Resistance without resistance: An anomaly

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The elementary two-terminal network consisting of a resistively (R) shunted inductance (L) in series with a capacitatively (C) shunted resistance (R) with $R = \sqrt{L/C}$, is known for its non-dispersive dissipative response, i.e. with the input impedance $Z_0(\omega) = R$, independent of the frequency (ω). In this communication, we examine the properties of a novel equivalent network derived iteratively from this two-terminal network by replacing everywhere the elemental resistive part R with the whole two-terminal network. This replacement suggests a recursion $Z_{n+1}(\omega) = f(Z_n(\omega))$, with the recursive function $f(z) = (i\omega Lz/i\omega L + z) + (z/1 + i\omega Cz)$. This recursive map has two fixed points – an unstable fixed point $Z_u^* = 0$, and a stable fixed point $Z_s^* = R$. Thus, resistances at the boundary terminating the infinitely iterated network can now be made arbitrarily small without changing the input impedance Z_∞ (= R). This, therefore, leads to realizing in the limit $n \to \infty$, an effectively dissipative network comprising essentially the non-dissipative reactive elements (L and C) only. Hence the oxymoron–resistance without resistance! This is best viewed as a classical anomaly akin to the one encountered in turbulence. Possible application as a formal decoherence device – the fake channel – is briefly discussed for its quantum analogue.

Keywords: Classical anomaly, dissipation, disorder, fake channels, fixed point, iteration, localization.

CONSIDER an elementary two-terminal LCR network shown in Figure 1. This series–parallel combination of the resistively (R) shunted inductance (L) in series with the capacitatively (C) shunted resistance (R) with, $R = \sqrt{L/C}$, has a dispersionless dissipative input impedance $Z_0(\omega) = R$, independent of the circular frequency (ω). This readily verifiable result is, of course, known, though not as commonly as one would have expected it to be. (The equivalence is detailed in that, e.g. the Nyquist–Johnson noise powers generated by the two shunt resistors (R) at temperature T, say, combine to give a noise output at the (1–2)-terminal equal to that for a single resistance R at temperature T.) The structure of this two-terminal network admits iteration generating an equivalent network as indicated in Figure 2, which is much familiar as a ladder network[1]. Consider such an iterated network, but now terminated arbitrarily at the boundary. With this, we can write the recursion relation

$$Z_{n+1} \equiv f(Z_n) = \frac{i\omega L Z_n}{i\omega L + Z_n} + \frac{Z_n}{1 + i\omega C Z_n}. \quad (1)$$

This recursion has two fixed points, $Z^* = f(Z^*)$, giving $Z^* = 0, R$. Linear stability analysis of these fixed points is readily done. A perturbation z_0 about the fixed point $Z^* = 0$, iterates away giving $|z_{n+1}| = 2|z_n|$, making $Z^* = 0$ an unstable fixed point Z_u^* (= 0) Next, consider the fixed point $Z^* = R$. A perturbation z_0 about R, iterates as

$$z_{n+1} = \frac{1 - \omega^2 LC}{(1 + i\omega\sqrt{LC})^2} z_n,$$

giving

$$\left|\frac{z_{n+1}}{z_n}\right|^2 = \left|\frac{1 - \omega^2 LC}{1 + \omega^2 LC}\right|^2 \le 1. \quad (2)$$

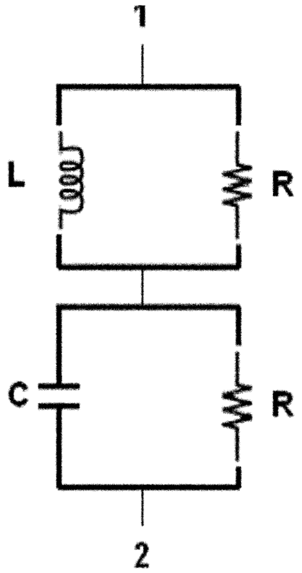

Figure 1. Dispersionless two-terminal LCR network with $R = \sqrt{L/C}$.

