ON RENORMALIZATION-GROUP THEORETIC APPROACH TO ANDERSON LOCALISATION

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ABSTRACT

Renormalization-Group theoretic approach to disorder induced electron localization is reconsidered. It is suggested that there is a naturally occurring lower wavevector cut-off in the problem such that one should eliminate the long-wavelength degrees of freedom in favour of the shorter ones, contrary to the usual procedure. One, indeed, obtains in this way a physical fixed point in that $u^* > 0$.

INTRODUCTION

THE suggestion[1-5] that, in a certain sense, the disorder induced transition[6] from the delocalized to the localized electron states can be viewed as a phase transition for an n-component classical field system in the limit $n \to 0$, is now known to be erroneous[4,7,8]. To be specific, consider the random lattice electronic Hamiltonian[6] :

$$H^e = \sum_{j=1}^{N} \epsilon_j |j\rangle\langle j| + \sum_{j \neq k}^{N} V_{jk} |j\rangle\langle k| \qquad (1)$$

with

$$\langle \epsilon_j \epsilon_k \rangle = \Sigma^2 \delta_{jk}, \quad \text{and} \quad V_{jk} = -V_0 v_{jk},$$

where the site-diagonal disorder is realized by regarding $\{\epsilon_j\}$ to be a set of identically independently distributed random gaussian variables having a common variance Σ. Here V_0 sets the scale of energy. It has been shown that the configurationally averaged electron propagator $\langle G^e_{jk}(E, \Sigma)\rangle_{\text{config.}}$ is then given as[9]

$$\langle G^e_{jk}(E, \Sigma)\rangle_{\text{config.}} \equiv \left\langle\!\!\left\langle j \left| \frac{1}{E - H^e} \right| k \right\rangle\!\!\right\rangle \propto G^{n=0}_{jk}(e, \sigma)$$

$$\equiv \lim_{n \to 0} \frac{1}{n} \sum_{\alpha=1}^{n} \langle S_{j\alpha} S_{k\alpha} \rangle_{\text{stat. mech}} \qquad (2)$$

where the statistical mechanical correlation function $G^n_{jk}(e, \sigma)$ is derived from the equivalent albeit complex, Hamiltonian (dimensionless)

$$H^n[S_{j\alpha}] = -\sum_{jk\alpha} i v_{jk} S_{j\alpha} S_{k\alpha} + i \sum_{j\alpha} S_{j\alpha}^2 + \frac{\sigma^2}{2} \sum_{j} \left(\sum_{\alpha} S_{j\alpha} \right)^2,$$

with

$$e = E/V_0, \quad \sigma = \Sigma/V_0, \quad i = \sqrt{-1}. \qquad (3)$$

The above equivalence can, of course, be generalized to arbitrary disorder[9]. It has been tempting, therefore, to interpret the divergence of the correlation length at the critical point of the equivalent statistical mechanical problem as the divergence of the localization length at the mobility edge[1-5]. One could, thus, hope to make the powerful formalism of the renormalization group (RG) to bear on the localization problem. It turned out, however, that the implicit assumption that there is a physical fixed point attracting the effective Hamiltonian in Eq. (3) is mistaken in that u^* (coefficient of the quartic term in the fixed-point Hamiltonian) comes out negative. It was concluded that something much more subtle happens at the mobility edge[7].

In the following we reconsider the RG-theoretic approach to the Anderson problem *via* the equivalence discussed above. We, however, take cognizance of the fact there is a natural lower wavevector cut-off in the problem necessitating a modification of the RG procedure in that we should now eliminate the low-wavevector degrees of freedom in favour of the higher ones. Interestingly enough, this yields a physical fixed point in that $u^* > 0$.

To bring out the essentials of the approach into sharp focus, we shall consider an electron moving in a d-dimensional continuum rather than on a lattice. The Gaussian white noise is now described by the potential $\epsilon(x)$. The continuum limit is formally equivalent to setting.

$$\sum_{k} V_{jk} a_k \to -\frac{\hbar^2}{2m} \int \nabla^2 \delta^d (x - x') a(x') d^d x'$$

$$\langle \epsilon_j \epsilon_k \rangle \to \langle \epsilon(x) \epsilon(x') \rangle = \Phi_0 \delta^d (x - x'), \qquad (4)$$

where m is electronic mass and $\delta^d(x)$ the d-dimensional Dirac delta-function.

