AVERAGES OF CHARACTER SUMS

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Suppose that \( \chi \) is a primitive residue character\(^1\) modulo \( k, k > 1 \), and that for \( y \) non-negative, \( S(y) = \sum_{0 \leq l \leq y} \chi(l) \). It is important [see, for example, 11] in the analytic theory of numbers to have as much information as possible about the sums \( S(y) \), in particular about their maximum order of magnitude; it is known (cf. [13; 14; 8]), for example, that \( S(y) < k^{1/2} \log k \), but unknown whether or not \( M(\chi) = o(k^{1/2} \log k) \) as \( k \) tends to infinity, where \( M(\chi) \) is the maximum of \( |S(1)|, \ldots, |S(k-1)| \). Hua [4; 5; 6] has shown that it is often helpful to consider the averages \( n^{-1} \sum_{n=0}^{n=m} S(m) \). In this paper we consider some further developments of this idea.

1. Preliminaries. We recall [7, pp. 483–486, 492–494] that if \( \chi \) is a primitive residue character mod \( k \) and if \( \tau(\chi) = \sum_{n=1}^{k} \chi(n)e^{2\pi i n/k} \), then \( |\tau(\chi)| = k^{1/2} \) and

\[
\sum_{n=1}^{k} \chi(n)e^{2\pi i mn/k} = \overline{\chi}(m)\tau(\chi)
\]

for any integer \( m, \overline{\chi} \) being the complex conjugate of \( \chi \).

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\(^1\) For the basic facts about residue characters see [7, pp. 401–414, 478–494]. Numbers in brackets refer to the bibliography.
The function $S(kx)$ is of bounded variation and period unity, and as such has an everywhere convergent Fourier series. It may be deduced from (1) (cf. [13, pp. 23–24] and [8, pp. 81–82]) that this Fourier series is

\[ A(x) = \frac{\tau(x)}{2\pi} \sum_{m=0}^{\infty} \frac{\bar{\chi}(m)}{m} e^{-2\pi imx}, \]

where $m$ runs over the positive and negative integers and

\[ A(x) = \int_{0}^{1} S(kx)dx = \frac{1}{k} \sum_{m=0}^{k-1} S(m). \]

Thus if we define a function $S^*(y)$ as $S(y)$ if $y$ is not an integer and as $S(y) - \chi(y) / 2$ if $y$ is an integer, we have

\[ S^*(kx) = \begin{cases} 
A(x) + \frac{\tau(x)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sin 2\pi nx & \text{if } \chi(-1) = 1 \\
A(x) - \frac{\tau(x)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \cos 2\pi nx & \text{if } \chi(-1) = -1 
\end{cases} \]

for all non-negative $x$.

2. Another proof of Paley's $\Omega$-result for $S(n)$. If we put $x=0$ in (4) we get a formula for the arithmetic mean $A(x)$ of $S(0), \cdots, S(k-1)$, namely

\[ A(x) = \begin{cases} 
0 & \text{if } \chi(-1) = 1 \\
\frac{\tau(x)}{\pi i} L(1, \bar{\chi}) & \text{if } \chi(-1) = -1,
\end{cases} \]

where $L(s, \bar{\chi} = \sum_{n=1}^{\infty} \bar{\chi}(n)n^{-s}$ for $R(s) > 0$. The second half of formula (5) can of course be proved by the method used in [9] to prove the second part of Satz 217 (of which the second half of (5) is a generalization), but some use of Fourier series is essential.

The second part of (5) enables us, when $\chi(-1) = -1$, to use information about the order of magnitude of $L(1, \bar{\chi})$ to get information about the order of magnitude of $A(x)$. For example, a slight change in the argument of [2] shows that for real primitive $\chi$ with $\chi(-1) = -1$ we have

\[ \limsup_{k \to \infty} \frac{L(1, \chi)}{\log \log k} \geq e^\gamma, \]

Naturally (5) can be proved in a roundabout way by showing that both sides are equal to $L(0, \chi)$. The argument indicated here essentially stems from Minkowski (cf. [13, p. 26]).
\(\gamma\) being Euler's constant. (The lim sup is unambiguous, since for each \(k\) there is at most one real primitive \(\chi\) with \(\chi(-1) = -1\). Now \(\tau(\chi) = tk^{1/2}\) for real primitive \(\chi\) with \(\chi(-1) = -1\) [9, Satz 215] and thus (6) gives the following result.

**Theorem 1.** If \(\chi\) runs through all real primitive characters with \(\chi(-1) = -1\) in order of increasing size of the modulus \(k\) of \(\chi\) and if \(A(\chi)\) is as defined in (3), then

\[
\limsup_{k \to \infty} \frac{A(\chi)}{k^{1/2} \log^2 k} \leq \frac{e^\gamma}{\pi}.
\]

This theorem is a slightly stronger form of Paley's result (see [12; 10; 1]) that if \(\chi\) is a primitive character modulo \(k\), then \(M(\chi) = \Omega(k^{1/2} \log \log k)\), where \(M(\chi)\) is the maximum of \(|S(0)|, \cdots, |S(k-1)|\); for our theorem shows that this \(\Omega\)-result is true even for the arithmetic mean of \(S(0), \cdots, S(k-1)\).

