AULUCK'S GENERALIZATION OF THE SIMSON LINE PROPERTY.¹

BY S. CHOWLA.

(From the Government College, Lahore.)

Received June 4, 1937.

1. Let \( l_m = p_m - x \cos a_m - y \sin a_m = 0 \) (1 ≤ m ≤ 5) be the equations of 5 lines in a plane. The foot of the perpendicular from \((x, y)\) on \( l_m = 0 \) is the point \( x + l_m \cos a_m, y + l_m \sin a_m \). Hence the point \((x, y)\) lies on the conic which passes through the feet of the perpendiculars from \((x, y)\) on the five lines if the sixth order determinant

\[
\begin{vmatrix}
x^2 & xy & y^2 & x & y & 1 \\
(x + l_1 \cos a_1)^2 & \cdot & \cdot & \cdot & y + l_1 \sin a_1 & 1 \\
(x + l_5 \cos a_5)^2 & \cdot & \cdot & \cdot & y + l_5 \sin a_5 & 1
\end{vmatrix} = 0
\]

which immediately reduces to the 5th order determinant

\[
l_1 l_2 l_3 l_4 l_5 \times \begin{vmatrix} l_1 \cos^2 a_1 & l_1 \cos a_1 \sin a_1 & l_1 \sin^2 a_1 & \cos a_1 & \sin a_1 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
l_5 \cos^2 a_5 & l_5 \cos a_5 \sin a_5 & l_5 \sin^2 a_5 & \cos a_5 & \sin a_5
\end{vmatrix} = 0
\]

or, by addition of the 3rd column, \( x \) times the 4th column and \( y \) times the 5th column to the first column,

\[
l_1 l_2 l_3 l_4 l_5 \begin{vmatrix} p_1 & l_1 \cos a_1 \sin a_1 & l_1 \sin^2 a_1 & \cos a_1 & \sin a_1 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
p_5 & l_5 \cos a_5 \sin a_5 & l_5 \sin^2 a_5 & \cos a_5 & \sin a_5
\end{vmatrix} = 0,
\]

a seventh degree curve consisting of the 5 lines themselves and a conic. Let \( M_1 \) be the Miquel point of the lines \( l_m = 0 \) (2 ≤ m ≤ 5), etc. We show that this conic passes through \( M_1, \cdots, M_5 \) and hence the conic is a circle. Since \( M_1 \) is the focus of the parabola touching the lines \( l_m = 0 \) (2 ≤ m ≤ 5), the feet of the perpendiculars from \( M_1 \) lie on a straight line (directrix of the

¹ This note contains a slight change in the proof of Auluck's theorem I below (he omits the proof of the clause that the locus which is the subject of Theorem I, also consists of the 5 lines themselves). We also give two additional theorems (II and III).
parabola). Hence $M_1$ has the property that the 5 feet of the perpendiculars from it on the 5 lines $l_m = 0$ lie on a conic (which degenerates into a pair of straight lines) through it. Hence

**Theorem I.** Given 5 lines in a plane, the locus of a point with the property that the feet of the perpendiculars from this point on the 5 lines lie on a conic through the point itself, is a 7th degree curve consisting of the 5 lines themselves and the circle which passes through the 5 Miquel points of the lines taken 4 at a time.

2. From this and the sequence of Miquel's theorems we easily obtain

**Theorem II.** Given 6 lines in a plane, there are in general at most 13 points with the property that the feet of the perpendiculars on the 6 lines from any of these points lie on a conic passing through the point from which the perpendiculars are dropped. These 13 points are

(i) The Miquel Point associated with the 6 lines.

(ii) The points (12 in all) where each of the lines meets the Miquel circle associated with the other 5 lines.

3. The method used to prove Theorem I generalizes to prove (theorem I is the case $n = 2$).

**Theorem III.** The locus of a point which moves so that the feet of the perpendiculars from this point on to $\left(\frac{n^2 + 3n}{2}\right)$ hyper-planes (in space of $n$ dimensions) is a hyper-quadric passing through the point itself, is a hyper-surface of degree $\frac{n^2 + 3n}{2} + \frac{(n - 1)(n + 2)}{2}$ consisting of the $\frac{n^2 + 3n}{2}$ hyper-planes and a hyper-surface of degree $\frac{(n - 1)(n + 2)}{2}$. 