Noting that Φ_0 has the dimensionality energy squared times volume, one can construct a characteristic length l_0 from $\hbar^2/m$ and Φ_0 for $d \neq 4$[10]. Expressing all lengths in the units of l_0 and all energies in terms of l_0^2/ml_0^2, the equivalent Hamiltonian may now be written as (in the reciprocal space)

$$H^n[S_{\alpha q}] \frac{i}{2} \int dq (q^2 + R) \tilde{S}_{\alpha q} \tilde{S}_{\alpha -q} + U \int_{q_1} \int_{q_2} \int_{q_3} \tilde{S}_{\alpha q_1} \tilde{S}_{\alpha q_2} S_{\beta q_3} S_{\beta -q_1 -q_2 -q_3} \qquad (5)$$

where

$$\int_q \equiv (2\pi)^{-d} \int d^d q \quad \text{for} \quad 0 < q < \infty,$$

and R and U are the dimensionless coefficients analogous to e and σ^2 of Eq. (3). Let it be noted that q is the dimensionless wavevector measured in the units of $1/l_0$.

Now the physical significance of the length l_0 is simply this. For $2 < d < 4$, it can be shown[10] that, for weak scattering, the mean-free path for the electron scattered by the random potential is smaller than its de Broglie wavelength for low energies while reverse is the case for the higher electron energies. The characteristic wavelength at which the transition takes place is just the length l_0. This implies that the degrees of freedom corresponding to higher spatial frequencies are relatively undamped while those with lower spatial frequencies behave like the diffusive modes. This is just the opposite of the usual hydrodynamic picture of Ginzburg–Landau–Wilson. This naturally suggests a coarse-graining in the reciprocal space where $\mathbf{S}_q$ is assumed to have been averaged over reciprocal space cells of linear dimension ~ 1. This sets a lower wavevector cut-off and motivates us to study the effective Hamiltonian

$$H^n_{eff}[S_{\alpha q}] = -\frac{i}{2} \int_{q>1} (q^2 + r) S_{\alpha q} S_{\alpha -q} + u \int\int\int_{q_1 q_2 q_3 > 1} S_{\alpha q_1} S_{\alpha q_2} \times S_{\beta q_3} S_{\beta -q_1-q_2-q_3} \qquad (6)$$

a la renormalization group. One proceeds exactly as in the usual case[11] except that now one splits

$$S_{\alpha q} = S_{\alpha_0 q} + S_{\alpha_1 q},$$

where

$S_{\alpha_0 q} = S_{\alpha q}$ for $b < q < \infty$ and zero otherwise

$S_{\alpha_1 q} = S_{\alpha q}$ for $1 < q < \infty$ " "

with $b > 1$,

and performs the partial trace over the low-spatial-frequency degrees of freedom S_{α_0}. One must only remember that now the unperturbed propagator is

$$\frac{i\delta^d(q_1 + q_2)}{(q_1^2 + r)},$$

The approximate recursion relations for $n \to 0$ are now

$$r_{l+1} = b^{-2}(r_l - 2u_l)$$
$$u_{l+1} = b^{-\epsilon}(u_l + 8u_l^2)$$
$$\text{with } \epsilon = 4 - d. \qquad (7)$$

Now recalling that $b > 1$, one readily sees that for $d \lesssim 4$, Eq. (7) yields a positive u^*. This is the main result obtained by us.

While the physics associated with this fixed point is not very clear yet, the following points are evident. There is no divergence of the correlation (localization) length at the fixed point. This would imply finite localization length even at the mobility edge. Physically this is quite understandable since the localization length cannot meaningfully exceed the mean free path that remains finite even for the 'extended' states just above the mobility edge. Mathematically this would imply analyticity of the averaged propagator at the mobility edge and in particular no kink in the density of states. This also provides an explicit answer to the general criticism of the RG approach to the localization problem based on the equivalence (2), namely, that it is predicated on the averaged propagator and hence inadequate to treat localization. The point is that the equivalence (2) is made use of only to the extent that it helps us find an appropriate free energy functional[5]. The latter contains much more information than the averaged propagator, and there is no *a priori* reason why it should not bear on the localization problem.

Finally, we would like to point out that the present inverse procedure of renormalization is very reminescent of the situation encountered in the theory of turbulence where also the universal features lie in the direction of the small scale eddies and, accordingly, one eliminates the smaller spatial frequency components in favour of the higher ones[12].

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