We remark in passing that Parseval's formula applied to (2) or (4) gives

\[
\frac{1}{k} \sum_{m=0}^{k-1} |S(m) - A(\chi)|^2 = \int_0^1 |S(kx) - A(\chi)|^2 \, dx = \frac{k}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{k}{12} \prod_{p|k} \left(1 - \frac{1}{p^2}\right),
\]

where the ' indicates summation over the positive integers relatively prime to \(k\). Alternatively we have, in view of (5),

\[
\frac{1}{k} \sum_{m=0}^{k-1} |S(m)|^2 = \begin{cases} \frac{k}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} & \text{if } \chi(-1) = 1 \\ \frac{k}{\pi^2} \left(\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} + |L(1, \chi)|^2\right) & \text{if } \chi(-1) = -1. \end{cases}
\]

3. **Estimation of certain partial averages of \(S(n)\).** We saw above that if \(\chi\) is a primitive character with \(\chi(-1) = 1\), then the arithmetic mean of the numbers \(S(0), \cdots, S(k-1)\) is zero. Hence in this case we should expect a fairly good estimate of the arithmetic mean of \(S(0), \cdots, S(n-1)\) for \(n < k\). And in fact Hua [4; 5; 6] has proved that we have

\[
\frac{1}{n} \sum_{m=0}^{n-1} |S(m)| \leq \frac{1}{2} \left( k^{1/2} - \frac{n}{k^{1/2}} \right) \quad \text{for } 0 \leq n \leq k, \chi(-1) = 1.
\]

If \(\chi\) is a primitive character with \(\chi(-1) = -1\), such a neat result
cannot be expected, since the arithmetic mean of $S(0), \cdots, S(k-1)$ is not zero. However we prove the following result, which is valuable for $n$ not too small relative to $k$, especially if $|L(1, \chi)|$ is large.

**Theorem 2.** If $\chi$ is a primitive character modulo $k$ with $\chi(-1) = -1$ and if $\alpha k \leq n \leq k$, where $\alpha$ is a number between 0 and 1, then

$$\left| \frac{1}{n} \sum_{m=0}^{n-1} S(m) - \frac{\tau(\chi)}{\pi i} L(1, \chi) \right| < \left( 2.1 + \frac{\pi}{4} \log \frac{1}{\alpha} \right) k^{1/2}. $$

**Proof.** Put $q = [k/2]$, $\rho = e^{2\pi i/k}$. Then by (1) we have

$$\tau(\chi)S(m) = \sum_{l=0}^{m} \chi(l)\tau(\chi) = \sum_{l=0}^{m} \sum_{h=q}^{q} \chi(h)\rho^{hl} $$

$$= \sum_{h=1}^{q} \chi(h) \sum_{l=0}^{m} (\rho^{hl} - \rho^{-hl}) $$

$$= \sum_{h=1}^{q} \chi(h) \frac{\rho^{h(m+1)} - 1 - \rho^h + \rho^{-hm}}{\rho^h - 1} $$

$$= \sum_{h=1}^{q} \chi(h) \frac{\rho^{h(m+1)} + \rho^{-hm}}{\rho^h - 1} $$

$$+ i \sum_{h=1}^{q} \chi(h) \cot \frac{\pi h}{k}. $$

Hence

$$\frac{\tau(\chi)}{n} \sum_{m=0}^{n-1} S(m) = \frac{1}{n} \sum_{h=1}^{q} \chi(h) \frac{\rho^{h(n+1)} - \rho^{-h(n-1)}}{(\rho^h - 1)^2} $$

$$+ i \sum_{h=1}^{q} \chi(h) \cot \frac{\pi h}{k} $$

$$= \frac{1}{2in} \sum_{h=1}^{q} \chi(h) \frac{\sin (2\pi hn/k)}{\sin^2 (\pi h/k)} $$

$$+ i \sum_{h=1}^{q} \chi(h) \cot \frac{\pi h}{k}. $$

Now $\tau(\chi)\tau(\chi) = -\tau(\chi)\tau(\chi) = -k$ for $\chi(-1) = -1$ and thus

$$\tau(\chi) \left\{ \frac{1}{n} \sum_{m=0}^{n-1} S(m) - \frac{\tau(\chi)}{\pi i} L(1, \chi) \right\} $$

$$= \tau(\chi) \ln \sum_{m=0}^{n-1} S(m) - \frac{ik}{\pi} L(1, \chi)$$
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(8) \[ \frac{1}{2\pi n} \sum_{h=1}^{q} \frac{\chi(h)}{\sin^2(\pi h/k)} \sin(2\pi hn/k) + \frac{i}{k} \sum_{h=q+1}^{\infty} \frac{\chi(h)}{h} \left( \cot \frac{\pi h}{k} - \frac{1}{\pi h/k} \right) - \frac{i k}{\pi} \sum_{h=q+1}^{\infty} \frac{\chi(h)}{h} = R_1 + R_2 + R_3, \]

say. If we put \( S_1(h) = \sum_{n=q+1}^{h} \chi(n) \) and use the fact that \( |S_1(h)| < 2^{-k} \), we have

\[ |R_3| = \left| \frac{k}{\pi} \sum_{h=q+1}^{\infty} \frac{S_1(h)}{h(h+1)} \right| \leq \frac{2^{-1}k^2}{\pi} \sum_{h=q+1}^{\infty} \frac{1}{h(h+1)} \]

