Bose–Einstein condensation: Where many become one *and, therefore, there is plenty of room at the bottom*

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Abstract. Bose–Einstein condensation and superfluidity are well known to occur in the dilute gaseous as well as in the dense liquid state of matter having a fixed number of Bose particles. Very recently, experimental evidence has been obtained for the probable realization of BEC and superfluidity in ^{4}He in the solid state too, as revealed through its non-classical rotational moment of inertia – smaller than that for the solid. Such a solid that can also subtend a flow as superfluid – and hence a supersolid – is indeed a surprise of condensed matter physics. In this conversation, an order-parameter description for the supersolid state will be given in which the superfluid flow is decoupled from the crystalline density modulation which remains at rest in the laboratory frame, thus giving it a non-classical inertia.

Keywords. Bose–Einstein condensation; Bose–Einstein statistics; indistinguishability; Bosonic stimulation; condensate; quantized vortex; de Boer quantum parameter; non-classical rotational inertia; supersolid.

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1. Introduction

Classically identical particles become quantum mechanically indistinguishable. Satyendra Nath Bose taught us, in 1924, how to correctly count the indistinguishables – the distinct microstates corresponding to a given macroscopic state – and for the gas of light quanta, whose number is not conserved, e.g., can change with temperature, he gave a proper derivation of Planck's law of black-body radiation. Einstein, in 1925, generalized the Bose statistics for a quantum gas of particles whose number is conserved (e.g., ^{4}He), and found that below a critical temperature (~ 1 K), a macroscopic and extensive number of particles must accumulate at the bottom, i.e., in the lowest one-particle state – the Bose–Einstein condensation (BEC). Initially thought to be a pathology of the ideal Bose gas, the BEC turned out to be robust against interactions, and to underlie superfluidity, though the exact connection is still unclear. Seventy years later, BEC came alive with the breakthrough in 1995 when near-ideal BEC was created in dilute alkali gases cooled down to nanokelvins in a trap. BEC has since become an ideal laboratory

for basic and condensed matter physics. Very recently [1], BEC seems to have been realized in ^{4}He in the solid state too that nevertheless sustains a superfluid flow – a supersolid – which is a surprise of condensed matter physics!

All earlier attempts at detecting the supersolid state had failed [2]. Thus, BECs (and superfluidity) are now possible in all the three states of matter – gas, liquid and the solid. In this talk, I will focus on the supersolid, and present an order-parameter description for the supersolid state, and derive its non-classical inertia. A timeline of Bose–Einstein condensation is, however, appended at the end to give a sense of the progress of BEC research over the past 80 years or so.

2. Supersolid: A surprise of condensed matter physics

- What is it?

  Geometrically it is an X-ray solid that can also flow as a superfluid while the solid-like density modulation remains at rest in the laboratory frame (or in the frame of the container). Thermodynamically, however, it is a pure Gibbs phase, and not a two-phase state. By an X-ray solid we mean operationally that it has the sharp Bragg diffraction peaks that reveal its crystalline order, i.e., periodicity of the matter density.

- Can a solid be a superfluid [3]?

  Yes, a quantum solid can, for large enough de Boer quantum parameter: $\Lambda = \hbar/\sigma(2m\varepsilon)^{1/2} \sim 1$, i.e., because of its large zero-point atomic amplitude of delocalization comparable to the inter-atomic separation. For ^{4}He (bcc), $\Lambda = 0.28$. In comparison, for Xe (a heavier rare gas crystal) $\Lambda = 0.0065$. The parameter Λ is essentially the ratio of the de Broglie wavelength associated with the zero-point motion to the minimum inter-atomic separation in the solid. It is thus also a measure of the zero-point energy relative to the binding energy of the solid. Here ε and σ are the depth and the width parameters of the Lennard–Jones 12-6 inter-atomic potential. Large lambda means a quantum (non-classical) and highly anharmonic solid.

- Has it been observed in the lab?

  Most probably, Yes! In helium-4 [1], some 30 years after it was predicted by Leggett [3].

- How to detect a *supersolid*?

  Through its diminished (non-classical) inertia as the superfluid flow is decoupled from the solid-like modulation which remains at rest relative to (fixed to) the moving container.

- Do we understand it?

