HEILBRONN'S CLASS-NUMBER THEOREM (II).

BY S. CHOWLA,
Andhra University, Waltair.

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1. LET $h(d)$ denote the number of primitive classes of binary quadratic forms of negative discriminant d. Heilbronn[1] has recently proved the result (conjectured by Gauss):

Theorem

$$h(d) \to \infty$$
$$as \quad -d \to \infty.$$

In proving his result Heilbronn makes use of a theorem of Hecke.[2] This note gives a slight variation of Heilbronn's argument which makes it unnecessary[3] to refer to Hecke's theorem.

2. With the notation in K, let[4]

$$L_0(s) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s},$$

$$L_1(s) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s},$$

$$L_2(s) = \sum_{n=1}^{\infty} \chi(n) \left(\frac{d}{n}\right) n^{-s}.$$

It is proved in K that for $\sigma > \frac{1}{2}$, $s \neq 1$, we have

$$L_0(s)\, L_2(s) = \zeta(2s) \left\{ \prod_{p/m} (1 - p^{-2s}) \right\} \left\{ \sum_a \chi(a) a^{-s} \right\} + o(1)$$

if $-\mathrm{d} \to \infty$ *and* $\mathrm{H} = \mathrm{h(d)}$ *is bounded.*

With $m = 1$ this becomes

$$(1) \quad \zeta(s)\, L_1(s) = \zeta(2s) \left(\sum_a a^{-s} \right) + o(1).$$

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[1] "On the class-number in imaginary quadratic fields", to be published shortly, hereafter referred to as K.

[2] In a paper by Landau, *Göttinger Nachrichten*, 1918.

[3] In a paper to appear in *Jour. Indian Math. Soc.* I have shown how to avoid using the theory of ideals. It follows from my two papers that we can avoid using Hecke's theorem as well as the theory of ideals in proving Heilbronn's theorem.

[4] $\chi(n)$ is a character (mod m).

Now let $^2/_3 < s \leqslant {^4/_5}$. Since $\zeta(s) < 0$ in this interval while the right hand side of (1) is obviously positive,[5] it follows that

(2) $L_1(s) < 0 \quad [^2/_3 < s \leqslant {^4/_5}]$.

It is well known that $L_1(1) > 0$, and hence it follows from (2) that $L_1(s) = 0$ for some s in $^4/_5 < s < 1$. Hence if $h(d) \to \infty$ is false for $-d \to \infty$, then we can find infinitely many $K > o$ such that

$$\sum_{n=1}^{\infty} \left(\frac{-K}{n}\right) n^{-s}$$

vanishes in the half-plane $\sigma > \frac{1}{2}$. But on the latter assumption Heilbronn proves without reference to Hecke that

$$h(d) \to \infty \text{ as } -d \to \infty,$$

and hence our theorem is proved.

[5] as $-d \to \infty$.