THE LATTICE POINTS IN A HYPERSPHERE.

By S. Chowla,
Andhra University, Waltair.

Received January 24, 1935.

Let $A_k(x)$ denote the number of lattice points inside and on the hypersphere

$$x_1^2 + x_2^2 + \cdots + x_k^2 = x,$$

and let $V_k(x)$ denote its "volume". We denote by $P_k(x)$ the "Gitterrest", so that

$$P_k(x) = A_k(x) - V_k(x).$$

We show here that

I. $\sum_{x \leqslant R} P_k(x) \sim \frac{1}{2} V_k(R) \quad [k \geqslant 2].$

1. In this section we prove that if $k > 0$ then

$$(1\cdot1) \qquad \sum_{n \leqslant \sqrt{R}} (R - n^2)^{\frac{k}{2}} = \tfrac{1}{2}\sqrt{\pi}\, \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k}{2}+\frac{3}{2})} R^{\frac{k+1}{2}} - \tfrac{1}{2} R^{\frac{k}{2}} + O\left(R^{\frac{k}{4}}\right).$$

Let $\mu > 0$; we have for $a > 0$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{xs}ds}{s^{\mu+1}} = \frac{x^{\mu}}{\Gamma(\mu+1)} \text{ if } x > 0;$$

$$= 0 \text{ if } x \leqslant 0.$$

It follows that

$$(1\cdot2) \qquad \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{xs}\left(\sum_{1}^{\infty} e^{-n^2 s}\right)}{s^{\mu+1}} ds = \frac{1}{\Gamma(\mu+1)} \sum_{n \leqslant \sqrt{x}} (x - n^2)^{\mu}.$$

For $R(s) > 0$ it is known that

$$(1\cdot3) \qquad \sum_{-\infty}^{\infty} e^{-n^2\pi s} = \frac{1}{\sqrt{s}} \sum_{-\infty}^{\infty} e^{-\frac{n^2\pi}{s}}$$

From (1·2) and (1·3)

$$\frac{1}{\Gamma(\mu+1)} \sum_{n \leqslant \sqrt{x}} (x - n^2)^{\mu}$$

$$= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{xs}}{s^{\mu+1}}\left(-\tfrac{1}{2} + \tfrac{1}{2}\frac{\sqrt{\pi}}{\sqrt{s}} + \frac{\sqrt{\pi}}{\sqrt{s}} \sum_{1}^{\infty} e^{\frac{-n^2\pi^2}{s}}\right) ds$$

$$= -\frac{1}{2}\frac{x^{\mu}}{\Gamma(\mu+1)} + \frac{1}{2}\sqrt{\pi}\frac{x^{\mu+1/2}}{\Gamma(\mu+\frac{3}{2})}$$

$$+\frac{1}{2\pi i}\sqrt{\pi}\int_{a-i\infty}^{a+i\infty}\frac{e^{xs}}{s^{\mu+3/2}}\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^m n^{2m}\pi^{2m}}{s^m\Gamma(m+1)}\right)ds$$

$$=\frac{1}{2}\sqrt{\pi}\frac{x^{\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{3}{2})} - \frac{1}{2}\frac{x^{\mu}}{\Gamma(\mu+1)}$$

$$+\frac{1}{2\pi i}\sqrt{\pi}\sum_{n=1}^{\infty}\left(\int_{a-i\infty}^{a+i\infty}e^{xs}\sum_{m=0}^{\infty}\frac{(-1)^m(n\pi)^{2m}}{s^{\mu+m+3/2}\Gamma(m+1)}ds\right)$$

$$(1\cdot4)\quad = \frac{1}{2}\sqrt{\pi}\frac{x^{\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{3}{2})} - \frac{1}{2}\frac{x^{\mu}}{\Gamma(\mu+1)}$$

$$+\sqrt{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m(n\pi)^{2m}x^{\mu+m+1/2}}{\Gamma(m+1)\Gamma(\mu+m+\frac{3}{2})}.$$

Also we have

$$(1\cdot5)\quad J_{\mu}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+1)\Gamma(m+1+\mu)}\left(\frac{x}{2}\right)^{2m+\mu}.$$

It follows that for $\mu>0$,

$$(1\cdot6)\quad \sum_{n\leqslant\sqrt{x}}(x-n^2)^{\mu} = \frac{1}{2}\sqrt{\pi}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\frac{3}{2})}x^{\mu+1/2} - \frac{1}{2}x^{\mu}$$

$$+\sqrt{\pi}\Gamma(\mu+1)x^{\frac{\mu}{2}+\frac{1}{4}}\sum_{n=1}^{\infty}\frac{J_{\mu+1/2}(2\pi n\sqrt{x})}{(n\pi)^{\mu+1/2}}.$$

Since for $\mu>0$,

$$J_{\mu+\frac{1}{2}}(x) = O\left(\frac{1}{\sqrt{x}}\right),$$

(1·1) follows from (1·6) and (1·7).