  At a phenomenological level, perhaps, Yes! And in the following, I will propose that, in terms of its order-parameter, a supersolid $\sim$ *square-root* of a quantum solid [4]! But, let me first briefly refer to some of the attempts at understanding the supersolid at a microscopic level [5–10]. It follows from a general go/no-go theorem due to Penrose and Onsager [5] that superfluidity is not possible in a classical crystal in which a particle is necessarily

confined (localized) to a particular cell – superfluidity can occur only in a system of bosons in which the wave function has no zeroes (nodes) anywhere, except at the boundaries. Thus, clearly it does not rule out superfluidity in a quantum solid (crystal) such as ^{4}He (one with a large de Boer quantum parameter) wherein a particle is de-localized over more than one cells. It was then shown by Chester [6] that Bose condensate can in fact occur in a state with crystalline order. His argument was based on the observation that the probability density associated with a correlated bosonic many-body wave function (of the Jastrow type) is formally a classical Gibbs distribution which is known to show a liquid–solid transition, while the same form is also known to give the Bose condensation, as shown by Reatto and Chester [7].

Conclusion – Crystalline order and superfluidity are not mutually exclusive. The models studied, however, suggested crystals with defects – the number of sites exceeding the number of atoms – and it was concluded that Bose–Einstein condensation was possible in crystals with a finite concentration of vacancies. It was thus essentially the vacancies that are Bose-condensed! This was precisely the problem that was treated in a seminal paper by Andreev and Lifshitz [8], who derived the possibility of zero-point defects (defectons) in solid ^{4}He that could move coherently as a superfluid. Leggett [3] had discussed the question of superfluidity as distinct from Bose condensation in a crystal. The wave function considered by him had a phase ascribed to the motion of the particle that led to rotational properties similar to those of a conventional superfluid. This would, of course, require a crystal with defects inasmuch as exchange of lattice sites by particles does not carry any momentum [9]. Recent variational Monte Carlo calculations of Galli and Reatto using many-body trial wave functions (in which correlations are build in through the technique of shadow wave functions) support the idea of BEC obtaining in a solid with delocalized vacancies [10]. They make several predictions – among them, no BEC in a perfect crystal!

Let us get acquainted with helium-4 (^{4}He) [11] before we turn to the order-parameter description of supersolid. It is known to be a noble inert gas; but

- It is also a Nobel active gas (count the Nobel Prizes given away!).
- Low-temperature physics since 1908 has been dominated by the helio-centric research.
- ^{4}He atom is a scalar (zero-spin) Bose particle (with 2 protons + 2 neutrons + 2 electrons adding up to zero-spin). Unlike the fermionic ^{3}He, the bosonic helium-4 undergoes Bose–Einstein condensation at 2.17 K.
- It is a highly quantum fluid: remains liquid down to 0 K under its saturation vapour pressure. Solidifies at 25 atm at 0 K. For ^{4}He, the de Boer quantum parameter $\Lambda = \hbar/\sigma(2m\varepsilon)^{1/2} \sim 0.28$, as noted earlier.

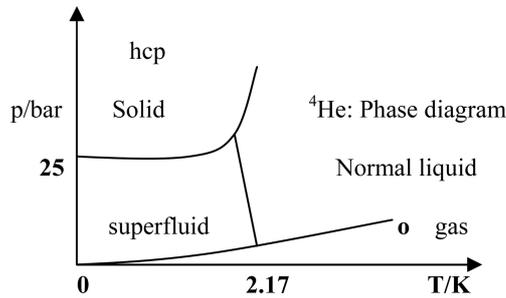The well-known phase diagram of ^{4}He is shown in figure 1 for ready qualitative reference.
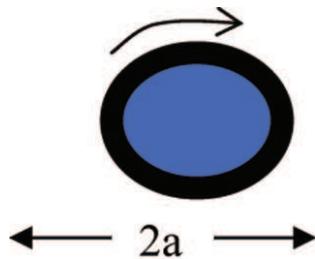
Figure 1.

Figure 2.

