INVARIANT IMBEDDING APPROACH TO THE LENGTH-SCALE DEPENDENT CONDUCTANCE OF A ONE-DIMENSIONAL DISORDERED MEDIUM

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ABSTRACT

The invariant-imbedding equation for the complex amplitude reflection coefficient $\rho(L) \equiv \rho_1 + i\rho_2$ evolving stochastically in the length L of a disordered one-dimensional conductor is reduced to a linear second order *deterministic* equation for the probability density $P(\rho_1, \rho_2; L)$. The latter can be used to calculate any averaged quantity of interest, in particular the average conductance using the Landauer formula, and to discuss the fluctuations about the average. The latter has been a subject of much discussion in the recent past.

THE one-electron eigenstates of a one-dimensional (1-D) disordered system are known to be exponentially localized for arbitrarily smaller disorder. This is an exact theorem proved by several workers with varying degree of rigour[1-4]. Exponential localization implies that the typical resistance $R(L)$ of the sample *should increase exponentially with the length* (L). This was shown to be indeed so for the ensemble averaged DC resistance $\langle R(L)\rangle$ by Landauer[5] who derived a remarkable expression for $R(L)$ in terms of the reflection (transmission) coefficient $\rho^*\rho$ $(\tau^*\tau)$ as

$$R(L) = \frac{\pi\hbar}{e^2}\frac{\rho^*(L)\rho(L)}{\tau^*(L)\tau(L)}, \; \rho^*\rho + \tau^*\tau = 1, \qquad (1)$$

where the sample is assumed to be terminated by two metallic reservoirs ensuring phase randomization and hence irreversibility. This also avoided the problem of non-zero level spacing due to finite sample size L. (This expression has been the subject of intense discussion and debate in recent years and seems well established for the strictly 1-D system, *i.e.* for 1-D, 1-channel case. Generalization to the N-channel case has been possible[6]. The large N limit may describe a real thin wire)[6]. An exponentially decreasing DC conductance $\langle G(L)\rangle$ was obtained by Abrikosov and Ryzhkin[7] *starting with* a microscopic Hamiltonian with white-noise disorder. Here the granularity of the finite size (L) level spacing was treated by a certain limiting and averaging procedure. It was, however, soon realized that the conductance depended on the mode of averaging as well as on the definition[8]. Thus the arithmetic, the geometric and the harmonic averages are all different implying large *fluctuations that grow with* L and dominate the average, *i.e.*, the central limit does not obtain for $R(L)$ or $G(L)$. It has been suggested and numerically supported that the geometric mean is the only physically meaningful quantity implying that $\ln G$ obeys the central limit theorem and is, therefore, the right scaling variable[8]. Physically, the large fluctuations for the 1-D system are understandable. Exponential increase of resistance with L is due to the interference of the incident wave with the waves reflected from the successive parts of the serially random conductor. This is a coherent effect as the scattering is elastic, and therefore, different parts of the system may not simulate different 'instances' of the system in the sense of ensemble theory. Thus the resistance is *not* a self-averaging quantity. (Of course, finite inelastic mean-free path due to scattering by thermal phonons can destroy the phase coherence and then the resistance will be additive.)

These considerations have motivated us to look for an analytic solution for the probability density for the resistance $R(L)$ itself rather than evaluate some moments. This should constitute a comprehensive solution to the problem of fluctuations. Following the invariant imbedding[9] approach natural to such a situation, the present author has been able to obtain a solution to this problem, and a brief account of the ideas and the derivations involved are reported in this paper.

The one-electron problem is completely described by the Hamiltonian in

$$\frac{\partial^2\Psi}{\partial x^2} + k^2(x)\Psi = 0, \; k^2(x) = \frac{2m}{\hbar^2}(E - V(x)), \qquad (2)$$

where $k(x)$ is the local wavenumber assumed to be random and L the sample length. In the invariant-imbedding approach one directly addresses the emergent quantity, namely the amplitude reflection coefficient $\rho(L)$ that obeys the Riccati equation[9]

$$\frac{\partial \rho}{\partial L} = f_1(L) + 2if_0\,\rho(L) - f_1(L)\rho^2(L), \qquad (3)$$

where $f_1(L) = 2\partial \ln k/\partial L$ and $f_0(L) = k(L)$. Here we shall always consider the case when $k^2(L)$ is positive (i.e. the 'refractive' index is real and random so that any localization is due entirely to the non-trivial wave interference and not due to barrier penetration when $E < V(x)$. Since $\rho(L)$ is complex it is convenient to write $\rho(L) = \rho_1(L) + i\rho_2(L)$. Then, we have

$$\partial \rho_1(L)/\partial L = f_1(L) - 2f_0(L)\rho(L) - f_1(L)(\rho_1^2 - \rho_2^2), \quad (4)$$

$$\partial \rho_2(L)/\partial L = 2f_0(L)\rho_1(L) - 2f_1(L)\rho_1(L)\rho_2(L). \quad (5)$$