2. Now assume that (I) has been proved for some k. We shall **then** show it is true for $(k+1)$. Write

$$(2\cdot1)\quad V_k(x) = \frac{\pi^{k/2}x^{k/2}}{\Gamma(k/2+1)} = C_k x^{k/2}.$$

We have

$$A_{k+1}(R) = V_{k+1}(R) + P_{k+1}(R)$$

$$= A_k(R) + 2\sum_{1}^{\sqrt{R}}A_k(R-n^2)$$

$$= V_k(R) + P_k(R) + 2\sum_{1}^{\sqrt{R}}V_k(R-n^2) + 2\sum_{1}^{\sqrt{R}}P_k(R-n^2)$$

$$= V_k(R) + P_k(R) + 2C_k\sum_{1}^{\sqrt{R}}(R-n^2)^{k/2} + 2\sum_{1}^{\sqrt{R}}P_k(R-n^2).$$

Hence, using (1·1),

$$P_{k+1}(x) = C_k x^{k/2} + P_k(x) - C_k x^{k/2} + O(x^{k/4}) + 2\sum_{1}^{\sqrt{x}} P_k(x-n^2)$$

$$= P_k(x) + 2\sum_{1}^{\sqrt{x}} P_k(x-n^2) + O(x^{k/4})$$

Now assume (I) and we get

$$\sum_{x \leqslant R} P_{k+1}(x) = \{\tfrac{1}{2} C_k R^{k/2} + o(R^{k/2})\}$$

$$+ O\left(R^{\frac{k}{4}+1}\right) + 2\sum_{1}^{\sqrt{R}} \{\tfrac{1}{2} C_k (R - n^2)^{k/2} + o(R^{k/2})\}$$

$$= \tfrac{1}{2} C_{k+1} R^{\frac{k+1}{2}} + o\left(R^{\frac{k+1}{2}}\right) \quad [k \geqslant 3].$$

3. It remains to prove the case $k = 3$. This is deduced from the case* $k = 2$ as follows:

We write, for $|x| < 1$

(3·1) $$\left(\sum_{-\infty}^{\infty} x^{n^2}\right)^2 = 1 + 4\sum_{n=1}^{\infty} r(n)\, x^n,$$

and

(3·2) $$\left(\sum_{-\infty}^{\infty} x^{n^2}\right)^3 = 1 + \sum_{n=1}^{\infty} r_3(n)\, x^n.$$

Let $c(n) = 1$ if n is a perfect square, $c_n = 0$ otherwise. It follows from (3·1) and (3·2) that

(3·3) $$r_3(n) = 8\sum_{1}^{n-1} c(m)\, r(n-m) + 4\, r(n) + 2\, c(n).$$

Hence

(3·4) $$\sum_{n \leqslant R} r_3(n) = 8\sum_{m=1}^{R-1} c_m \left[\frac{\pi}{4}(R-m) + P(R-m)\right]$$

$$+ 4\sum_{n \leqslant R} r(n) + 2\sum_{n \leqslant R} c(n).$$

Write $f(x)$ for the fractional part of x. Then

(3·5) $$\sum_{m \leqslant x} c(m) = \sqrt{x} - f(\sqrt{x}).$$

It follows that if $\theta = f(\sqrt{R})$, then

(3·6) $$\sum_{m=1}^{R-1} [\sqrt{m} - f(\sqrt{m})] = \sum_{m \leqslant R-1} (R-m)\, c(m) = \sum_{m \leqslant R} (R-m)\, c(m).$$

$$= R(\sqrt{R} - \theta) - \frac{(\sqrt{R} - \theta)^3}{3} - \frac{(\sqrt{R} - \theta)^2}{2} - \frac{\sqrt{R} - \theta}{6}$$

---

* The case $k = 2$ is also an easy deduction from known results. In this paper a completely elementary proof is indicated (see section 4).

whence

$$(3\cdot 7)\quad \sum_{m\leqslant R} f(\sqrt{m}) = \theta + \left[\tfrac{2}{3}R^{3/2}+\tfrac{1}{2}\sqrt{R}+\frac{1}{24\sqrt{R}}+\ldots\right]$$
$$+\zeta(-\tfrac{1}{2})-\sqrt{R}-\left[\tfrac{2}{3}R^{3/2}-\tfrac{1}{2}R+(-\theta^2+\theta-\tfrac{1}{6})\sqrt{R}+\frac{\theta^3}{3}-\frac{\theta^2}{2}+\frac{\theta}{6}\right]$$
$$=\tfrac{1}{2}R+\sqrt{R}(\theta^2-\theta-\tfrac{1}{3})+\zeta(-\tfrac{1}{2})-\frac{\theta^3}{3}+\frac{\theta^2}{2}+\frac{5\theta}{6}+\frac{1}{24\sqrt{R}}+\ldots.$$