Onsager–Feynman quantization of circulation in a superfluid and the non-classical moment of inertia [12]

In a superfluid, the circulation is quantized as $\int v(r) \cdot \mathrm{d}\ell = (h/m)m, n = $ integer, $v = (\hbar/m)$ grad ϕ, where ϕ is the phase of the complex order-parameter. Thus, with superfluid helium in a rotating cylinder (see figure 2) for $\Omega < \hbar/(4\pi ma^2)$, the helium will stay at rest in the lab frame, and hence will not contribute to the moment of inertia of the total system. For such a simply connected topology and at higher angular velocities, quantized vortices will enter in the otherwise irrotational fluid flow, and the moment of inertia will increase towards the value for solid-body rotation. Thus, for liquid helium contained in a cylinder of inner diameter ~ 10 cm, held vertically, the Earth's rotations will introduce a few (~ 5) vortices at high enough latitudes.

Similar phenomenon will also occur in multiply connected topology, e.g., helium contained in the annular space between two co-axial cylinders. Here the flow remains irrotational (potential) but the circulation around the annulus is quantized as above.

3. Detection of non-classical moment of inertia of supersolid ^{4}He [1]

Here, high-Q torsional oscillator (of spring constant G) with ^{4}He cooled and solidified under pressure in the annulus of a co-axial cylindrical bob was used. A drop in

the resonant period $(2\pi(I/G)^{1/2})$ indicated a reduction of the moment of inertia (I) at low enough temperatures (below $\sim$175 mK), and hence a decoupling of superflow from the solid modulation that co-moves with the container – the supersolid phase. The superfluid fraction has been estimated [1] to be 5–25 parts in 10^3. Actually, in these experiments, helium-4 was frozen in nanometric pores of a Vycor glass filling the annular space, rather than as bulk crystal.

4. Order-parameter description for supersolid [4]

Crystalline order means a periodic (density-wave) modulation of matter. It implies a diagonal long-range order. Can it simultaneously exhibit an off-diagonal long-range order too? That is the question. Now, given the periodicity, the particle number density $n(\mathbf{x})$ may be Fourier-series analysed in the reciprocal lattice vector $\mathbf{g}$ as

$$n(\mathbf{x}) = n_0 + \sum_{\mathbf{g} > 0} n_{\mathbf{g}} \cos \mathbf{g} \cdot x \,, \tag{1}$$

where the reciprocal lattice vectors $\mathbf{g}$ correspond to the crystal lattice anticipated. The density-wave amplitudes $\eta_{\mathbf{g}}(=n_{\mathbf{g}}/n_0)$ and the order-parameters for the crystal, are to be chosen eventually so as to minimize the Gibbs free energy, in principle over all possible lattices. A great simplification, however, results from the following physical considerations. For a classical solid, the atoms are of course well-localized at the respective lattice sites, and strictly so at the absolute zero of temperature. This, therefore, would require summing over many (in principle infinitely many) reciprocal lattice vectors in eq. (1). However, as the solid is approached from the liquid state at the liquid–solid first-order phase transition, there is appreciable thermal spread of the atomic positions about the nominal direct lattice sites, and so one needs to sum over a relatively much smaller set of the reciprocal lattice vectors. Thus, in the conventional mean-field theories of the liquid–solid transition, only the dominant reciprocal lattice vectors are chosen, namely those that correspond in magnitude to the peak in the static structure factor of the liquid, i.e., with $|\mathbf{g}| = 2\pi/$(mean inter-particle distance) [13]. The peak in the static structure factor anticipates the instability of the liquid towards the crystalline state. Now, a quantum system such as ^{4}He remains liquid down to the absolute zero of temperature, and solidifies then only at a sufficiently high pressure that offsets the zero-point quantum pressure. Here too then a similar situation is obtained at the liquid–solid transition (at the melting line) in which the role of temperature is now effectively played by the zero-point fluctuations as far as the atomic positional spread is concerned. At $T = 0$ and the pressure (p) just above the melting line, the atomic positions are delocalized about the nominal lattice sites over distances of the order of the lattice spacing. Thus, we can again approximate the density-wave expansion in eq. (1) by retaining only the dominant reciprocal lattice vectors $\mathbf{g}$, with $|\mathbf{g}| = 2\pi/$(mean inter-particle distance) that correspond to the peak in the liquid-state static structure factor. (For the superfluid helium, this peak also corresponds to the roton minimum.) Thus, the zero-point atomic delocalization makes

plausible at once the dominant reciprocal lattice-vector approximation as well as the superfluid solid. This makes our phenomenology tractable so as to clearly bring out the essential points of our order-parameter description of the supersolid, as we approach the melting line from the liquid side. The zero-point delocalization of the atoms almost re-approximates the solid helium as liquid helium across the melting line, and makes the transition effectively a weakly first-order transition. Indeed, the near equality of, e.g., the refractive indices makes it difficult to distinguish between the solid and the liquid helium.