(9) \[ \leq \frac{2^{-1}k^2}{\pi(q+1)} \leq \frac{k}{\pi}. \]

Using the fact that \( 0 < x^{-1} - \cot x \leq 2/\pi \) for \( 0 < x \leq \pi/2 \), we find

(10) \[ |R_2| \leq \frac{2}{\pi} \frac{k}{q} \leq \frac{1}{\pi} k. \]

We divide the summation in \( R_1 \) into two parts, namely \( 1 \leq h \leq [\alpha^{-1}+1] \) and \([\alpha^{-1}+2] \leq h \leq q\). In the first part we use \( \sin(2\pi hn/k) \leq 2\pi hn/k \), in the second part we use \( \sin(2\pi hn/k) \leq 1 \), and in both parts we use \( \sin(\pi h/k) \geq 2h/k \). Thus

\[ |R_1| \leq \frac{1}{2n} \sum_{h=1}^{[\alpha^{-1}+1]} \frac{2\pi hn/k}{4h^2/k^2} + \frac{1}{2n} \sum_{h=[\alpha^{-1}+2]}^{q} \frac{1}{4h^2/k^2} \]

(11) \[ \leq \frac{\pi}{4} k \left[ \frac{3}{2} + \log \frac{1}{\alpha} \right] + \frac{k^2}{8n} \frac{1}{\alpha} \]

\[ \leq \left( \frac{3\pi}{8} + \frac{\pi}{4} \log \frac{1}{\alpha} + \frac{1}{8} \right) k. \]

Combining (8), (9), (10), and (11), we get our theorem.

4. A theorem of Davenport. Davenport [3] has proved that if \( s \) is a fixed complex number with \( 0 < \sigma = R(s) < 1 \), then for any primitive \( \chi \) we have \( |L(s, \chi)| \leq C k^{(1-\sigma)/3} \) where \( C \) is a constant depending on \( s \). In this section we show how Hua's inequality (7) gives a very simple proof of this result in the case \( \chi(-1)=1 \), with a specific value of the
constant. Unfortunately our Theorem 2 is too weak to give Davenport's result for the case \( \chi(-1) = -1 \).

**Theorem 3.** If \( \chi \) is a primitive character with \( \chi(-1) = 1 \) and if 

\[
0 < \sigma = R(s) < 1,
\]

then

\[
|L(s, \chi)| \leq \frac{|s(s+1)|}{\sigma(1-\sigma)} k^{(1-\sigma)/2}.
\]

**Proof.** Put \( T(n) = \sum_{m=0}^n S(m) \). Then we get by two partial summations

\[
L(s, \chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s} = \sum_{n=1}^\infty S(n)s \int_0^1 \frac{dy}{(n+y)^{s+1}}
\]

\[
= \sum_{n=1}^\infty T(n)s(s+1) \int_0^1 \int_0^1 \frac{dxdy}{(n+x+y)^{s+2}}.
\]

For \( n \leq [k^{1/2}] \) clearly \( |T(n)| \leq \sum_{m=0}^n m \leq n^2 \), while for \( n \geq [k^{1/2}+1] \) we have by (7)

\[
|T(n)| \leq 2^{-1} k^{1/2} \{ n+1 - (n+1)^2/k \} \leq 2^{-1} k^{1/2} n.
\]

Thus

\[
|L(s, \chi)| \leq |s(s+1)| \sum_{n=1}^\infty \frac{|T(n)|}{n^{\sigma+2}}
\]

\[
= |s(s+1)| \left\{ \sum_{n=1}^{[k^{1/2}]} \frac{1}{n^\sigma} + \sum_{n=[k^{1/2}]+1}^\infty \frac{2^{-1} k^{1/2}}{n^{\sigma+1}} \right\}
\]

\[
\leq |s(s+1)| \left\{ 1 + \int_1^{[k^{1/2}]} \frac{dx}{x^\sigma} + 2^{-1} k^{1/2} \int_{[k^{1/2}]}^\infty \frac{dx}{x^{\sigma+1}} \right\}
\]

\[
\leq |s(s+1)| \left\{ 1 + \frac{k^{(1-\sigma)/2} - 1}{1 - \sigma} + \frac{2^{-1} k^{1/2}}{\sigma [k^{1/2}]^{\sigma}} \right\}
\]

\[
\leq |s(s+1)| \left\{ \frac{k^{(1-\sigma)/2}}{1 - \sigma} + \frac{k^{(1-\sigma)/2}}{\sigma} \right\} = \frac{|s(s+1)|}{\sigma(1-\sigma)} k^{(1-\sigma)/2},
\]

which proves our theorem.

**Bibliography**
