ON SUMS OF POWERS.

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1. We denote by N(k) the least value of m such that the equation

(I) $a_1^n + \cdots + a_m^n = b_1^n + \cdots + b_m^n \quad (1 \leqslant n \leqslant k)$ has a non-trivial solution in which a_r $(1 \leqslant r \leqslant m)$ and b_s $(1 \leqslant s \leqslant m)$ are positive integers.

Pillai[1] has recently shown that

(1) $\mathrm{N}(k) = \mathrm{O}\left(\frac{2^k}{\sqrt{k}}\right)$,

and Wright[2] (independently) has proved the stronger result:

(2) $\mathrm{N}(k) = \mathrm{O}\left((160)^{k/19}\right)$.

On the other hand, it is trivial that $\mathrm{N}(k) \geqslant k+1$. It is known that

$$\mathrm{N}(k) = k+1 \quad (k = 2, 3, 5, 7),$$
$$\mathrm{N}(6) \leqq 8.$$

I shall show here that

(3) $\mathrm{N}(18) \leqslant 68$.

From (3) and a process described by Wright[3] we at once deduce that

(4) $\mathrm{N}(k) = \mathrm{O}\left((136)^{k/19}\right)$,

which is an improvement of (2).

In the sequel we shall frequently employ the following:

Lemma 1. If[4]

(5) $a_1, \cdots, a_m \overset{k}{=} b_1, \cdots, b_m$

then

(6) $a_1, \cdots, a_m, b_1+x, \cdots, b_m+x \overset{k+1}{=\!=} b_1, \cdots, b_m, a_1+x, \cdots, a_m+x$.

Suppose that in (5) the a's and b's are written in ascending order of magnitude. Further let d_1 be the number of solutions of $y = a_r - a_s$ $(r > s)$, d_2 the number of solutions of $y = b_r - b_s$ $(r > s)$, $d = d_1 + d_2$.

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[1] Pillai, **1**. See also a paper by A. Moessner in the same issue.

[2] Wright, **1**.

[3] *Loc. cit.*

[4] (5) is an abbreviated form of (I).

Then the number of terms in (6) when x is put equal to y and terms common to both sides are cancelled, is $2m-d$.

Lemma 2.

$$1, 5, 10, 24, 28, 42, 47, 51 \overset{7}{=} 2, 3, 12, 21, 31, 40, 49, 50.$$

Lemmas 1 and 2 are due to Tarry.[5]

We apply lemma 1 to lemma 2 with

$y = 9,\ d_1 = 2,\ d_2 = 4\ (d = 6)$. This gives

Lemma 3.

$$1, 5, 11, 24, 28, 30, 42, 47, 58, 59 \overset{8}{=} 2, 3, 14, 19, 31, 33, 37, 50, 56, 60.$$

whence $N(8) \leqslant 10$.

We now apply lemma 1 in succession, starting with lemma 3.

We use

(7)	$y = 4$,	$d_1 = 2$,	$d_2 = 2$	$(d = 4)$	on lemma 3.	Thus $N(9) \leqslant 16$.
(8)	$y = 1$,	$d_1 = 4$,	$d_2 = 7$	$(d = 11)$	on (7).	Hence $N(10) \leqslant 21$.
(9)	$y = 6$,	$d_1 = 6$,	$d_2 = 6$	$(d = 12)$	on (8).	Hence $N(11) \leqslant 30$.
(10)	$y = 7$,	$d_1 = 14$,	$d_2 = 16$	$(d = 30)$	on (9).	Hence $N(12) \leqslant 30$.
(11)	$y = 5$,	$d_1 = 13$,	$d_2 = 13$	$(d = 26)$	on (10).	Hence $N(13) \leqslant 34$.
(12)	$y = 2$,	$d_1 = 14$,	$d_2 = 8$	$(d = 22)$	on (11).	Hence $N(14) \leqslant 46$.
(13)	$y = 3$,	$d_1 = 24$,	$d_2 = 24$	$(d = 48)$	on (12).	Hence $N(15) \leqslant 44$.
(14)	$y = 1$,	$d_1 = 18$,	$d_2 = 16$	$(d = 34)$	on (13).	Hence $N(16) \leqslant 54$.
(15)	$y = 19$,	$d_1 = 19$,	$d_2 = 19$	$(d = 38)$	on (14).	Hence $N(17) \leqslant 70$.
(16)	$y = 17$,	$d_1 = 34$,	$d_2 = 38$	$(d = 72)$	on (15).	Hence $N(18) \leqslant 68$.

Our final results expressing (15) and (16) are,

(17) 1, 3, 4, 4, 4, 8, 8, 12, 14, 14, 20, 21, 21, 21, 25, 26, 26, 31, 35, 35, 37, 37, 37, 37, 38, 38, 41, 42, 47, 51, 51, 52, 54, 54, 54, 55, 55, 55, 57, 58, 58, 62, 67, 68, 71, 71, 72, 72, 72, 72, 74, 74, 78, 83, 83, 84, 88, 88, 88, 89, 95, 95, 97, 101, 101, 105, 105, 105, 106, 108 $\overset{17}{=}$ 2, 2, 2, 6, 6, 7, 7, 10, 16, 17, 19, 19, 19, 22, 23, 28, 29, 32, 33, 33, 36, 36, 36, 39, 39, 39, 40, 44, 48, 49, 50, 53, 53, 53, 53, 56, 56, 56, 56, 59, 60, 61, 65, 69, 70, 70, 70, 73, 73, 73, 76, 76, 77, 80, 81, 86, 87, 90, 90, 90, 92, 93, 99, 102, 102, 103, 103, 107, 107, 107.

(18) 1, 3, 4, 4, 4, 8, 8, 12, 14, 14, 23, 24, 24, 26, 26, 27, 34, 35, 35, 37, 37, 37, 41, 45, 46, 47, 50, 51, 51, 55, 57, 57, 58, 62, 66, 67, 67, 70, 72, 73, 74, 78, 78, 82, 83, 83, 87, 87, 88, 93, 94, 95, 97, 97, 98, 101, 104, 108, 109, 110, 116, 119, 119, 120, 120, 124, 124, 124 $\overset{18}{=}$ 2, 2, 2, 6, 6, 7, 7, 10, 16, 17, 18, 22, 25, 28, 29, 29, 31, 32, 33, 38, 39, 39, 43, 43, 44, 48, 48, 52, 53, 54, 56, 59, 59, 60, 64, 68, 69, 69, 71, 75, 75, 76, 79, 80, 81, 85, 89, 89, 89, 91, 91, 92, 99, 100, 100, 102, 102, 103, 112, 112, 114, 118, 118, 122, 122, 122, 123, 125.

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[5] See Dickson's *History of the Theory of Numbers*, II, 705-713 (710).

The latter result implies

(19) $(x+1)^{18} + \cdots + (x+124)^{18} = (x+2)^{18} + \cdots + (x+125)^{18}.$

Integrating twice we get

(20) $\{(x+1)^{20} + \cdots + (x+124)^{20}\} - \{(x+2)^{20} + \cdots + (x+125)^{20}\} = Cx + D$

where there are 136 terms on the left side.

As with Wright (20) implies

(21) $N(k) = O\left((136)^{k/19}\right),$

which is (4).

REFERENCES.

Pillai, **1,** *Mathematics Student* (Madras), Sept. 1934.

Wright, **1,** *Jour. Lond. Math. Soc.*, 1934, **9,** 267-272.