Now we come to the idea which is central to this phenomenology, namely, that underlying the real classical-looking matter density wave $n(\mathbf{x})$ in eq. (1), directly observable by X-ray diffraction, there is the coherent complex order-parameter $\psi(\mathbf{x})$ that subsumes its quantum features: the zero-point kinetic energy and the superflow (the particle current density $\mathbf{j}(\mathbf{x})$) associated with the gradient of ψ. Thus we have

$$\psi(\mathbf{x}) = \sqrt{n(\mathbf{x})}\exp(i\phi(\mathbf{x})), \tag{2}$$

$$\mathbf{j}(\mathbf{x}) = \frac{\hbar}{m}n(\mathbf{x})\nabla\phi, \tag{3}$$

and

$$\nabla\phi = \left(\frac{m}{\hbar}\right)\left(\frac{\mathbf{j}(\mathbf{x})}{n_0 + \sum_{\mathbf{g}} n_{\mathbf{g}}\cos\mathbf{g}\cdot\mathbf{x}}\right), \tag{4}$$

where $\mathbf{j}(\mathbf{x})$ is the particle current density assumed to be uniform $\mathbf{j}_0$.

The total kinetic (quantum) energy E_{Q} associated with the gradient can be written as

$$E_{\mathrm{Q}} = \frac{\hbar^2}{2m}\int |\nabla\psi|^2(\mathrm{d}\mathbf{x}) \equiv E_{\text{zero-point}} + E_{\text{flow}}, \tag{5}$$

where E_{flow} involves the current density $\mathbf{j}_0$ quadratically, and $E_{\text{zero-point}}$ is the zero-point energy. This kinetic energy (of purely quantum origin) must be added to the usual Gibbs free energy (energy at zero temperature, without the entropy term) associated with the density waves as given in eq. (1). This quantum-corrected Gibbs energy has to be then minimized with respect to $\eta_{\mathbf{g}}$. This summarizes our mean-field order-parameter phenomenology.

So far it is general, and one has to now specialize to the case of helium-4. We approach the melting line from the liquid side along $T = 0$ as the helium is solidified under pressure in the close-packed lattice (hcp) structure. The reciprocal lattice being a hexagonal one, it requires considering at least six shortest reciprocal lattice vectors in eq. (1). This considerably complicates the algebra. We will, however, illustrate the essential points of and the results from our order-parameter description for the algebraically simplest case of a one-dimensional geometry, where the set of the reciprocal lattice vectors contains just one value of $\mathbf{g}$ ($\sim 2\pi/a$, with a the lattice constant). Of course, there is no phase transition, or superfluidity, in one dimension. Thus, integrating eq. (4), we at once obtain for the phase

$$\phi(\mathrm{x}) = \left(\frac{2m\mathrm{j}_0}{\hbar g n_0}\right) \ \frac{1}{\sqrt{1 - \eta_\mathrm{g}^2}} \mathrm{arctg} \left(\frac{\sqrt{1 - \eta_\mathrm{g}^2} tg(gx/2)}{1 + \eta_\mathrm{g}}\right). \tag{6}$$

Similarly, from eq. (6), we obtain for the kinetic energy E_Q (in the lab frame in which the density-wave modulation is at rest)

$$E_\mathrm{Q} = \frac{\pi^2 \hbar^2}{2ma^2} N(1 - \sqrt{1 - \eta_\mathrm{g}^2}) + \frac{N\mathrm{j}_0^2 m}{2n_0^2} \ \frac{1}{\sqrt{1 - \eta_\mathrm{g}^2}}$$

$$= E_\text{zero-point} + E_\text{flow}. \tag{7}$$

Re-writing $E_\text{flow} = P^2/(2M\sqrt{1 - \eta_\mathrm{g}^2})$, with $P = $ momentum of the flow $= mL\mathrm{j}_0$, and $M = mn_0L = $ total mass, we obtain

$$M_\text{flow} = M\sqrt{1 - \eta_\mathrm{g}^2}. \tag{8a}$$

This is the mass of the superfluid flow decoupled from the solid-like density modulation, the latter being at rest in the lab frame. Thus, the inertia $M_\text{non-classical}$ of this supersolid is given by

$$M_\text{non-classical} = M - M_\text{flow} = M(1 - \sqrt{1 - \eta_\mathrm{g}^2}). \tag{8b}$$

