A THEOREM ON SUMS OF POWERS WITH APPLICATIONS TO THE ADDITIVE THEORY OF NUMBERS (II).

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1. LET $v(k)$ denote the least value of s such that every integer n (positive or negative) can be expressed in the form

$$n = \epsilon_1 m_1^k + \cdots + \epsilon_s m_s^k$$

where m_i is a positive integer or zero, and $\epsilon_i = \pm 1$. This function has been studied by Wright.[1] He proves

$$v(k) = O\ (2^{\cdot 385k}).$$

The principal result of this paper is

Theorem 1. *For every* k > 1 *there exists a* g = g(k) *such that*

$$(1) \quad k < g \leqslant 2k+1$$

and

$$(2) \quad v(g) \leqslant k^2+9k+6.$$

From this follows immediately

Theorem 2. *There are infinitely many* k *such that*

$$(3) \quad v(k) \leqslant (k-1)^2+9(k-1)+6 = k^2+7k-2.$$

2. *Notation.* We write

$$(4) \quad a_1, \cdots, a_m \overset{k}{=} b_1, \cdots, b_m$$

when

$$(5) \quad \sum_a a^\theta = \sum_b b^\theta \ (\theta = 1, 2, 3, \cdots, k),$$

and the b's are not merely a permutation of the a's. N(k) denotes the least value of m such that (4) is possible. The trivial result N(k) $\geqslant k+1$ is a consequence of the impossibility of

$$a_1, \cdots, a_k \overset{k}{=} b_1, \cdots, b_k.$$

In what follows the B's *are positive numbers* (whose values differ from place to place) *depending only on* k *and* s. Further

$$p_m = \frac{s}{k} - 1 - \sum_{r=1}^{m} \frac{r}{k}.$$

[1] *Journ. London Math. Soc.*, 1934, **9**, 267-272.

Two sets of integers a_r $(r \leqslant h)$ and b_r $(r \leqslant h)$ are said to be different if the b's are not a permutation of the a's.

We deduce Theorem 1 from

Theorem 3.

$$(6) \quad \mathrm{N}(k) \leqslant \frac{k^2+k}{2} + 1.$$

We prove this result in the next section.

3. (6) is a consequence of

Theorem 4. *Let* $\mathrm{m} < \mathrm{k}, \mathrm{s} \geqslant \frac{k^2+k}{2} + 1$. *Then there are infinitely many sets of positive integers* (m+1 *in number*)

$$\{n = n^{(k)}, n^{(1)}, n^{(2)}, \cdots, n^{(m)}\}$$

such that the system of (m+1) *equations*

$$(\mathrm{A}) \quad n^{(r)} = \sum_{t=1}^{s} n_t^{\ r} \quad [r = k;\ 1 \leqslant r \leqslant m].$$

has more than $\mathrm{B}n^{p_m}$ *solutions.*

Proof: (1) The case $m = 1$.

It is known that we can find infinitely many positive integers $n = n^{(k)}$ such that the equation

$$(7) \quad n^{(k)} = \sum_{t=1}^{s} n_t^k$$

has more than $\mathrm{B}n^{\frac{s}{k}-1}$ solutions in positive integers $n_r (r \leqslant s)$. In (7) we must have

$$(8) \quad 0 < \sum_{t \leqslant s} n_t \leqslant s.n^{\frac{1}{k}}.$$

Let us arrange the solutions of (7) according to the value of $\sum_{t \leqslant s} n_t$. It follows from (7) and (8) that there is a positive integer $n^{(1)}$ such that the equations

$$n^{(r)} = \sum_{t \leqslant s} n_t^r \quad (r = k;\ r = 1)$$

have more than $\mathrm{B}n^{\frac{s}{k}-1-\frac{1}{k}} = \mathrm{B}n^{p_1}$ solutions in $n_r (r \leqslant s)$. This completes the proof of Theorem 4 for $m = 1$.

(2) Assume the theorem proved for m where $m \leqslant k-2$. We shall then prove it for $m+1$.

