THE LAST ENTRY IN GAUSS'S DIARY

BY S. CHOWLA

The Institute for Advanced Study

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Herglotz, in his paper "Zur letzten Eintragung im Gaußschen Tagebuch" [Leipsig. Ber., 73, 271–276 (1921)] proves by means of the complex multiplication of elliptic functions, the following conjecture of Gauss (entered in his Tagebuch on July 9, 1814):

For a complex primary prime modulus \( \pi \equiv 1 \mod (1 - i)^2 \) the number of solutions of the congruence (including the solutions \( x, y \equiv \pm i, \infty \); \( \infty, \pm i \)):

\[
x^2y^2 + x^2 + y^2 \equiv 1 \pmod{\pi}
\]

is equal to the norm of \( \pi - 1 \). Dedekind verified this for \( N(\pi) < 100 \). We give a proof (Theorem 1 below) based on elementary number theory.

Notation: \( p \) denotes a prime \( \equiv 1 \pmod{4} \); \( g \) is a primitive root of \( p \);
(\(n/p\)) is the Legendre symbol of quadratic residuacity; \(\theta\) is defined uniquely by \(p = \theta^2 + \phi^2, \theta \equiv 1 \text{ or } 3 \pmod{4}\) according as \(p \equiv 5 \text{ or } 1 \pmod{8}\); finally

\[
S_m = \sum_{x=1}^{p-1} \left( \frac{x^4 - g^m}{p} \right) \quad [m = 0, 1, 2, 3].
\]

**Theorem 1:**

\[S_0 = 2\theta - 2.\]

**Proof:** We observe that our theorem follows immediately from the 5 relations:

(A) \(S_1 = -S_3\)

(B) \(S_0 + S_2 = -4\)

(C) \(S_0^2 + S_1^2 + S_2^2 + S_3^2 = 8p + 8\)

(D) \(p = \theta^2 + \phi^2, \theta \equiv 1, 3 \pmod{4} \pmod{p = 5, 1 \pmod{8}}\)

(E) \(p = \theta^2 + \phi^2, \theta \equiv 1, 3 \pmod{4} \pmod{p = 5, 1 \pmod{8}}\)

(A) \(\sum_{1}^{p-1} \left( \frac{x^4 - g}{p} \right) = \sum_{1}^{p-1} \left( \frac{1 - gy^4}{p} \right) = -\sum_{1}^{p-1} \left( \frac{g^3 - g^4}{p} \right) = -\sum_{1}^{p-1} \left( \frac{g^3 - t^4}{p} \right) \left[ x = \frac{1}{y} = \frac{g}{t} \right]\)

(B) \(\frac{p-1}{4} (S_0 + S_1 + S_2 + S_3) = \sum_{a=1}^{p-1} \sum_{x=1}^{p-1} \left( \frac{x^4 - a}{p} \right) = \sum_{x=1}^{p-1} \sum_{a=1}^{p-1} \left( \frac{x^4 - a}{p} \right) = -(p-1)\)

(C) \(\frac{p-1}{4} (S_0^2 + S_1^2 + S_2^2 + S_3^2) = \sum_{a=1}^{p-1} \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} \frac{x^4 - a}{p} \left( \frac{y^4 - a}{p} \right) = \sum_{x=1}^{p-1} \left\{ \sum_{y=1}^{p-1} \sum_{a=1}^{p-1} \left( \frac{a^2 - a(x^4 + y^4) + x^4 y^4}{p} \right) \right\}\)

Consider the sum inside the braces. If \(y^4 = x^4\), the innermost sum = \(p - 2\); for the remaining \((p - 5)\) values of \(y\) the innermost sum is \(-2\) since writing \(f = \frac{1}{2}(x^4 - y^4)\) this sum easily transforms into

\[-1 + \sum_{a=1}^{p-1} \left( \frac{a^2 - f^2}{p} \right) = -1 + \sum_{t=1}^{p-1} \left( \frac{t^2 + 2}{p} \right) = -1 + \sum_{t=1}^{p-1} \left( \frac{2t + 1}{p} \right) = -2[a = t + 1; tl = 1; f \not\equiv 0].\]

(D) This follows at once by noting that \(x^4\) has the same non-zero value mod \(p\) for exactly 4 incongruent values of \(x\), and these \(x\) are equidistant from \(1/2p\); further exactly 4 terms of \(S_0\) vanish, while the rest are \(+1\) or \(-1\). Similarly we prove:
Theorem 2:

$$\sum_{x=1}^{p-1} \left( \frac{x^{\theta} - 1}{p} \right) = -2 - 4\theta$$

where $\theta$ is defined uniquely by $p = \theta^2 + 3\phi^2$, $\theta \equiv 1 \pmod{3}$ and $p$ is a prime $\equiv 1 \pmod{12}$.

I should like to thank Professor Siegel (who found slightly more general results) and Professor Weyl for their interest in this note.

Note added in proof: Results similar to Theorems 1 and 2 have been found recently by E. Lehmer and A. L. Whiteman. Deuring, in a paper dedicated to Herglotz, finds more general results by advanced methods [see Hamburg Abhandlungen, 14, 1941, 197–272].