This is the mass that would co-move with the container (i.e., be at rest with respect to it) while the superfluid flow is decoupled from it. Clearly, the non-classical inertia increases monotonically with increasing density modulation η_g, and vanishes as η_g tends to zero. This is as expected physically.

The above calculation can be readily generalized to the three-dimensional case. Thus, for a simple cubic lattice, we have

$$\frac{M_\text{nc}}{M} = 1 - \frac{1}{\int_0^\infty \mathrm{e}^{-t} I_0^3(2\eta t)\mathrm{d}t}. \tag{8c}$$

Now we turn briefly to the phase transition itself. The order-parameter $\eta_\mathbf{g}$ occurring in the expression (8b) for the non-classical inertia has to be determined, of course, by minimizing the Gibbs free energy. This will require the explicit order-parameter ($\eta_\mathbf{g}$) expansion of the free energy for a specific choice of the lattice at the liquid–solid transition, which is a first-order transition, as indeed all phase transitions at a finite wave vector from the isotropic liquid state to the crystalline solid state necessarily must be (the so-called Brazovskii effect). We can, however, draw certain definite conclusions from the general structure of these order-parameter expansions. Here it is known that the free energy expansion should have either (1) a term cubic in $\eta_\mathbf{g}$, along with a positive quartic term and a quadratic term which is the tunable parameter, or (2) a negative quartic term along with a positive sextic term as well as a quadratic term which is again the tunable parameter. In the case of helium-4, the hexagonal reciprocal lattice symmetry allows a cubic term in the free-energy expansion as it is possible then to find three shortest reciprocal lattice

vectors that satisfy the triangular closure condition [13]. Now the important point relevant to the supersolid is that to the classical free-energy expansion, we must add the quantum energy term E_Q as in eq. (7). The latter too must be expanded in powers of $\eta_{\mathbf{g}}$. Thus, we have

$$E_Q = u_{2Q}\eta_g^2 + u_{4Q}\eta_g^4 + u_{6Q}\eta_g^6 + \cdots. \tag{9}$$

For the simple 1D case we already have

$$u_{2Q} = \frac{\pi^2 \hbar^2}{4ma^2},$$
$$u_{4Q} = \frac{1}{16}\frac{\pi^2 \hbar^2}{ma^2} + \frac{3m j_0^2}{16 n_0^2},$$
$$u_{6Q} = \frac{\pi^2 \hbar^2}{32ma^2} + \frac{5m j_0^2}{32 n_0^2}. \tag{10}$$

From eq. (5), for the 3D case clearly only the numerical coefficients in the above expressions will be somewhat different. This, however, does not change the discussion qualitatively. The positive quantum contribution $u_{2Q}\eta_g^2$ to the quadratic term in the quantum-corrected free energy clearly suggests that one must apply an external pressure (introduce a PV term in the Gibbs free energy) to offset this quantum pressure in order to reach the liquid–solid transition. This is consistent with the fact that ^{4}He remains liquid down to $T = 0$ under its vapour pressure, which of course vanishes at $T = 0$, and solidifies only under pressure of about 25 atm (see figure 1). Another significant point that emerges is that the quantum-corrected free energy also involves a positive quadratic term containing the particle current density $\mathbf{j}_0$ as in eq. (10). This makes it possible to tune the liquid–solid transition by varying the flow too, as well as by varying the external pressure. This is a general prediction following from our order-parameter description for the supersolid.

