EXACT SOLUTION OF d-DIMENSIONAL ISING MODEL FOR FINITE CONTINUOUS SPINS (D) AND NON-ZERO EXTERNAL FIELD (B)

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ABSTRACT

Making use of the Stratonovich linearization procedure for bilinear exponents and certain eigenvalue properties of the generalised Jacobi matrices, an exact expression has been obtained, in a closed form, for the partition function of the d $(=3)$ dimensional, nearest-neighbour coupled, Ising model for "finite" continuous spins $(= \mathrm{D})$ in the presence of an external field. A Gaussian measure has been chosen for integration over the spin space. The method admits complete generalisation with respect to the dimensionality, the range of interaction and the sign of the interaction constant J (ferro- or antiferromagnetic). All thermodynamical quantities of interest can be obtained by taking suitable partial derivatives of the partition function. The critical indices can be ascertained for the cases of $d = 3$ for which the theory predicts a second-order phase transition. For the ferromagnetic case $(\mathrm{J} > \mathrm{O})$, the spontaneous magnetisation vanishes identically the temperature $\mathrm{T} > \mathrm{T}_c$ (the transition temperature) and the magnetic susceptibility follows essentially the Curie Weiss Law.

THE exact solution of the three-dimensional Ising model for the case of "finite", even if continuous, spins and non-zero external field has been of considerable interest ever since the famous Onsager[1] solution for the zero-field two-dimensional Ising model and its subsequent extension by Yang[2] and others[3-5]. In this preliminary communication we shall report an exact solution for this problem in that the partition function has been obtained in a closed form exactly. While the method is completely general with respect to the dimensionality (d) of the problem, we shall for specificity, consider the case of three-dimensional Ising ferromagnet $(\mathrm{J} > \mathrm{O})$, with nearest-neighbour interactions only. The spins μ_i are taken to be continuous, and a Gaussian measure $p(\mu_i)$ has been used for integration over the spin spaces. Thus the effective "finite" value of the spin

$$\int_{-\infty}^{+\infty} |\mu_i| \, p(\mu_i) \, d\mu_i.$$

... are taken to be distributed over a simple-cubic lattice having N $(= n \times n \times n)$ sites and the periodic boundary condition has been imposed.

The partition function Z is then given by

$$\mathrm{Z} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp.\left(\beta J \sum_{i,\Delta} \mu_i \mu_{i+\Delta}\right) \times \exp.\left(\beta \mathrm{B}_0 \sum_i \mu_i\right) \Pi_i p(\mu_i)\, d\mu_i, \qquad (1)$$

where the Gaussian measure $p(\mu)$ is given by

$$p(\mu) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp.\left(-\mu^2/2\sigma^2\right)$$

and

$$\mathrm{D} = \int_{-\infty}^{+\infty} |\mu| \, p(\mu) \, d\mu = \sqrt{\frac{\pi}{2}}\,\sigma. \qquad (2)$$

Here the field B_0 is in the units of energy, and $\beta = 1/k\mathrm{T}$. Δ spans the nearest neighbours, that number $z(= 6)$.

Writing

$$\mu_i \mu_{i+\Delta} = \tfrac{1}{2}(\mu_i + \mu_{i+\Delta})^2 - \tfrac{1}{2}\mu_i^2 - \tfrac{1}{2}\mu_{i+\Delta}^2, \qquad (3)$$

and substituting this in Eq. (1), and making use of the Stratonovich[6] identity

$$\exp.(\mu_i + \mu_{i+\Delta})^2 = \int_{-\infty}^{+\infty} \exp.\left\{-\pi x_i^2 - 2\sqrt{\pi}\, x_i (\mu_i + \mu_{i+\Delta})\right\} \times dx_i, \qquad (4)$$

we get

$$\mathrm{Z} = \left(\frac{1}{\pi}\right)^{N/2} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(-\sum_i x_i^2\right) \cdot \exp.\left(-\frac{z}{2}\beta J \sum_i \mu_i^2\right) \cdot \exp.\left[\sum_i \mu_i \left\{\sum_{\Delta} \sqrt{2\beta J}\,(\; x_{i+\Delta}) + \beta \mathrm{B}_0\right\}\right] \pi_i \, dx_i \, d\mu_i. \qquad (5)$$

