THE REPRESENTATION OF A LARGE NUMBER AS A SUM OF "ALMOST EQUAL" SQUARES.

By F. C. Auluck
(*Punjab University*)
AND
S. Chowla
(*Government College, Lahore*).

Received June 8, 1937.

1. Wright has announced that[1]

Theorem I. *Every positive integer* $N \not\equiv 0 \pmod 8$ *is expressible in the form*

(1) $$N = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

where the m's *are integers satisfying*

$$\frac{N}{4} - m_i^2 = O(N^{\frac{5}{6}}) \quad [1 \leqslant i \leqslant 4]$$

and he has proved (*loc. cit.*) that this result is true with

$$\frac{N}{4} - m_i^2 = 0\,(N).$$

By an entirely elementary method we prove the sharper

Theorem II. *If* $N \not\equiv 0 \pmod 8$, *then* (1) *is true with*

(2) $$\frac{N}{4} - m_i^{\,2} = O\,(N^{\frac{3}{4}}) \quad [i = 1, 2, 3, 4].$$

Proof.

Case I. $N \equiv 2$ or $6 \pmod 8$

Write

(3) $$N - 4m^2 = P$$

where $4m^2$ is the nearest even perfect square below N. Then $P \equiv 2, 6 \pmod 8$. Hence

(4) $$P = x_1^2 + x_2^2 + x_3^2$$

where $x_1 + x_2 + x_3 \equiv 0 \pmod 2$. Hence we can write

(5) $$x_1 = a + b,\ x_2 = b + c,\ x_3 = c + a$$

where a, b, c are integers. From (3), (4), (5),

(6) $$N = (m + a)^2 + (m + b)^2 + (m + c)^2 + (m - a - b - c)^2$$

where, since

(7) $$P = O\,(N^{\frac{1}{2}})$$

[1] This is a special case of his result in *Q. J. M.* (Oxford), 1936. Tchudakoff has recently proved a similar result for cubes.

we have from (4) and (5),

(8) $$a, b, c = O(N^{\frac{1}{4}})$$

Hence

(9) $$\begin{aligned}(m + a)^2 &= m^2 + 2ma + a^2 \\ &= \frac{N}{4} + O(\sqrt{N}) + O(N^{\frac{3}{4}}) + O(N^{\frac{1}{2}}) \\ &= \frac{N}{4} + O(N^{\frac{3}{4}}),\end{aligned}$$

and the same is true of $(m + b)^2$, $(m + c)^2$, $(m - a - b - c)^2$. Thus Theorem II is proved in Case I [from (6) and (9)].

Case II. $N \equiv 1 \pmod 2$.

Let $4m^2 - 2m$ be the nearest integer of this form below N. Put

(10) $$N - 4m^2 + 2m = P$$

so that $P \equiv 1 \pmod 2$ and $4P - 1 \equiv 3 \pmod 8$.

Hence

(11) $$4P - 1 = x_1^2 + x_2^2 + x_3^2$$

where x_1, x_2, x_3 are all odd and can therefore be chosen so that (*by changing all their signs if necessary*)

(12) $$x_1 + x_2 + x_3 \equiv 1 \pmod 4)$$

If we put

(13) $x_1 = 2b + 2c - 1,\ x_2 = 2c + 2a - 1,\ x_3 = 2a + 2b - 1$

it follows from (12) and (13) that a, b, c are all integers. From (11) and (13),

(14) $$\begin{aligned}4P - 1 = 8(a^2 + b^2 + c^3 + ab + bc + ca) \\ - 8(a + b + c) + 3.\end{aligned}$$

or

(15) $$\begin{aligned}P = 2(a^2 + b^2 + c^2 + ab + bc + ca) \\ - 2(a + b + c) + 1.\end{aligned}$$

From (10) and (15),

(16) $$\begin{aligned}N = (m - a)^2 + (m - b)^2 + (m - c)^2 \\ + (m + a + b + c - 1)^2\end{aligned}$$

Since a, b, c are $O(N^{\frac{1}{4}})$ the rest of the proof proceeds as in Case I.

Case III. $N \equiv 4 \pmod 8$.

This follows from Case II by multiplication by 4. Theorem II is now completely proved.

Note added 10–7–37.

Wright's proof of Theorem I has appeared in *Proc. London Math. Soc.*, **42** (1937), 481–500. The error term is improved to $O(N^{\frac{4}{5} + \epsilon})$ for any positive ϵ. Our index is slightly better.