ON A CERTAIN ARITHMETICAL FUNCTION.

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Received April 1, 1935.

1. We define $\epsilon(k)$ as the least value of s such that the equation

$$\sum_{n=1}^{s} \epsilon_n m_n^k = 0$$

[where each $\epsilon_n = +1$ or -1 and the m's are positive integers] has infinitely many solutions in m_r $(r \leqslant s)$ with

$$(m_1, \cdots, m_s) = 1.$$

[The symbol $(a_1, \cdots, a_t)$ denotes the greatest common divisor of $a_1, \cdots, a_t$].

We observe that

(1) It is not yet known whether

$$\epsilon(k) \geqslant 4$$

for all positive integers $k \geqslant 5$.

(2) It is trivial that

$$\epsilon(k) \leqslant 2k+2.$$

(3) The conjecture that

$$\epsilon(k) \to \infty \text{ as } k \to \infty$$

is a natural generalisation of the Thue-Siegel theorem.[1]

(4) We have[2]

(1) $\epsilon(3) = \epsilon(4) = 4.$

(2) $4 \leqslant \epsilon(5) \leqslant 6,\ 4 \leqslant \epsilon(8) \leqslant 16.$

(3) $4 \leqslant \epsilon(6) \leqslant 12.$

(4) $4 \leqslant \epsilon(7) \leqslant 13,\ 4 \leqslant \epsilon(9) \leqslant 17.$

In all these cases $\epsilon(k) \leqslant 2k$. We do not know whether this inequality is true for $k \geqslant 10$.

In what follows we say that k is 'exceptional' when it belongs to any of the following four forms:

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[1] See Landau, *Vorlesungen über Zahlentheorie*, Bd. 3.

[2] That $\epsilon(k) \geqslant 4$ for $3 \leqslant k \leqslant 9$ is a consequence of the impossibility of $x^k + y^k = z^k$ $(xyz \neq 0)$ for these values of k. The right hand side inequalities in (2), (3), (4) are due to Sastry, Subba Rao and S. Chowla respectively. See *Journ. London Math. Soc.*, 1934, **9**, 172-73, 242-46, and *Proc. Ind. Acad. Sci.* (A), 1935, **1**, 590-591.

I. $k = 2^{\vartheta}$ $(\vartheta > 1)$.

II. $k = 2^{\vartheta}\cdot 3$ $(\vartheta > 1)$.

III. $k = \pi^{\vartheta}(\pi - 1)$ $(\vartheta > 0)$.

IV. $k = \frac{1}{2}\pi^{\vartheta}(\pi - 1)$ $(\vartheta > 0)$.

Here π denotes any odd prime.

Our main object is to prove

Theorem 1. *If* k *is not 'exceptional', then*

$$\epsilon(k) \leqslant 2k + 1.$$

A special case is

Theorem 2. *If* k *is odd and not of the form* IV *above* (the forms I, II, III can only represent even numbers), *then there is an* $s \leqslant 2k+1$ *such that the diophantime equation*

$$\sum_{m=1}^{s} x_m{}^k = 0$$

has infinitely many solutions with $(x_1, \cdots, x_s) = 1$.

It seems not impossible that our theorems should be capable of *elementary proof* for *all integers k,* but our proof of Theorem 1 is based on the following deep result of Hardy and Littlewood[3]:

Theorem A. *If 'Hypothesis* K*' is true and* k *is not exceptional then almost all positive integers are expressible as the sum of* (k+1) *non-negative* k*th powers of integers.*

We write[4]

$$(5)\quad (m)^k = (n)^k \quad i.\ o.$$

when there exist infinitely many sets of positive integers x_s $(s \leqslant m)$, y_t $(t \leqslant n)$ such that

$$(6)\quad \sum_{s \leqslant m} x_s^k = \sum_{t \leqslant n} y_t^k$$

where

$$(7)\quad (x_1, \cdots, x_m, y_1, \cdots, y_n) = 1.$$

2. If

$$(8)\quad (k)^k = (k)^k \quad i.\ o.$$

is true then $\epsilon(k) \leqslant 2k$, and hence our theorem is proved. Hence we may assume that (8) is false and hence that[5]

$$(B)\quad r_{k,k}(n) = 0(1),$$

[3] P. N. (VI) in *Math. Ztschr.*, 1925, **23**, 1-37.
See also P. N. (VIII) in *Proc. London Math. Soc.*, 1928, **27**, 518-42.

[4] '*i. o.*' ≡ 'infinitely often'.

[5] $r_{k,k}(n)$ is the number of representations of n as a sum of k positive kth powers of integers.

so that ' Hypothesis K ' is certainly true. But from (B),

$$\text{(C)} \qquad \sum_{\substack{n \leqslant x \\ n = n_1^k + \cdots + n_k^k}} 1 > c\,x$$

where c is a positive constant independent of x. In words (C) means that the sequence of integers which are sums of k non-negative kth powers (of integers) has ' positive density '.

From theorem (A) if k is not ' exceptional ' then in virtue of (B),

$$\text{(D)} \qquad \sum_{\substack{n \leqslant x \\ n = n_1^k + \cdots + n^k_{k+1} \\ n_1, \cdots, n_{k+1} \geqslant 0}} 1 \sim x.$$

From (C) and (D) it follows that if k is not ' exceptional ' then

$$(k)^k = (k+1)^k \quad i.\ o.$$

i.e.

$$\epsilon\,(k) \leqslant 2k+1.$$

Hence our Theorem 1.