Some general remarks are now in order. First, the order-parameter η_g, to be determined from the minimization of the free energy, depends on the flow j_0, and enters the non-classical inertial mass as in eq. (8b). Inasmuch as for the potential flow in an annular geometry (1D flow with periodic boundary condition), the circulation is quantized to $\nu h/m$, with ν an integer (the Onsager–Feynman quantization), this inertia will change when the rim velocity of the containing cylinder corresponds to $v = 1$ as indeed observed experimentally [1]. Secondly, we have considered here only a potential (irrotational) flow and derived the translational non-classical inertia associated with the superflow ($\mathbf{j}_0$) relative to the density modulation, at rest in the laboratory frame. Experimentally, however, it is obviously convenient to have a bounded (confined) motion, which is readily realized in rotation – hence the (non-classical) rotational inertia measured usually in experiments by confining the ^{4}He-liquid (solidified under pressure) in an annulus which is then made to oscillate about its axis in a torsional mode. It is, however, to be noted that for an annular geometry, the motion can be irrotational inasmuch as the annular region is not simply connected. Of course, we can have a situation where the motion has local circulation distributed in the form of vortices for a simply connected system – ^{4}He after all is a type II superfluid! In any case, fundamentally the

translational inertia is well-defined, calculable, and turns out to be non-classical as derived above.

It is apt to recall here that a key point in the microscopic theory of superfluidity and supersolidity in the strongly interacting [4]He is the role of the condensate, namely the hybridization of the single-particle excitation and the collective mode caused by the condensate [14,15]. This involves the physics of rotons. Rotons have been invoked recently in the context of supersolids [16]. In the present case, the complex-modulated order-parameter underlying the mass density wave is to be viewed as a macroscopically occupied single-particle state – the eigenstate of the one-particle reduced density matrix. This too should subtend interesting hybridization effects.

The problem of supersolids in rotation does raise certain general questions of interest about the partitioning of a given amount of angular momentum among the different possible modes of motion (degrees of freedom) so as to minimize the free energy. Thus we note, that the angular momentum may be shared between the orbital and the spin motions for a system such as [3]He, assuming unpaired spins, giving a rotation-induced spin polarization. In the context of rotating superfluid [4]He in a simply connected region, it may be taken up by the quantized vortices in the otherwise irrotational flow. In a multiply connected (e.g., annular) region, however, the angular momentum may also be taken up by the translational flow as possibly in the present case [1]. For a supersolid spinning about an axis, the angular momentum may be viewed as being taken up by delocalized point defects (the defectons where the number of lattice sites exceeds the number of [4]He atoms [8]. These defectons may have local ring-like exchange motions. For a supersolid confined to a thin rotating annulus, however, the quantum defects may carry the angular momentum by translating coherently around the annulus. We believe that atomistically this may be the case. The present single order-parameter based phenomenological theory, however, cannot address these atomistic details.

In conclusion, we have described the supersolid in terms of a complex order-parameter underlying the real mass–density wave characteristic of a crystalline quantum solid. This simple model gives a non-classical inertia smaller than the literal mass. It also gives the possibility of a flow-tunable supersolid-to-superfluid transition. Of course, for a strongly interacting Bose system such as [4]He the above phenomenology is a drastic approximation. Finally, the supersolid regarded as a non-uniform (modulated) density $n(\mathbf{x})$ results from the fact that underlying the classical looking density modulation there is the quantum complex order-parameter $\psi(\mathbf{x})$ with $|\psi|^2 = n(\mathbf{x})$. Thus, a quantum supersolid is in a sense the square-root of a classical solid!

Annexure

Timeline: Bose–Einstein condensation (BEC)

[1900]   Planck's quantum hypothesis; Planck's w of black body radiation ('... happy guesswork...').

[1924]   Planck's law and light quantum hypothesis; Satyendra Nath Bose, *Zeit. f. Phys.* **26**, 178 (1924). The pre-factor $2 \times (4\pi\nu^2/c^3)$ also derived.

