ON A PROBLEM OF ARRANGEMENTS

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The problem is to arrange the numbers 1, 2, 3, $\cdots$, upto $(2n+1)$ in a circle in n different ways so that no number has the same neighbours in different arrangements. We shall call this problem P_n. My attention was directed to this problem by Dr. Vijayaraghavan.

In recent issues of these *Proceedings* the problem has been dealt with in special cases by Gul Abdulla, Lal Bahadur, and myself; it was pointed out that P_n *is soluble when* $(2n+1)$ *is a prime*; Gupta has developed a general method for attacking the problem. We use his method to prove the

Theorem. P_{n+1} *is soluble when* $(2n+1)$ *is a prime.*

(That the " 10–21 " problem is soluble is the special case $n=10$).

Gupta shows how we can attempt to solve P_{n+1}, in case P_n is solved; he shows that P_{n+1} also can be solved (when P_n is solved) provided we can solve another problem in permutations, whose solution " seems always to exist ", but he was unable to give a " formal proof of this statement ".

Let $(a_1, a_2, a_3, \cdots, a_t)$ denote the arrangement of these natural numbers round a circle, in the order indicated. We know that the solution of P_n is given by the n arrangements A_m $(1 \leqslant m \leqslant n)$ where A_m is the arrangement

$$(1, 1+m, 1+2m, 1+3m, \cdots)$$

[the arrangement contains n numbers; numbers greater than $2n+1$ are represented by their least positive residues mod $\overline{(2n+1)}$].

Let B_m $(1 \leqslant m \leqslant n+1)$ denote the different arrangements in the solution of P_{n+1}. We shall show that all these arrangements, except one (which we call B_{n+1}), are obtained from the A_m $(1 \leqslant m \leqslant n)$ by the introduction of the 2 numbers $(2n+2)$ and $(2n+3)$ at suitable places in A_m (the order of the numbers in an A_m is not disturbed, only at two suitable places we insert the two new numbers between the old ones). In this way we obtain B_m from A_m for $1 \leqslant m \leqslant n$. We denote by (C) the arrangement

$$\begin{aligned}(C) &= (1, 2n, 2, 2n-1, 3, 2n-2, \cdots, n, n+1)\\ &= (\theta_1, \theta_2, \theta_3, \theta_4, \cdots, \theta_{2n-1}, \theta_{2n})\end{aligned}$$

It is clear that $(\theta_1 = 1$, etc.).

(1) $\theta_1 + \theta_2 = \theta_3 + \theta_4 = \theta_5 + \theta_6 = \cdots = \theta_{2n-1} + \theta_{2n} = 2n + 1$

We take

$$B_{n+1} = (\theta_1, \theta_2, \cdots, \theta_{2n-1}, 2n+2, \theta_{2n}, 2n+3, 2n+1),$$

i.e., as we shall put it : B_{n+1} is obtained from (C) by "inserting" $(2n+2)$ between θ_{2n-1} and θ_{2n} [in (C)], and "inserting" $(2n+3)$ and $(2n+1)$ after θ_{2n} [in (C)]. B_1 is obtained from A_1 by inserting $(2n+2)$ and $(2n+3)$ at the end of A_1; before proceeding further we must explain our terminology ; we shall say that the pair (a, b) occurs in an arrangement like $(d_1, d_2, \cdots, d_m)$ when a and b are consecutive d's, *i.e.*, $a = d_t$, $b = d_{t+1}$ for some t. Further d_m is regarded as also consecutive to d_1 (since the d's are supposed to be arranged in a circle). Now our θ's ($\theta_1 = 1$, $\theta_2 = 2n$, $\cdots$, etc., $\theta_{2n-1} = n$, $\theta_{2n} = n+1$) have been chosen so that each of the pairs (θ_1, θ_2), (θ_3, θ_4), (θ_5, θ_6), etc., upto $(\theta_{2n-3}, \theta_{2n-2})$ occurs in *exactly one* of the arrangements A_m $(2 \leqslant m \leqslant n)$. The same is true of the pairs (θ_2, θ_3), (θ_4, θ_5), etc., upto $(\theta_{2n-2}, \theta_{2n-1})$, *i.e.*, each of these pairs "occurs" *exactly once* in the arrangements A_m $(2 \leqslant m \leqslant n)$.

To get B_m from A_m, we insert $(2m+2)$ between $(\theta_{2k-1}, \theta_{2k})$ where k is chosen so that the latter pair "occurs" in A_m ; we also insert $(2m+3)$ between the numbers θ_{2l} and θ_{2l+1}, where l is chosen so that $(\theta_{2l}, \theta_{2l+1})$ is a pair which occurs in A_m. It is thus that we get B_m from A_m, for $2 \leqslant m \leqslant n$.

The proof is easy and is left to the reader : the arrangements B_m $(1 \leqslant m \leqslant n+1)$ defined above are a solution of P_{n+1}.

We illustrate the theorem and method by an example, $n = 10$.

If, for example, 4 and 7 are neighbours in an arrangement, then $(1, \overbrace{4, 7}, 10, \cdots)$ will mean that 20 is inserted between 4 and 7 ; a similar sign *below* will mean that 21 is to be inserted between the numbers thus connected.

Then (here the "bars" mean nothing ; they merely help to construct the remaining B's).

$B_{10} = (\overline{1, 18}, \overline{2, 17}, \overline{3, 16}, \overline{4, 15}, \overline{5, 14}, \overline{6, 13}, \overline{7, 12}, 8, 11, 9, 20, 10, 21, 19)$.

The B_m $(2 \leqslant m \leqslant 9)$ are simply A_m $(2 \leqslant m \leqslant 9)$ with the connecting signs at two places in each B.

$B_1 = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)$

$*B_2 = (3, 5, 7, \underbrace{9, 11}, 13, 15, 17, 19, 2, 4, 6, 8, 10, 12, 14, 16, \overbrace{18, 1})$

$B_3 = (1, 4, 7, 10, 13, 16, 19, 3, 6, 9, 12, 15, \underbrace{18, 2}, 5, \overbrace{8, 11}, 14, 17)$

$$B_4 = (1, 5, 9, 13, 17, 2, 6, 10, 14, 18, 3, 7, 11, 15, 19, 4, 8, 12, 16)$$

$$B_5 = (1, 6, 11, 16, 2, 7, 12, 17, 3, 8, 13, 18, 4, 9, 14, 19, 5, 10, 15)$$

$$B_6 = (1, 7, 13, 19, 6, 12, 18, 5, 11, 17, 4, 10, 16, 3, 9, 15, 2, 8, 14)$$

$$B_7 = (1, 8, 15, 3, 10, 17, 5, 12, 19, 7, 14, 2, 9, 16, 4, 11, 18, 6, 13)$$

$$B_8 = (1, 9, 17, 6, 14, 3, 11, 19, 8, 16, 5, 13, 2, 10, 18, 7, 15, 4, 12)$$

$$B_9 = (1, 10, 19, 9, 18, 8, 17, 7, 16, 6, 15, 5, 14, 4, 13, 3, 12, 2, 11)$$

* Note, in regard to B_2, that

$$(a_1, a_2, \cdots, a_m) \text{ is the same as } (a_2, a_3, \cdots, a_m, a_1)$$

on account of the "circular" arrangement.

In B_2 to B_9, *note that* 20 *is inserted between a pair of consecutive numbers whose sum is* 19 ; 21 *is inserted between a pair of consecutive numbers whose sum is* 20 ; *this remark generalizes to the general case.*

Note added May 15, 1939.—

Prof. Levi has obtained a remarkably simple solution of P_n for all n.