ON SOME INFINITE SERIES INVOLVING ARITHMETICAL FUNCTIONS (II).

BY S. CHOWLA.

(From the Government College, Lahore.)

Received May 29, 1937.

1. We have

\[ \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\log \left( \frac{2}{\sin \frac{\theta}{2}} \right) \]

if \( \theta \) is not an integral multiple of \( 2\pi \). This leads us to the formal identity

\[ \sum_{n=1}^{\infty} \frac{c_n \log \left( \frac{2}{\sin n\pi\theta} \right)}{n} = -\sum_{n=1}^{\infty} \frac{G_n \cos (2n\pi\theta)}{n} \]

where \( G_n = \sum_{d|n} c_d \) and \( d/n \) means that \( d \) is a divisor of \( n \). We shall investigate the truth of (2) for irrational values of \( \theta \). Following Davenport (\( Q.J.M. \), Oxford, March 1937) we prove.

**Theorem I.** If \( c_n = O(n^\phi) \) where \( \phi < \frac{1}{2} \), then (2) is true for almost all \( \theta \). In particular the series on the left-hand side of (2) is convergent for almost all \( \theta \).

2. We need the following definite integrals (we evaluate them formally but it is easy to give rigorous proofs).

From (1) if \( r \) and \( s \) are positive integers

\[ \int_0^1 \log \left( \frac{2}{\sin r\pi\theta} \right) \log \left( \frac{2}{\sin s\pi\theta} \right) d\theta \]

\[ = \int_0^1 \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\cos 2mr\pi\theta \cos 2ns\pi\theta}{mn} \right\} d\theta \]

\[ = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \]

\[ = \frac{\pi^2}{12} \left( \frac{r}{s} \right)^2 \]

where \( (r, s) \) is the g.c.d. of \( r \) and \( s \). Again, from (1) it is easy to see that
(4) \[
\int_0^1 \cos (2\pi n \theta) \log (2 - \sin \pi \theta) \, d\theta \\
= - \frac{1}{2} \sum_{d|n} \left\{ \frac{d}{n} \right\} \\
= O \text{ if } d + n
\]

3. Consider

(5) \[
R_n(\theta) = \sum_{n=1}^N \left\{ c_n \log (2 - \sin \frac{n \pi \theta}{n}) + \frac{G_n}{n \cos (2n \pi \theta)} \right\}
\]

From (3) and (4),

(6) \[
\int_0^1 R_n^2(\theta) \, d\theta = \frac{\pi^2}{12} \sum_{r=1}^N \sum_{s=1}^N \frac{c_r c_s (r, s)^2}{r^2 s^2} - \frac{1}{2} \sum_{n=1}^N \frac{G_n^2}{n^2}
\]

In what follows we shall use

\[G_n = O (n^\phi + \epsilon)\]

for any \(\epsilon > 0\). This follows from \(c_n = O(n^\phi)\) and \(\sum 1 = O(n \epsilon)\).

4. \[
G_n = \sum_{d|n} c_d
\]

This gives

(7) \[
G_n^2 = \sum_{d|n} h(d)
\]

where

(8) \[
h(v) = \sum c_m c_n
\]

where \(m, n\) take all values such that \(\frac{mn}{(m, n)} = v\).

Hence we have

(9) \[
(\sum n^{-s}) \times \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{c_m c_n (m, n)^2}{m^s n^s} = \sum_{n=1}^\infty \frac{G_n^2}{n^s}
\]

Putting \(s = 2\),

(10) \[
\frac{\pi^2}{6} \sum_{m=1}^\infty \frac{c_m c_n (m, n)^2}{m^2 n^2} = \sum_{n=1}^\infty \frac{G_n^2}{n^2}
\]

5. From (10) the right-hand side of (6) is of the order

\[
\sum_{N+1}^\infty n^{2\phi - 2 + \epsilon} + \sum_{m > N} \sum_{n > N} \frac{|c_m| |c_n| (m, n)^2}{m^2 n^2}
\]

Put \(n = n'd\) where \(d = (m, n)\) and the second expression is of the order

\[
\sum_{m > N} d \phi m^{\phi - 2} \sum_{d|m} (n')^{\phi - 2} = O \left( \sum_{m > N} m^{2\phi - 2 + \epsilon} \right) = O \left( N^{2\phi - 1 + \epsilon} \right)
\]

Hence the r.h.s. of (6) is \(0 (N^{2\phi - 1 + \epsilon})\).
6. Let $E_m$ denote the set of points $\theta$ at which

$$|R_m^2(\theta)| > \frac{1}{\log m}$$

From the result of the last section, measure $E_m < c \cdot m^{m(\phi^2-1+\epsilon)} \log^2(m) < c \cdot m^{m \phi - 3 + \epsilon}$

where $c$ denotes an absolute positive constant. Write

$$E = \lim_{n \to \infty} \sum_{m > n} E_m$$

Then

(12) Measure $E = O$ since $\phi < \frac{1}{2}$.

If $m^3 \leq n < (m + 1)^3$, then

$$|R_n(\theta) - R_{m^3}(\theta)| \leq \left( \sum_{t = m^3}^{(m+1)^3} t^{\phi-1 + \epsilon} \right)$$

is true for almost all $\theta$ since $\log |\sin n\pi\theta| = O(\log n)$ for almost all $\theta$.

From (13), for almost all $\theta$,

(14) $|R_n(\theta) - R_{m^3}(\theta)| \to 0$ as $m \to \infty$ if $m^3 \leq n < (m + 1)^3$.

Thus, as in Davenport,

$$R_n(\theta) \to O \quad \text{as} \quad n \to \infty,$$

for almost all $\theta$. Since $G_n = O(n^{\frac{3}{4}})$ it follows from a theorem of Kolmogoroff that

$$\sum_{n=1}^{\infty} \frac{G_n}{n} \cos (2n\pi\theta)$$

is convergent for almost all $\theta$. Hence the first half of the expression (5) for $R_N(\theta)$ tends to a definite limit as $N \to \infty$ for almost all $\theta$. This proves our theorem.