Now performing the μ_i integrals, we get

$$Z = \left\{\frac{1}{\pi(1 + z\beta J\sigma^2)}\right\}^{N/2}$$

$$\times \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp.\left(-\sum_i x_i^2\right)$$

$$\times \exp.\left\{\sum_i \frac{\left(\sum_\Delta 2\beta J\,(+x_{i+\Delta}) + \beta B_0\right)^2}{\left(\frac{1}{2\sigma^2} + \frac{z}{2}\beta J\right)}\right\}$$

$$\times \Pi_i dx_i. \qquad (6)$$

The exponent in Eq. (6) involves a bilinear form of the type $\sum_{i,j} A_{ij} x_i x_j$. The coefficient matrix "A" of this bilinear form is a generalised Jacobi matrix. The bilinear form can be brought into the canonical (quadratic) form $\sum_{i=1}^{N} \lambda_i X_i^2$ by an orthogonal transformation. Here λ_i's are the eigenvalues of the matrix "A" and X_i's are the new variables. These eigenvalues are readily written down if we note the analogy with the energy eigenvalue problem of the band structure calculation in the tight-binding scheme for a simple-cubic lattice with one Wannier orbital per site. Then the eigenvalues turn out to be given by

$$\lambda_{l_i m_i n_i} = -\left[\left\{1 - \frac{4\beta Jz}{\left(\frac{1}{2\sigma^2} + \frac{z}{2}\beta J\right)}\right\} - \frac{8\beta J}{\left(\frac{1}{2\sigma^2} + \frac{z}{2}\beta J\right)} (\cos l_i\theta + \cos m_i\theta + \cos n_i\theta)\right] \qquad (7)$$

with $\theta = \pi/(n+1)$ and $l_i, m_i, n_i = 0, 1, 2 \ldots\ldots n$. Substituting these eigenvalues and performing the resulting Gaussian integrals over the new coordinates X_i and recalling that the Jacobian of the orthogonal transformation is unity, we get

$$Z = \{1/(1 + z\beta J\sigma^2)\}^{N/2}$$

$$\times \prod_{l_i, m_i, n_i}\left[\left(1 - \frac{4z\beta J}{\left(\frac{1}{2\sigma^2} + \frac{z}{2}\beta J\right)}\right) - \left(\frac{4z\beta J}{\frac{1}{2\sigma^2} + \frac{z}{2}\beta J}\right)\frac{1}{3}(\cos l_i\theta + \cos m_i\theta + \cos n_i\theta)\right]^{-\frac{1}{2}} \exp.(N\beta^2 B_0^2)$$

$$\times \exp.\left[32Nz^2\beta^3 B_0^2 J\Big/\left(\frac{1}{2\sigma^2} + \frac{z}{2}\beta J\right)^2 \times \left(1 - \left\{8Z\beta J\Big/\left(\frac{1}{2\sigma^2} + \frac{z}{2}\beta J\right)\right\}\right)\right]. \qquad (8)$$

Thus the Gibbs' free energy per spin G/N can be written as

$$G/N = \frac{1}{\beta}\ln(1 + z\beta J\sigma^2) - \beta B_0^2$$

$$-32Z^2\beta^2 B_0^2 J\Big/\left(\frac{1}{2\sigma^2} + \frac{z}{2}\beta J\right)^2 \times \left\{1 - \left(8z\beta J\Big/\frac{1}{2\sigma^2} + \frac{z}{2}\beta J\right)\right\}$$

$$-\frac{1}{2\pi^3\beta}\int_0^\pi\int_0^\pi\int_0^\pi \ln\left[1 - \frac{4z\beta J}{\left(\frac{1}{2\sigma^2} + \frac{z}{2}\beta J\right)} - \left(\frac{4z\beta J}{\frac{1}{2\sigma^2} + \frac{z}{2}\beta J}\right)\frac{1}{3}(\cos x + \cos y + \cos z)\right] dx\,dy\,dz. \qquad (9)$$

In writing the last equation we have taken the thermodynamic limit $N \to \infty$, and converted the summation over l_i, m_i, n_i into integrals. The above expression is valid for $T > T_c = 5\, z^2 D^2 J/\pi$ at which the system undergoes a second-order phase transition. It is readily seen that in this regime the spontaneous magnetisation is identically zero and the magnetic susceptibility obeys the Curie-Weiss law (aside from a slight departure away from T_c). The analytic properties of the partition function in the complex temperature plane and the detailed discussion of the critical indices will be published elsewhere.

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