ON A CERTAIN ARITHMETICAL FUNCTION.

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1. We define $\epsilon (k)$ as the least value of $s$ such that the equation

$$\sum_{n=1}^{s} \epsilon_n m_n^k = 0$$

[where each $\epsilon_n = +1$ or $-1$ and the $m$'s are positive integers] has infinitely many solutions in $m_r$ ($r \leq s$) with

$$(m_1, \ldots, m_s) = 1.$$  

[The symbol $(a_1, \ldots, a_i)$ denotes the greatest common divisor of $a_1, \ldots, a_i$].

We observe that

(1) It is not yet known whether

$$\epsilon (k) \geq 4$$

for all positive integers $k \geq 5$.

(2) It is trivial that

$$\epsilon (k) \leq 2k + 2.$$

(3) The conjecture that

$$\epsilon (k) \to \infty \text{ as } k \to \infty$$

is a natural generalisation of the Thue-Siegel theorem.\(^1\)

(4) We have\(^2\)

$$(1) \quad \epsilon (3) = \epsilon (4) = 4.$$

$$(2) \quad 4 \leq \epsilon (5) \leq 6, \quad 4 \leq \epsilon (8) \leq 16.$$

$$(3) \quad 4 \leq \epsilon (6) \leq 12.$$

$$(4) \quad 4 \leq \epsilon (7) \leq 13, \quad 4 \leq \epsilon (9) \leq 17.$$

In all these cases $\epsilon (k) \leq 2k$. We do not know whether this inequality is true for $k \geq 10$.

In what follows we say that $k$ is 'exceptional' when it belongs to any of the following four forms:

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\(^1\) See Landau, Vorlesungen über Zahlentheorie, Bd. 3.

\(^2\) That $\epsilon (k) \geq 4$ for $3 \leq k \leq 9$ is a consequence of the impossibility of $x^k + y^k = z^k$ ($xyz \neq 0$) for these values of $k$. The right hand side inequalities in (2), (3), (4) are due to Sastry, Subba Rao and S. Chowla respectively. See Journ. London Math. Soc., 1934, 9, 172-73, 242-46, and Proc. Ind. Acad. Sci. (A), 1935, 1, 580-591.

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I. \( \mathcal{F} = 2^2 \) (\( \mathcal{F} > 1 \)).

II. \( \mathcal{F} = 2^2 \cdot 3 \) (\( \mathcal{F} > 1 \)).

III. \( \mathcal{F} = \pi^2 \cdot (\pi - 1) \) (\( \mathcal{F} > 0 \)).

IV. \( \mathcal{F} = \frac{1}{2} \pi^2 \cdot (\pi - 1) \) (\( \mathcal{F} > 0 \)).

Here \( \pi \) denotes any odd prime.

Our main object is to prove

**Theorem 1.** If \( k \) is not ‘exceptional’, then

\[ \epsilon (k) \leq 2k + 1. \]

A special case is

**Theorem 2.** If \( k \) is odd and not of the form IV above (the forms I, II, III can only represent even numbers), then there is an \( s \leq 2k + 1 \) such that the diophantine equation

\[ \sum_{m=1}^{s} x_m^k = 0 \]

has infinitely many solutions with \( (x_1, \ldots, x_s) = 1 \).

It seems not impossible that our theorems should be capable of elementary proof for all integers \( k \), but our proof of Theorem 1 is based on the following deep result of Hardy and Littlewood³:

**Theorem A.** If ‘Hypothesis K’ is true and \( k \) is not exceptional then almost all positive integers are expressible as the sum of \( (k + 1) \) non-negative \( k \)th powers of integers.

We write⁴

\[ (m)^k = (n)^k \quad \text{i. o.} \]

when there exist infinitely many sets of positive integers \( x_s \) (\( s \leq m \)), \( y_t \) (\( t \leq n \)) such that

\[ \sum_{s \leq m} x_s^k = \sum_{i \leq n} y_i^k \]

where

\[ (x_1, \ldots, x_m, y_1, \ldots, y_n) = 1. \]

2. If

\[ (k)^k = (k)^k \quad \text{i. o.} \]

is true then \( \epsilon (k) \leq 2k \), and hence our theorem is proved. Hence we may assume that (8) is false and hence that⁵

\[ r_{k, k} (n) = 0 \quad (1), \]


⁵ *r* \( k, k \) (n) is the number of representations of \( n \) as a sum of \( k \) positive \( k \)th powers of integers.
so that 'Hypothesis K' is certainly true. But from (B),

\[ \sum \frac{1}{n} > c \sqrt[3]{x} \]

where \( c \) is a positive constant independent of \( x \). In words (C) means that the sequence of integers which are sums of \( k \) non-negative \( k \)th powers (of integers) has 'positive density'.

From theorem (A) if \( k \) is not 'exceptional' then in virtue of (B),

\[ \sum \frac{1}{n} \sim x. \]

From (C) and (D) it follows that if \( k \) is not 'exceptional' then

\[ (k)^k = (k+1)^k \quad i.e. \]

\[ e(k) \leq 2k+1. \]

Hence our Theorem 1.