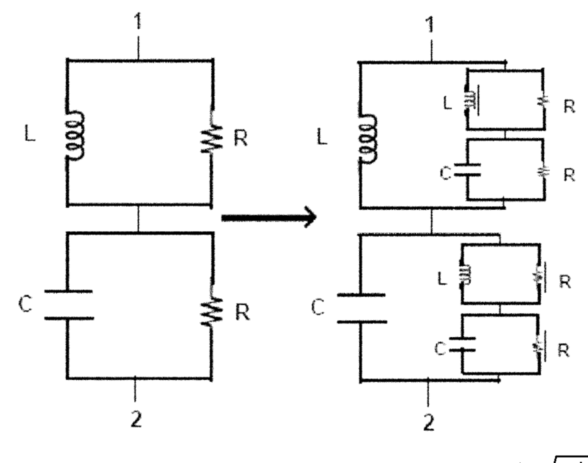

Figure 2. Iteration of the two-terminal network with $R(= \sqrt{L/C})$ replaced recursively by the whole two-terminal network. Shown here is one stage of iteration.

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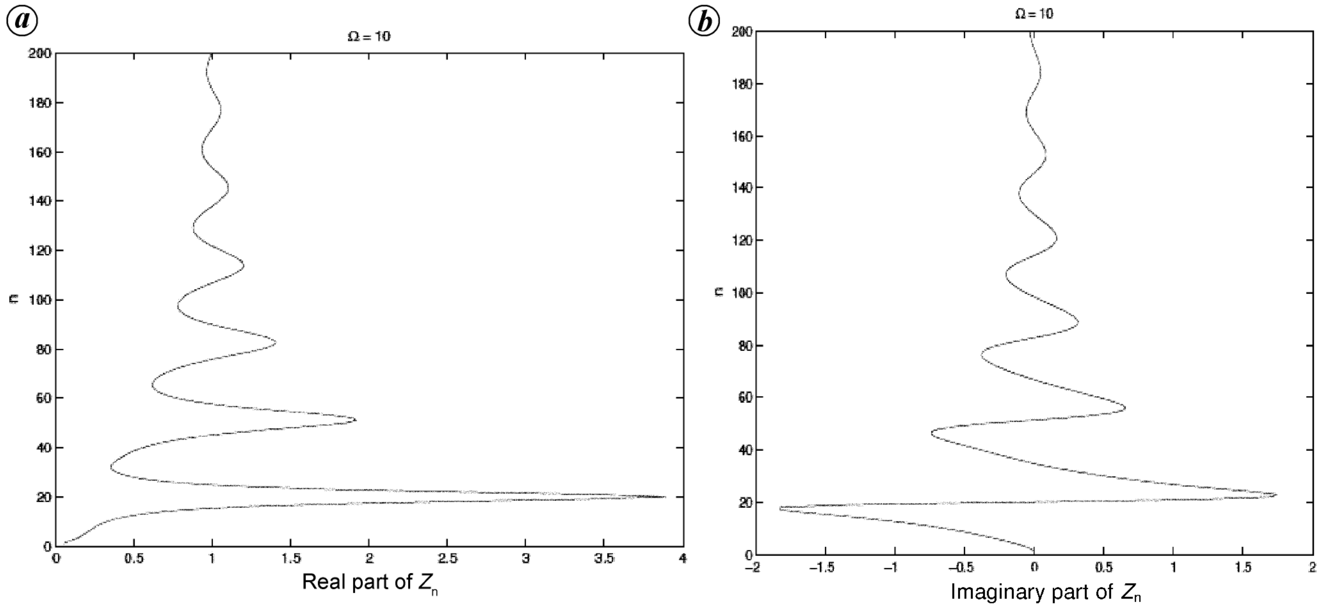

Figure 3. ***a***, Iteration of the two-terminal network impedance (Re $Z_n(\omega)$) initialized at $Z_0 = 0.05 + i0$. Note the fast convergence to the stable fixed point $Z_s^* = 1$. Here $R = L = C = 1$, and $\omega = 10$. ***b***, Iteration of the two-terminal network impedance (Im $Z_n(\omega)$) initialized at $Z_0 = 0.05 + i0$. Note the fast convergence to the stable fixed point $Z_s^* = 1$. Here $R = L = C = 1$ and $\omega = 10$.

