

IRRATIONAL INDEFINITE QUADRATIC FORMS.

BY S. CHOWLA,
Andhra University, Waltair.

Received July 13, 1935.

1. I have recently proved the

*Theorem.*¹ If the c 's are not all of one sign and if all the ratios $\frac{c_s}{c_t}$ ($s \neq t$) are irrational, we can find integers n_1, \dots, n_r (not all zero) such that

$$\left| \sum_{s=1}^r c_s n_s^2 \right| < \epsilon$$

where ϵ is an arbitrary positive number and $r \geq 9$.

This was deduced from a theorem of Jarnik and Walfisz in the theory of lattice points.

In the same direction we have

Theorem 1. Let $c_1, c_2 > 0$, $\sqrt{\frac{c_1}{c_2}}$ irrational,

$$(1) \quad \sqrt{\frac{c_1}{c_2}} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}$$

Then, to every positive ϵ , we can find infinitely many pairs of positive integers n_1 and n_2 , such that

$$(2) \quad |n_1^2 c_1 - n_2^2 c_2| < \epsilon$$

whenever $a_n \neq 0$ (1), but not otherwise.

Theorem 2. If the c 's are not all of one sign, then for 'almost all' sets (c_1, c_2, \dots, c_r) we can find integers n_1, \dots, n_r (not all zero) such that

$$\left| \sum_{s=1}^r c_s n_s^2 \right| < \epsilon$$

where ϵ is an arbitrary positive number and $r \geq 2$.

2. Let $\frac{p_n}{q_n}$ be the n th convergent to (1).

Then

$$(3) \quad \left| \frac{p_n}{q_n} - \sqrt{\frac{c_1}{c_2}} \right| < \frac{1}{a_n q_n^2}.$$

¹ *Jour. London Math. Soc.*, 1934, 9, 162-63.

Further, (2) requires,

$$(4) \quad \left| \frac{n_2}{n_1} - \sqrt{\frac{c_1}{c_2}} \right| = O\left(\frac{1}{n_1^2}\right)$$

From (3) and (4) we obtain Theorem 1, since a_n is unbounded.

To prove Theorem 2, we can assume without loss of generality that c_1 and c_2 have opposite signs. Let $c_1 = b_1 > 0$, $c_2 = -b_2$. Then $b_1 > 0$, $b_2 > 0$.

We know that if

$$\theta = d_1 + \frac{1}{d_2 +} \frac{1}{d_3 + \dots}$$

where the d 's are positive integers, then $d_n \neq O(1)$ for almost all θ . It now follows from Theorem 1 that for almost all (b_1, b_2) we can find positive integers n_1 and n_2 such that

$$(5) \quad |b_1 n_1^2 - b_2 n_2^2| < \epsilon$$

where ϵ is an arbitrary positive number. Hence for almost all sets (b_1, b_2) we have from (5),

$$(6) \quad |b_1 n_1^2 - b_2 n_2^2 + c_3 \cdot 0^2 + c_4 \cdot 0^2 + \dots + c_r \cdot 0^2| < \epsilon [n_1, n_2 > 0].$$

Hence Theorem 2 follows from (6), as we can take $n_3 = \dots = n_r = 0$.

If in Theorem 2 we require that the integers n_1, \dots, n_r are *all different from zero*, then Theorem 2 is no longer easy to prove -- there is, however, little doubt that the theorem is true even with this restriction.