. \end{aligned}$$

Combining (8), (9), (10), and (11), we get our theorem.

4. **A theorem of Davenport.** Davenport [3] has proved that if s is a fixed complex number with $0 < \sigma = R(s) < 1$, then for any primitive χ we have $|L(s,\chi)| \leqq Ck^{(1-\sigma)/2}$ where C is a constant depending on s. In this section we show how Hua's inequality (7) gives a very simple proof of this result in the case $\chi(-1) = 1$, with a specific value of the

constant. Unfortunately our Theorem 2 is too weak to give Davenport's result for the case $\chi(-1)=-1$.

THEOREM 3. *If* χ *is a primitive character with* $\chi(-1)=1$ *and if* $0<\sigma=R(s)<1$, *then*

$$|L(s,\chi)| \leqq \frac{|s(s+1)|}{\sigma(1-\sigma)} k^{(1-\sigma)/2}.$$

PROOF. Put $T(n)=\sum_{m=0}^{n} S(m)$. Then we get by two partial summations

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\sum_{n=1}^{\infty}S(n)s\int_0^1\frac{dy}{(n+y)^{s+1}}$$
$$=\sum_{n=1}^{\infty}T(n)s(s+1)\int_0^1\int_0^1\frac{dxdy}{(n+x+y)^{s+2}}\cdot$$

For $n\leqq[k^{1/2}]$ clearly $|T(n)|\leqq\sum_{m=0}^{n} m\leqq n^2$, while for $n\geqq[k^{1/2}+1]$ we have by (7)

$$|T(n)|\leqq 2^{-1}k^{1/2}\{n+1-(n+1)^2/k\}\leqq 2^{-1}k^{1/2}n.$$

Thus

$$|L(s,\chi)|\leqq|s(s+1)|\sum_{n=1}^{\infty}\frac{|T(n)|}{n^{\sigma+2}}$$
$$=|s(s+1)|\left\{\sum_{n=1}^{[k^{1/2}]}\frac{1}{n^{\sigma}}+\sum_{n=[k^{1/2}]+1}^{\infty}\frac{2^{-1}k^{1/2}}{n^{\sigma+1}}\right\}$$
$$\leqq|s(s+1)|\left\{1+\int_1^{[k^{1/2}]}\frac{dx}{x^{\sigma}}+2^{-1}k^{1/2}\int_{[k^{1/2}]}^{\infty}\frac{dx}{x^{\sigma+1}}\right\}$$
$$\leqq|s(s+1)|\left\{1+\frac{k^{(1-\sigma)/2}-1}{1-\sigma}+\frac{2^{-1}k^{1/2}}{\sigma[k^{1/2}]^{\sigma}}\right\}$$
$$\leqq|s(s+1)|\left\{\frac{k^{(1-\sigma)/2}}{1-\sigma}+\frac{k^{(1-\sigma)/2}}{\sigma}\right\}=\frac{|s(s+1)|}{\sigma(1-\sigma)}k^{(1-\sigma)/2},$$

which proves our theorem.

BIBLIOGRAPHY

1. S. Chowla, *On the k-analogue of a result in the theory of the Riemann zeta function*, Math. Zeit. vol. 38 (1934) pp. 483–487.

2. ———, *On the class-number of the corpus* $P((-k)^{1/2})$, Proceedings of the National Institute of Sciences, India vol. 13 (1947) pp. 197–200.

3. H. Davenport, *On Dirichlet's L-functions*, J. London Math. Soc. vol. 6 (1931) pp. 198–202.

4. L. K. Hua, *On character sums*, Science Record. Academia Sinica vol. 1 (1942) pp. 21–23.

5. ———, *On the least primitive root of a prime*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 726–730.

6. ———, *On the least solution of Pell's equation*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 731–735.

7. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 1, Leipzig, 1909.

8. ———, *Abschätzungen von Charaktersummen, Einheiten, und Klassenzahlen*, Nachr. Ges. Wiss. Göttingen (1918) pp. 79–97.

9. ———, *Vorlesungen über Zahlentheorie*, vol. 1, Leipzig, 1927.

10. ———, *Der Paleysche Satz über Charaktere*, Math. Zeit. vol. 37 (1933) pp. 28–32.

11. U. V. Linnik and A. A. Rényi, *On certain hypotheses in the theory of Dirichlet characters*, Bull. Acad. Sci. URSS Sér. Math. vol. 11 (1947) pp. 539–546.

12. R. E. A. C. Paley, *A theorem on characters*, J. London Math. Soc. vol. 7 (1932) pp. 28–32.

13. G. Pólya, *Über die Verteilung der quadratischen Reste und Nichtreste*, Nachr. Ges. Wiss. Göttingen (1918) pp. 21–29.

14. J. Schur, *Einige Bemerkungen zu den vorstehenden Arbeit des Herrn G. Pólya*, Nachr. Ges. Wiss. Göttingen (1918) pp. 30–36.

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