HEILBRONN’S CLASS-NUMBER THEOREM.

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1. Let \( h(d) \) denote the number of non-equivalent forms of the type

\[
Q(X, Y) = aX^2 + bXY + cY^2, \quad (a, b, c) = 1, \quad a > 0
\]
of the discriminant

\[
d = b^2 - 4ac.
\]

Heilbronn\(^1\) has recently proved the beautiful result (which includes a classical conjecture of Gauss):

**Theorem.** If \( d \) runs over all negative fundamental discriminants, then

\[
h(d) \to \infty
\]

for \( d \to -\infty \).

2. I shall now introduce the problem discussed in this paper.

There are examples of negative discriminants (e.g., \( d = -232 \) and \( d = -1848 \)) with the following property: all non-equivalent reduced primitive forms of negative discriminant \( -d \) have their middle coefficients equal to zero.

In such cases, we shall say, briefly, that \( "d \) (or \(-d\)) has the property \( P \). It seems natural to inquire whether the number of negative discriminants \( d \) with the property \( P \) is finite or not. It will be proved here by the methods developed by Heilbronn in \( K \) that:

**Theorem 1.** There is only a finite number of negative discriminants \( d \), with \( \frac{d}{d} \) quadratfrei, having the property \( P \).

It is clear that if \( d \) has the property \( P \), then

\[
h(d) = H = 2^{t-1}
\]

where \( t \) is the number of different prime factors contained in \( \frac{d}{d} \).

In what follows we shall assume that \( d \) has the property \( P \), and then prove that this assumption leads to a contradiction for all large \(-d\).

\(^1\)“On the class-number in imaginary quadratic fields”, hereafter referred to as \( K \). The success of Heilbronn’s argument rests essentially on—

(i) a theorem of Hecke (see Landau, *Gött. Nachr.*, 1918);


I am indebted to Dr. Heilbronn for a copy of his manuscript.

\(^2\)The numbers 232 and 1848 are “idoneal”. See Dickson’s *History of the Theory of Numbers*, I, 361–65.
Notation.

The following notation is almost identical with that used in $K$.

All roman letters except $o$, $O$, $s$, $L$, $Q$ ($P$ and $K$) denote rational integers. $\chi(n)$ denotes a real character mod $m$ ($m > 0$) such that

$$L_{\chi}(s) = L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

vanishes\(^3\) for at least one $\rho$ in the halfplane $\sigma > \frac{1}{2}$; $m$, $\chi$ and $\rho$ are fixed throughout the paper;

(4) $\rho = \theta + i \phi$ ($\theta > \frac{1}{2}$, $\phi \equiv 0$).

We introduce the following Dirichlet series:

(5) $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$,

(6) $L_{\mu}(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$,

(7) $L_{1}(s) = \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) n^{-s}$,

(8) $L_{2}(s) = \sum_{n=1}^{\infty} \chi(n) \left( \frac{d}{n} \right) n^{-s}$,

where $\left( \frac{d}{n} \right)$ is the Kronecker symbol.

The constants implied in the signs $O$ and $o$ depend only on $m$, $s$, $\rho$, but they are independent of $d$, $Q$ and (here we differ from $K$) $H$.

Proof of Theorem 1.

Lemma 1: If $1 \leq l_{2} \leq m$ and if for $\sigma > 1$

$$\psi(s) = \psi(s, m, l_{1}, l_{2}, Q) = \sum_{Y=1}^{\infty} \sum_{X=-\infty}^{\infty} Q^{-s} (X, Y),$$

then

$$\psi(s)$$

is regular, and

(9) $\psi(s) = O (\left| d \right|^{\frac{1}{2} - \frac{1}{2}\sigma} + \left| d \right|^{-\frac{1}{2}\sigma})$

for $d \to -\infty$ if $\sigma > \frac{1}{2}$, $s \neq 1$.

Proof: This result is contained in the proof of lemma 9 of $K$.

\(^3\) If this assumption were not true for any $m > 0$ it would follow, from a theorem of Hecke (in a paper by Landau previously cited), that

$$h(d) = \frac{c \sqrt{-d}}{\log(-d)},$$

so that (3) would be false for sufficiently large $-d$. 

Lemma 2: For $\sigma > \frac{3}{10}$, $s = \frac{1}{2}$,

\[ L_0(s) L_2(s) = \zeta(2s) \prod_{p/m} (1 - p^{-2s}) \sum \chi(a) a^{-s} + O(H/d^{\frac{1}{2}-\frac{1}{2}\sigma}) + O(H/d^{-\frac{1}{2}\sigma}). \]

Proof: This follows from (9) above and the proof of lemma 10 in K.

Lemma 3: If $\sigma \geq \frac{1}{2}$,

\[ |\sum \chi(a) a^{-s}| \geq \frac{1}{2} H^{-2} + O(H/d^{-\frac{1}{2}\sigma}). \]

Proof: Since $d$ has the property P,

\[ d = -4ac, \quad a \in \frac{d}{k}, \quad 0 < a \leq \frac{1}{2} \left\lfloor \sqrt{d} \right\rfloor. \]

Hence

\[ \sum \chi(a) a^{-s} = \prod_{p/kd} (1 + \chi(p) p^{-s}) - \sum_{r > \frac{1}{2} \left\lfloor \sqrt{d} \right\rfloor} \chi(r) r^{-s}. \]

As in K it is easy to see that

\[ |\prod_{p/kd} (1 + \chi(p) p^{-s})| \geq \frac{1}{2} H^{-2} \]

Further, from (12),

\[ \sum_{r > \frac{1}{2} \left\lfloor \sqrt{d} \right\rfloor} \chi(r) r^{-s} = O \left(2^d \cdot d^{-\frac{1}{2}\sigma}\right). \]

(11) follows from (13), (14), (15) and (3).

Proof of Theorem 1: We put $s = \rho = \theta + i\phi$ in lemma 2 and we let $d$ tend to $-\infty$. Then we get

\[ O = \lim_{d \to -\infty} \zeta(2\rho) \prod_{p/m} (1 - p^{-2\rho}) \sum \chi(a) a^{-\rho} + O(H/d^{\frac{1}{2}-\frac{1}{2}\theta}) + O(H/d^{-\frac{1}{2}\theta}), \]

and from lemma 3,

\[ |\sum \chi(a) a^{-\rho}| \geq \frac{1}{2} H^{-2} + O(H/d^{-\frac{1}{2}\theta}). \]

since $\theta > \frac{1}{2}$ we see that, if

\[ \rho < |d| \epsilon \]

for every $\epsilon = \epsilon(\theta) > 0$, then (16) and (17) contradict each other for $-d > d_0(\theta)$. Hence there is an $\epsilon = \epsilon(\theta) > 0$ such that

\[ H \geq |d| \epsilon \quad \text{for} -d > d_0(\epsilon). \]

Now (19) contradicts (3) since $2^{\epsilon} < |d|^{\frac{1}{2}\epsilon}$ for all $-d > d_0(\epsilon)$. Hence (3) is false for large $-d$ and our theorem is proved.