	Beginning of quantum statistics: Loss of identity of light quanta. A new way of counting the indistinguishables.
	Photons indistinguishable but the phase space cells distinct.
	Photon number not conserved: Chemical potential $\mu = 0$.
[1925]	Quantum theory of the monatomic ideal gas, A Einstein, Prussian Academy of Sciences (1925), p. 3.
	Extension of the Bose statistics: Particle number conserved – Chemical potential $\mu \neq 0$, e.g., helium (^{4}He).
	Gibbs paradox (the tyranny of $N!$) resolved, and the Nernst Theorem obeyed.
	Startling consequences: Macroscopic occupation of the lowest single-particle state – Bose–Einstein condensation (BEC).
	Purely quantum statistical phase transition *sans* interaction.
	Initial reaction to BEC:
	Einstein ...'... that is only by the way...',
	Planck ... frankly disbelieved it,
	Schrödinger ... suspected an error in it.
[1926]	P A M Dirac gave antisymmetric wave function for fermions (^{3}He) obeying Pauli's exclusion principle with the occupation numbers restricted to 0 and 1 (the exclusive Fermi–Dirac statistics); and symmetric wave function for the bosons (^{4}He) obeying inclusive statistics with occupation numbers not restricted (the inclusive Bose–Einstein statistics). Matters of statistics were clarified by 1927.
[1928]	W Hendrik Keesom: He I (normal helium)–He II (superfluid helium) phase transition – the λ-transition at a critical temperature $T_c = 2.18$ K.
[1938]	Pyotr Kapitza: Discovers superfluidity of helium (^{4}He). (Earlier in 1911 Heike Kamerlingh Onnes had discovered superconductivity in Sn.)
	Fritz London hypothesis: Superfluidity of ^{4}He a manifestation of BEC. A macroscopic wave function proposed for this phase. Now called the order-parameter.
[1940]	W Pauli derived spin-statistics connection from special relativity and quantum mechanics: Bosons for integer spins and fermions for half-integer spins.
[1948]	N N Bogoliubov: First microscopic theory of interacting Bose gas: ^{4}He – Superfluidity and BEC connected. Depletion of BEC due to strong interactions in liquid ^{4}He.
[1956]	O Penrose and Lars Onsager: First estimation of BEC fraction $\sim 10\%$ for ^{4}He.
[1957]	Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity: Condensation of Bose-like Cooper pairs in the zero momentum state.
[1966]	Seminal suggestion of P Hohenberg and P Platzman initiates probing of the condensate fraction by high-energy (epithermal) neutron scattering–momentum distribution (Compton profile). But conflicting results for BEC fraction.
1972 millikelvin	Condensation of bosonic pairs of fermionic ^{3}He.

1980s millikelvin	Advances in laser cooling and trapping of neutral alkali atoms down to microkelvins; Steven Chu and William D Phillips; and Claude Cohen-Tannoudji.

BEC SAGA: 70 years after 1925 and end of helio-centricity

1995 ∼nanokelvin	*BEC resurrected miraculously.* Eric A Cornell (NIST), Wolfgang Ketterle (MIT) and Carl E Wieman (JILA +Univ. Colorado) obtain BEC in dilute gases of ^{87}Rb alkali atoms at ∼20 nK (0.00000002 K) and ^{23}Na. BEC fraction ∼100%, the ideal value.
∼1999–	New state of matter: *Tunable condensate* Coherent matter waves – atom laser Bosonic stimulation Non-linear matter–wave interaction: 4WM Quantum phase transition: BEC in optical lattice Interaction tunable through Feshbach resonance
∼2003–	Fermionic atom pairs (composite bosons): ^{40}K$_2$, ^{6}Li molecular condensates. Close encounters: Cold collisions for scattering length ≪ de Broglie wavelength.
∼2004–	BEC (real-space pairs)-to-BCS (momentum-space pairs) cross-over in fermionic systems. Molecular BEC: Chemistry with cold coherent matter; photo-association of atoms into molecules. Highest spatial and spectral resolutions; sensitive detectors (possibly for gravitational waves?). BEC on a microchip. BEC: A 'laboratory' for testing condensed matter. Models of strongly interacting systems, e.g., Mott insulator to superfluid transition.

Given below is the Zoo of BECs:

^{4}He (the inert noble, but Nobel-active gas)
$H \downarrow$ (spin-polarized hydrogen), an example of effectively spin-half Bose gas
Excitonic condensates
Composite bosons, e.g., $(e^- - e^-)$; $(^3$He–^{3}He) ...
Alkali atomic (bosonic) isotopes ^{87}Rb, ^{85}Rb, ^{7}Li, ^{23}Na, ...
Alkali molecules (fermionic-isotope pairs) ^{40}K$_2$, ^{6}Li$_2$
Protonic/neutronic and pion condensates – neutron star interior.
Cosmological condensates – field vacua.

Acknowledgements

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