From the case $r = k$ of (A) we obtain

$$(9) \quad 0 < \sum_{t \leqslant s} n_t^{m+1} \leqslant s\, n^{\frac{m+1}{k}}.$$

Now arrange the solutions $(n_1, \cdots, n_s)$ of (A) according to the value of

$$\sum_{t \leqslant s} n_t^{m+1}.$$

It follows from the theorem (proved for m) and from (9) that there exists an integer $n^{(m+1)}$ such that the number of solutions of (A) which also satisfy

$$n^{(m+1)} = \sum_{t \leqslant s} n_t^{m+1}$$

is more than Bn^{ϕ} where $\phi = p_m - \frac{m+1}{k} = p_{m+1}$. Hence the theorem is proved for $m+1$.

(3) From (1) and (2) Theorem 4 follows by mathematical induction.

Proof of Theorem 3 *:* Putting $s = \frac{k^2+k}{2} + 1$, $m = k-1$ in Theorem 4 we get

$$p_m = \frac{k+1}{2} + \frac{1}{k} - 1 - \frac{k-1}{2} = \frac{1}{k} > 0.$$

Hence there exist at least two different sets of integers $\{a_1, \cdots, a_s\}$ and $\{b_1, \cdots, b_s\}$ such that

$$\sum_a a^r = \sum_b b^r \quad (r = k\,;\, 1 \leqslant r \leqslant k-1)$$

which is the same as (4) with $s = m$. Hence Theorem 3.

4. Theorem 3 implies the existence of an $s \leqslant \frac{k^2+k}{2} + 1$ and two different sets of integers $a_r (r \leqslant s)$ and $b_r (r \leqslant s)$ such that

$$\sum_{r \leqslant s} a_r^m = \sum_{r \leqslant s} b_r^m \quad (1 \leqslant m \leqslant k).$$

The latter equation implies that

$$(10) \quad \sum_{r \leqslant s} (x+a_r)^k - \sum_{r \leqslant s} (x+b_r)^k = 0.$$

Let $g > k$ be the (least) integer such that continued integration of (10) gives

$$(11) \quad \sum_{r \leqslant s} (x+a_r)^g - \sum_{r \leqslant s} (x+b_r)^g = c$$

where $c \neq 0$. Then $g \leqslant s$ for

$$a_1, \cdots, a_s \overset{S}{=} b_1, \cdots, b_s$$

is impossible. Hence we obtain

Theorem 5. *There is an* $s \leqslant \frac{k^2+k}{2} + 1$, *two different sets of integers* $a_r(r \leqslant s)$ *and* $b_r(r \leqslant s)$, *and an integer* g *satisfying*

$$(12) \quad k < g \leqslant s$$

such that

$$(13) \quad \sum_{r \leqslant s} a_r^m = \sum_{r \leqslant s} b_r^m$$

is true for $m < g$ *but not true for* $m = g$.

5. From Theorem 5 we shall deduce the sharper

Theorem 6. *There is an* $s \leqslant \frac{k^2+k}{2}+1$, *two different sets of integers* $a_r(r \leqslant s)$ *and* $b_r(r \leqslant s)$, *and an integer* g *satisfying*

$$(14) \quad k < g \leqslant 2k$$

such that

$$(15) \quad \sum_{r\leqslant s} a_r^m = \sum_{r\leqslant s} b_r^m$$

is true for $m < g$ *but not true for* $m = g$.

Theorem 6 differs from Theorem 5 only in that (12) is replaced by the sharper inequality (14).

6. In this section we prove the

Lemma. If $a_r(r \leqslant s)$ *and* $b_r(r \leqslant s)$ *are two different sets of integers such that*

$$\sum_{r\leqslant s} a_r^m = \sum_{r\leqslant s} b_r^m$$

is true for $m \leqslant w$ *but not true for* $m = w+1$ *and if* y *is any integer* $\geqslant w+1$, *then we can find two different sets of integers* $c_r(r \leqslant s)$ *and* $d_r(r \leqslant s)$ *such that*

$$\sum_{r\leqslant s} c_r^m = \sum_{r\leqslant s} d_r^m$$

is true for $m \leqslant w$ *but not true for* $m = y$.

Proof. It follows from hypothesis that for arbitrary x,

$$(16) \quad \sum_{r\leqslant s} (x+a_r)^{w+1} - \sum_{r\leqslant s} (x+b_r)^{w+1} = c \neq 0.$$

Integrating (16) $(y-w-1)$ times we get

$$(17) \quad \sum_{r\leqslant s} (x+a_r)^y - \sum_{r\leqslant s} (x+b_r)^y = \sum_{r\leqslant y-w-1} a_r x^r$$

where the coefficient of the highest power is $a_{y-w-1} \neq 0$ since $c \neq 0$. Hence by proper choice of x, the right hand side of (17) is not zero. For such x put $c_r = a_r+x$, $d_r = b_r+x$, and the lemma is proved.