From (3·4) and (3·6) we obtain

$$(3\cdot 8)\quad \sum_{n\leqslant R} r_3(n) = 2\pi\left[\tfrac{2}{3}R^{3/2}-\tfrac{1}{2}R+\sqrt{R}\ (-\theta^2+\theta-\tfrac{1}{6})+\frac{\theta^3}{3}-\frac{\theta^2}{2}+\frac{\theta}{6}\right]$$
$$+\ 8\sum_{m\leqslant R-1} c(m)\,P(R-m)+[\pi R+4P(R)]+2\sum_{m\leqslant R} c(m).$$

It follows that

$$(3\cdot 9)\quad P_3(R) = 1+\sum_{n\leqslant R} r_3(n)-\tfrac{4}{3}\pi R^{3/2}$$
$$= 8\sum_{m<R} c(m)\,P(R-m)+4P(R)$$
$$+\ 2\sum_{m\leqslant R} c(m)+2\pi\sqrt{R}\,(-\theta^2+\theta-\tfrac{1}{6})$$
$$+\ 2\pi\left(\frac{\theta^3}{3}-\frac{\theta^2}{2}+\frac{\theta}{6}\right).$$

Now we have

$$(3\cdot 10)\quad \sum_{n\leqslant x} r(n) = \frac{\pi}{4}x+P(x) = \frac{\pi}{4}x+O(\sqrt{x}).$$

Where, assuming (I) for $k=2$.

$$(3\cdot 11)\quad \sum_{x\leqslant R} P(x) = \left(\frac{\pi}{8}-\tfrac{1}{4}\right)R+o(R).$$

From (3·9) and (3·11),

$$(3\cdot 12)\quad \sum_{x\leqslant R} P_3(x) = 8\sum_{m=1}^{R-1} c(m)\left[\left(\frac{\pi}{8}-\tfrac{1}{4}\right)(R-m)+o(R)\right]$$
$$+O(R)+2\pi\sum_{n=1}^{R}\sqrt{n}\,\{f(\sqrt{n})-f^2(\sqrt{n})-\tfrac{1}{6}\}$$
$$+O(R)+2\sum_{m\leqslant R} c(m)\,(R+1-m)$$
$$=\frac{2\pi}{3}R^{3/2}+2\pi\sum_{n\leqslant R}\sqrt{n}\,\{f(\sqrt{n})-f^2(\sqrt{n})-\tfrac{1}{6}\}+o(R^{3/2}).$$

We can prove in an elementary manner, similar to the proof of (3·7), that

$$(3\cdot 13)\quad \sum_{n\leqslant R}\sqrt{n}\,\{f(\sqrt{n})-f^2(\sqrt{n})-\tfrac{1}{6}\} = O(R).$$

From (3·12) and (3·13),

(3. 14) $\sum_{x \leqslant R} P_3(x) = \frac{2\pi}{3} R^{3/2} + o(R^{3/2})$,

which is the case $k = 3$ of (I).

4. We have

(4·1) $P_2(x) = 1 + 4 P(x)$.

To prove (3·11) assumed above, we first show, *similar to the proof of* (3·14), that

(4·2) $\sum_{x \leqslant R} P_2(x) = \frac{\pi R}{2} + o(R)$.

Thus we obtain completely elementary proofs of (I) in the cases $k=2$ and $k=3$. (I) then follows by mathematical induction for $k>3$, using the results of sections 1 and 2, which, however, involve a knowledge of Bessel functions. It is not difficult to construct a completely elementary proof of (I), similar to the cases $k=2$ and $k=3$ [*i.e., not using sections* 1 and 2].

It is also possible to prove (I) for $k \geqslant 2$ from known results in the theory of lattice points due to Walfisz and Landau.

5. Nothing is known about the signs of $P_3(n)$ and $P_4(n)$, n an *integer*. Thus (I) proves that the inequalities,

(5·1) $P_3(n) > 0, \quad P_4(n) > 0$

are true for infinitely many *integers n*. It seems not impossible that these inequalities are true for *all large positive integers n*. This, if true, must lie very deep.

6. It may be remarked that the method used above also applies to the ellipsoid of k dimensions:

$$a_1 x_1^2 + \cdots + a_k x_k^2 = x,$$

where $a_1, \cdots, a_k$ are positive integers. Here, also, (I)* holds.

* Credit is claimed for the completely elementary method of proving (I) suggested in this paper [details were given only for $k=3$, the case $k=2$ is considerably simpler].