AULUCK'S GENERALIZATION OF THE SIMSON LINE PROPERTY.[1]

By S. Chowla.

(From the Government College, Lahore.)

Received June 4, 1937.

1. Let $l_m \equiv p_m - x \cos \alpha_m - y \sin \alpha_m = 0 \quad (1 \leqslant m \leqslant 5)$ be the equations of 5 lines in a plane. The foot of the perpendicular from (x, y) on $l_m = 0$ is the point $x + l_m \cos \alpha_m, y + l_m \sin \alpha_m$. Hence the point (x, y) lies on the conic which passes through the feet of the perpendiculars from (x, y) on the five lines if the sixth order determinant

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ (x + l_1 \cos \alpha_1)^2 & \cdot & \cdot & \cdot & y + l_1 \sin \alpha_1 & 1 \\ & \cdot & \cdot & \cdot & & \\ (x + l_5 \cos \alpha_5)^2 & \cdot & \cdot & \cdot & y + l_5 \sin \alpha_5 & 1 \end{vmatrix} = 0$$

which immediately reduces to the 5th order determinant

$$l_1 l_2 l_3 l_4 l_5 \times \begin{vmatrix} l_1 \cos^2 \alpha_1 & l_1 \cos \alpha_1 \sin \alpha_1 & l_1 \sin^2 \alpha_1 & \cos \alpha_1 & \sin \alpha_1 \\ & \cdot & \cdot & \cdot & \\ l_5 \cos^2 \alpha_5 & l_5 \cos \alpha_5 \sin \alpha_5 & l_5 \sin^2 \alpha_5 & \cos \alpha_5 & \sin \alpha_5 \end{vmatrix} = 0$$

or, by addition of the 3rd column, x times the 4th column and y times the 5th column to the first column,

$$l_1 l_2 l_3 l_4 l_5 \begin{vmatrix} p_1 & l_1 \cos \alpha_1 \sin \alpha_1 & l_1 \sin^2 \alpha_1 & \cos \alpha_1 & \sin \alpha_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_5 & l_5 \cos \alpha_5 \sin \alpha_5 & l_5 \sin^2 \alpha_5 & \cos \alpha_5 & \sin \alpha_5 \end{vmatrix} = 0,$$

a seventh degree curve consisting of the 5 lines themselves and a conic. Let M_1 be the Miquel point of the lines $l_m = 0$ $(2 \leqslant m \leqslant 5)$, etc. We show that this conic passes through $M_1, \cdots, M_5$ and hence the conic is a circle. Since M_1 is the focus of the parabola touching the lines $l_m = 0$ $(2 \leqslant m \leqslant 5)$, the feet of the perpendiculars from M_1 lie on a straight line (directrix of the

[1] This note contains a slight change in the proof of Auluck's theorem I below (he omits the proof of the clause that the locus which is the subject of Theorem I, also consists of the 5 lines themselves). We also give two additional theorems (II and III).

parabola). Hence M_1 has the property that the 5 feet of the perpendicualrs from it on the 5 lines $l_m = 0$ lie on a conic (which degenerates into a pair of straight lines) through it. Hence

THEOREM I. *Given 5 lines in a plane, the locus of a point with the property that the feet of the perpendiculars from this point on the 5 lines lie on a conic through the point itself, is a 7th degree curve consisting of the 5 lines themselves and the circle which passes through the 5 Miquel points of the lines taken 4 at a time.*

2. From this and the sequence of Miquel's theorems we easily obtain

THEOREM II. *Given 6 lines in a plane, there are in general at most 13 points with the property that the feet of the perpendiculars on the 6 lines from any of these points lie on a conic passing through the point from which the perpendiculars are dropped. These 13 points are*

(i) *The Miquel Point associated with the 6 lines.*

(ii) *The points (12 in all) where each of the lines meets the Miquel circle associated with the other 5 lines.*

3. The method used to prove Theorem I generalizes to prove (theorem I is the case $n = 2$).

THEOREM III. *The locus of a point which moves so that the feet of the perpendiculars from this point on to* $\left(\frac{n^2+3n}{2}\right)$ *hyper-planes (in space of n dimensions) is a hyper-quadric passing through the point itself, is a hyper-surface of degree* $\frac{n^2+3n}{2} + \frac{(n-1)(n+2)}{2}$ *consisting of the* $\frac{n^2+3n}{2}$ *hyper-planes and a hyper-surface of degree* $\frac{(n-1)(n+2)}{2}$.