This makes the fixed point Z_s^* (= R) stable. The implication of this fixed-point analysis is now straightforward. Terminating the network at the boundary with $z_0 = r_0 + ix_0$, where r_0 can be made arbitrarily small (but non-zero positive), the impedance will iterate away to the stable fixed point $Z_s^* = R$ as $n \to \infty$. This is, however, so assuming that there are no other attractors. We have, therefore, carried out the recursion in eq. (1) numerically with different initializations, and a typical evolution is shown in Figure 3 *a* and *b*.

Again, note the fast recursive convergence to the fixed point $Z_u^*/R \to 1$. This is the all important point – an arbitrarily small resistive termination at the boundary generates a finite resistance $R = \sqrt{L/C}$ in the limit $n \to \infty$. And this result suffices for our purpose. (Inasmuch as the recursion holds for all values of the frequency ω, other attractors, if any, e.g. a period-doubling (two-cycle) attractor, would generate infinitely many isospectral networks. Such attractors, or indeed a strange attractor, should be interesting for network synthesis.) The physical picture, of course, is just this[1]. The energy fed at the input terminal into the infinitely iterated network appears to be absorbed effectively resistively at the input terminal. But, in fact, it is really not dissipated there instantaneously and locally – it is cascaded away to the distant boundary where it is ultimately dissipated. In a steady state a.c. response, for instance, much energy remains stored in the reactive elements. This is strongly reminiscent of what happens in fluid turbulence. There too, energy fed at the large-scale eddy (integral regime) is cascaded away progressively to smaller-scale eddies (inertial regime), and is ultimately dissipated at the distant smallest (Kolmogorov) scale – of viscosity. Indeed, the dissipation rate becomes independent of the viscosity in the limit of vanishingly small viscosity! This is a classic example of the classical dissipative anomaly[2] in the technical sense of the term – the time-reversible symmetry remains broken even as the symmetry-breaking parameter (the viscosity) tends to zero, giving dissipation without dissipation!

The conceptual similarity to our network is obvious (and not a little because of the inward-bound nature of our iterated network that makes the drawing in Figure 2 increasingly more difficult beyond even the second stage of iteration). It will be apt at this stage to note that yet another example of such an anomaly is encountered in the context of infinite-ladder networks of inductors and capacitors[3,4], where finite dissipation arises as a result of a subtle limiting procedure well known in the context of phase transitions. The networks are, however, essentially low-pass filters and the input impedance is not dispersionless. We may note in passing that the iterated network is hierarchical in its geometry.

Our analysis of the iterated network has implications for dissipative quantum mechanics. It is known that there is no simple way of introducing dissipation phenomenologically into a Hamiltonian quantum system without inconsistencies[5]. A way out in the context of quantum transport has been to introduce fake channels[5,6], such as transmission lines that outcouple part of the wave amplitude causing the so-called stochastic attenuation. Our infinitely iterated network is essentially a *lumped*-element transmission line where the reactive elements can be considered as part of the Hamiltonian system, and dissipation enters only through the anomaly discussed above. A quantum version of our iterated network is the Cayley tree composed of one-dimensional scatterer as introduced by Shapiro[7] in the context of quantum conduction in parallel resistors using splitters.

An interesting feature of our network is its invariance with respect to a certain correlated disorder, namely that the condition $R = \sqrt{L/C}$ (fixed) allows us to vary L and C for a given R at random with the strong correlation, without leading to Anderson wave-localization[8,9] that would have blocked energy cascading. This is a case of purely gauge disorder.

In conclusion, we have analysed a two-terminal *LCR* network which is dispersionless and admits hierarchical iteration. When infinitely iterated, it gives an essentially reactive (L and C) network and yet provides dissipation – through an anomaly. Possible application to dissipative quantum systems is pointed out. The network admits correlated disorder without localization.

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