In order to evaluate the probability distribution $P(\rho_1, \rho_2; L)$ we now introduce a dust of phase points of density $p(\rho_1, \rho_2; L)$ in the 2-D (ρ_1, ρ_2) phase space evolving with "time" L subject to the "initial" subsemble condition that $\rho(L) = 0$ for $L = 0$. As for the phase angle $\theta = \tan^{-1} \rho_2/\rho_1$, we may take it to be a circular ensemble uniform in the θ space. The phase fluid will evolve according to the stochastic Liouville equation[10].

$$\frac{\partial p}{\partial L} = -\frac{\partial}{\partial \rho_1}\left(p\frac{\partial \rho_1}{\partial L}\right) - \frac{\partial}{\partial \rho_2}\left(p\frac{\partial \rho_2}{\partial L}\right), \qquad (6)$$

where $\partial \rho_1/\partial L$ and $\partial \rho_2/\partial L$ are given by the stochastic differential equations (4) and (5). Now we have the well-known result[10]

$$P(\rho_1, \rho_2; L) = \langle p(\rho_1, \rho_2; L)\rangle_f, \qquad (7)$$

where the average is with respect to the basic randomness variables f_0 and f_1. Now the circular angular ensemble persists in L and the terms on the right side of (6) involving f_0 cancel out, leaving f_1 as the basic random variable in the problem. Without much ado we take f_0 to be a Gaussian white noise process. This is physically admissible since it is proportional to the derivative $\partial k(L)/\partial L$ rather than just $k(L)$. Thus, $p(\rho_1, \rho_2; L)$ is a functional of the Gaussian random variable and one has, therefore[11],

$$\langle f_1(L)\,p(\rho_1, \rho_2; L)\rangle = \int \mathrm{d}L' \langle f_1(L) f_1(L')\rangle \times \left\langle \frac{\delta p}{\delta f_1(L')\,\mathrm{d}L'} \right\rangle = \Delta^2 \left\langle \frac{\delta p}{\delta f_1(L)\,\mathrm{d}L} \right\rangle, \qquad (8)$$

where we have used

$$\langle f_1(L) f_1(L')\rangle = \Delta^2 \delta(L - L'). \qquad (9)$$

This enables us to average (6) and obtain a closed *deterministic* equation for $P(\rho_1, \rho_2; L)$. Following the earlier procedure[11], we obtain

$$\frac{\partial P}{\partial L} = \Delta^2 r(1-r)^2 \frac{\partial^2 P}{\partial r^2} + \Delta^2(1-r) \times (1-5r)\frac{\partial P}{\partial r} + 2\Delta^2(2r-1)P, \qquad (10)$$

where we have introduced the reflection coefficient $r = \rho_1^2 + \rho_2^2$ which is directly related to the resistance through (1). This is a *deterministic* equation for the quantity of interest $P(\rho_1, \rho_2; L)$, and constitutes the central result which the present author believes to be first of its kind. We note that (10) is essentially a Fokker-Planck equation. It has a regular singular point at $r = 1$ which dominates the behaviour for large L when the reflection is nearly complete. It also has a singularity (irregular) at $r = 0$. Thus the fundamental domain $0 < r < 1$ is bounded by two singularities. The equation can be solved in terms of the eigenfunctions of the associated eigenvalue problem. We will discuss the explicit solutions in a detailed paper elsewhere. But let us clarify a technical point here. The stochastic equations (4) and (5) have the usual ambiguity for the white noise randomness assumed. The above procedure corresponds to the Stratonovic procedure and thus we avoid the Ito calculus completely[12]. Another point, of physical interest, is that (10) does not admit a normalizable 'steady state' solution as $L \to \infty$, obtainable by setting $\partial P/\partial L = 0$. This is, of course, as it should be since the resistance cannot saturate as $L \to \infty$. Finally, we should note that the normalization condition is

$$\int_{\rho_1^2+\rho_2^2 \leq 1} P(\rho_1, \rho_2; L)\,\mathrm{d}\rho_1\,\mathrm{d}\rho_2 = \pi \int_{r \leq 1} P(r, L)\,\mathrm{d}r = 1. \qquad (11)$$

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