SOME INFINITE SERIES.

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1. Do there exist real a, b for which[1]

$$(1) \quad \sum_{1}^{\infty} e^{ian^2} \frac{\sin bn}{n} \neq O(1)\ ?$$

The present note arose out of an attempt to throw some light on this question. Though the results obtained here have little connection with the problem suggested, they seem to be of some interest in themselves.

Let k denote a positive integer. I show that[2]

$$\text{(I)} \quad \cos\frac{\pi n}{4} - \frac{1}{3}\cos\frac{9\pi n}{4} + \frac{1}{5}\cos\frac{25\pi n}{4} - + \cdots$$

$$= (-1)^{k+1}\frac{\pi\sqrt{n}}{4}\left[\cos\frac{\pi n}{4} - \cos\frac{9\pi n}{4} + \cos\frac{25\pi n}{4} - + \cdots \text{ to } \frac{1}{n} \text{ terms}\right]$$

where $n = \dfrac{1}{4k-1}$.

$$\text{(II)} \quad \cos\frac{\pi n}{4} - \frac{1}{3}\cos\frac{9\pi n}{4} + \frac{1}{5}\cos\frac{25\pi n}{4} - + \cdots$$

$$= (-1)^{k}\frac{\pi\sqrt{n}}{4}\left[\sin\frac{\pi n}{4} - \sin\frac{9\pi n}{4} + \sin\frac{25\pi n}{4} - + \cdots \text{ to } \frac{1}{n} \text{ terms}\right]$$

where $n = \dfrac{1}{4k+1}$.

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[1] This problem is due to Davenport and Heilbronn. After writing this note I learnt that Dr. Spaček of Prague has answered the problem in the negative. In fact he proves that

$$\sum_{1}^{N} \frac{1}{n} e^{2\pi i a n^2} \sin n\beta = O(1)$$

uniformly in α, β (real numbers) and N.

[2] In the series on the left side of (I) the signs are alternately + and −. The numbers 1, 9, 25, ⋯ are the squares of odd numbers.

2. Ramanujan[3] has proved that

$$\int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \cos \pi n x^2 \, dx$$

$$= \left[e^{-t} \cos \frac{\pi n}{4} - e^{-3t} \cos \frac{9\pi n}{4} + e^{-5t} \cos \frac{25\pi n}{4} - + \cdots \right]$$

$$+ \frac{1}{\sqrt{n}} \left[e^{-t/n} \cos \left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{\pi}{4n} \right) - e^{-3t/n} \right.$$

$$\left. \cos \left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{9\pi}{4n} \right) + - \cdots \right]$$

when $n = \dfrac{1}{4k-1}$ this becomes

$$\int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \cos \pi n x^2 \, dx = \left[e^{-t} \cos \frac{\pi n}{4} - e^{-3t} \cos \frac{9\pi n}{4} + \cdots \right]$$

$$+ \frac{(-1)^k}{\sqrt{n}} \frac{\cos (t^2/\pi n)}{2 \cosh t/n}.$$

Integrating the last result between the limits $t = O$ and $t = T$ $(T \to \infty)$ we obtain

$$\lim_{T \to \infty} \int_0^\infty \frac{\sin 2Tx}{2x} \frac{\cos \pi n x^2}{\cosh \pi x} dx$$

$$= \sum_{s=1}^{\infty} \frac{1}{s} \cos \frac{\pi n s^2}{4} \sin \frac{\pi s}{2}$$

$$+ \frac{(-1)^k}{2\sqrt{n}} \int_0^\infty \pi n \frac{\cos \pi n t^2}{\cosh \pi t} dt$$

But the limit on the left side of the above equation is equal to (put $2Tx = y$)

$$\frac{1}{2} \int_0^\infty \frac{\sin y}{y} dy = \frac{\pi}{4}.$$

Hence we obtain

$$(2) \quad \sum_{s=1}^{\infty} \frac{1}{s} \cos \frac{\pi n s^2}{4} \sin \frac{\pi s}{2}$$

$$= \frac{\pi}{4} + (-1)^{k+1} \frac{\pi \sqrt{n}}{2} \int_0^\infty \frac{\cos \pi n x^2 \, dx}{\cosh \pi x}.$$

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[3] *Collected Papers*, Cambridge, 1927, p. 62, formula (19).

Again, by Ramanujan,[4]

$$(3)\quad 2\int_0^\infty \frac{\cos \pi n x^2}{\cosh \pi x}\, dx$$

$$= \left[\cos \frac{\pi n}{4} - \cos \frac{9\pi n}{4} + \cos \frac{25\pi n}{4} - + \cdots \text{ to } \frac{1}{n} \text{ terms}\right] + \frac{(-1)^k}{\sqrt{n}}.$$

(I) follows immediately from (2) and (3).

3. The proof of (II) is similar, and details are omitted.

[4] *Loc. cit.*, this is a special case of formula (21), p. 62.