7. We shall now deduce Theorem 6 from Theorem 5.

From (12) we see that there is a t such that

$$(18) \quad tk < g \leqslant (t+1)k \quad \left[1 \leqslant t \leqslant \frac{k+1}{2}\right].$$

Having found t there is an m such that

$$(19) \quad t(k+m) < g \leqslant t(k+m+1)$$

where $0 \leqslant m < k$. Using our lemma (Section 6) and Theorem 5 we deduce from (19) that we can find two different sets of integers $c_r(r \leqslant s)$ and $d_r(r \leqslant s)$ such that

$$(20) \quad \sum_{r\leqslant s} c_r^\theta = \sum_{r\leqslant s} d_r^\theta$$

is true for $\theta \leqslant (k+m)$ but not true for $\theta = (k+m+1)$.

Hence putting

$$e_r = c_r^t,\ f_r = d_r^t$$

we have 2 different sets of integers $e_r (r \leqslant s)$ and $f_r (r \leqslant s)$ such that

$$\sum_{r \leqslant s} e_r^\theta = \sum_{r \leqslant s} f_r^\theta$$

is true for $\theta \leqslant k+m$ but not true for $\theta = k+m+1$. Since $m < k$, Theorem 6 is now proved.

8. From Theorem 6 we can find $s \leqslant \frac{k^2+k}{2} + 1$, two different sets $a_r (r \leqslant s)$ and $b_r (r \leqslant s)$, and a g satisfying

$$(21) \quad k+1 < g \leqslant 2k+1$$

such that

$$(22) \quad \sum_{r \leqslant s} (x+a_r)^g - \sum_{r \leqslant s} (x+b_r)^g = cx+d \ (c \neq 0)$$

whence (as with Wright)

$$(23) \quad v(g) \leqslant 2s + \Gamma(g) \leqslant 2s + 4g \leqslant k^2+k+2+4(2k+1) = k^2+9k+6.$$

From (21) and (23) Theorem 1 follows.

9. Denote by $\gamma(k)$ the least value of n such that

$$\sum_{s \leqslant m} x_s^k = \sum_{t \leqslant n} y_t^k$$

has infinitely many solutions in positive integers $x_s (s \leqslant m)$, $y_t (t \leqslant n)$ with

$$\text{(B)} \quad x_1, \cdots, x_m, y_1, \cdots, y_n) = 1$$

and with $m < n$.

Now from Theorem 4, given any arbitrary r, we can find an $s \leqslant \frac{k^2+k}{2} + 1$ and r sets (all different from each other) of s integers each, namely

$$\{a_{11}, a_{12}, \cdots, a_{1s}\};$$
$$\{a_{21}, a_{22}, \cdots, a_{2s}\};$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\{a_{r1}, a_{r2}, \cdots, a_{rs}\};$$

such that

$$(24) \quad \sum_{t \leqslant s} a_{it}^m = \sum_{t \leqslant s} a_{jt}^m \quad (1 \leqslant m \leqslant k)$$

is true for any i, j $(i, j \leqslant r)$.

From (24),

$$(25) \quad \sum_{t \leqslant s} (x+a_{it})^k = \sum_{t \leqslant s} (x+a_{jt})^k$$

for any $i, j \leqslant r$. Now for fixed i, j suppose that a_{i1} is the smallest of the positive numbers a_{it}, a_{jt}. Then putting $s = -a_{i1}$ in (25) we get a solution of

$$(26) \quad \sum_{s \leqslant m} x_s^k = \sum_{t \leqslant n} y_t^k \quad (m < n \leqslant s)$$

with x_s and y_t positive and satisfying (B) above.

Since i, j can be given any values upto r we get as many *different* solutions of (26) as we like. Hence

Theorem 7. $\gamma(k) \leqslant s \leqslant \frac{k^2+k}{2} + 1.$

10. From Theorem 6 and a process described by Wright (*loc. cit.*, page 271) we get, easily,

Theorem 8. $v(k) < \exp(A \sqrt{k} \log k)$,

where A is a positive constant independent of k.