A DEFINITE INTEGRAL

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THEOREM: *If* n *and* r *are non-negative integers, where* r < n, *then*

$$(r+1)\,K(r) = (-1)^{n-1}(n-r)\,K(n-r-1)$$

where

$$K(r) = \int_0^1 \frac{1}{u}(\log u)^r \left\{\log\left(\frac{1+u}{1-u}\right)\right\}^{n-r} du$$

Proof: Integration by parts, gives

$$K(r) = -\int_0^1 \log u \left(\log\frac{1+u}{1-u}\right)^{n-\nu} \frac{r(\log u)^{r-1}}{u}\,du$$

$$-\int_0^1 (\log u)(\log u)^r \left(\log\frac{1+u}{1-u}\right)^{n-r-1} (n-r)\left(\frac{1-u}{1+u}\right)\frac{2}{(1-u)^2}\,du$$

$$= -r\,K(r) - 2(n-r)\int_1^0 \left(\log\frac{1-x}{1+x}\right)^{r+1}\left(\log\frac{1}{x}\right)^{n-r-1}\frac{(1+x)^2}{4x}\left(\frac{-2}{(1+x)^2}\right)dx$$

by means of the substitution $u = \dfrac{1-x}{1+x}$.

Hence

$$K(r) = -r\,K(r) - (-1)^n (n-r)\,K(n-r-1)$$

Corollary. If n is an even positive integer, then

$$\int_0^1 \frac{1}{u}\left[\left\{\log u + \log\frac{1+u}{1-u}\right\}^n - (\log u)^n\right] du = 0.$$

(See *American Mathematical Monthly*, November 1938.)

Correction to a previous paper. My paper "A remark on $g(n)$" in these *Proceedings* for January 1939 contains some trivial blunders, but the whole argument is rendered correct by replacing "$\epsilon = \frac{1}{100}$" by "$\epsilon = \frac{1}{200}$". When this change is made the argument reads correctly.