Motion of Space Curves in Three-dimensional Minkowski Space R_1^3, SO(2,1) Spin Equation and Defocusing Nonlinear Schrödinger Equation

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Abstract. We consider the dynamics of moving curves in three-dimensional Minkowski space R_1^3 and deduce the evolution equations for the curvature and torsion of the curve. Next by mapping a continuous SO(2,1) Heisenberg spin chain on the space curve in R_1^3, we show that the defocusing nonlinear Schrödinger equation(NLSE) can be identified with the spin chain, thereby giving a geometrical interpretation of it. The associated linear eigenvalue problem is also obtained in a geometrical way.

Keywords: Minkowski space, Frenet equations, SO(2,1) Heisenberg spin equation, defocusing nonlinear Schrödinger equation

1. INTRODUCTION

Modelling of physical systems by curves, surfaces and other differential geometric objects is highly rewarding, see for example the pioneering work of Hasimoto [1] on vortex filaments and the one-dimensional continuum Heisenberg ferromagnetic spin equation by Lakshmanan, Ruijgork and Thompson [2,3]. In both cases the systems were shown to be equivalent to the integrable nonlinear Schrödinger equation (NLSE) of the focusing type [1-5]. In recent times the relation between differential geometry and certain dynamical systems described by nonlinear evolution equations in (1+1) and (2+1) dimensions, especially the integrable systems , has come into sharp focus [6-13].

Integrable nonlinear evolution equations occur in many branches of physics and applied mathematics. Such equations possess a number of interesting properties such as soliton solutions, infinite number of conservation laws, infinite number of symmetries, Bäcklund and Darboux transformations, bi-Hamiltonian structures and so on, see [7, 8].

Now it is well known that a class of important soliton equations can be interpreted in terms of moving space curves in R^3 and the linear eigen value problems of the soliton equations can be obtained from the defining Serret-Frenet equations of space curves [6, 9-13]. Extension to (2+1) dimensions is also possible. A brief survey of the developments up to more recent times can be seen in [13].

Our focus in this Letter is on the defocusing NLSE, in which the sign of the nonlinear term is negative, which is encountered in many physical problems. It was shown to be integrable by the inverse scattering transform method[14] and admits dark soliton solutions. However, there does not seem to be available a simple differential geometric model and its equivalent spin system for the defocusing case such as available for the focusing NLSE as demonstrated in [1,2].

In this Letter we wish to address this problem. We have found the rich curve theory available in three-dimensional Minkowski space is more suited to this problem. Nakayama [16] has used the geometry of curves on a three-dimensional ellipsoid in a four-dimensional Minkowski space to obtain a model for the defocusing NLSE. For other related works see also [17,18]. But we feel our approach is simpler and more direct and can be extended profitably to other nonlinear evolution equations also.

In the next section we give the basic equations of curves in $\mathrm{R}_1^3$and fix the notation closely following [19] and [20]. Subsequent sections establish the curve model of the solution of the defocusing NLSE with the tools and techniques similar to the ones found in [3] and [20-23] and its connection to the SO (2,1) continuous Heisenberg spin chain.

We hope to extend similar treatment of integrable and nonintegrable systems to surfaces in R_1^3 and higher dimensional Minkowski geometries in our subsequent work.

2. Motion of curves in the Minkowski Space $\mathbf{R}_1^3$

The nature of the metric in a Minkowski space induces a rich geometry of curves and surfaces. For instance the familiar Serret-Frenet equations in the Euclidean space R^3 give way to four such systems in R_1^3. In this section we give the basic curve geometry apparatus in R_1^3, see [19], and we closely follow [20] in writing down the curve evolution equations. The metric on the Minkowski space R^3_1 is given by ds^2 = -dx_1^2 + dx_2^2 + dx_3^2. We note here that the scalar and vector products of two vectors $a = a_1 i + a_2 j + a_3 k$, $b = b_1 i + b_2 j + b_3 k$ in R^3_1, where i, j, k are unit vectors along the x, y, z axes respectively, are given as follows:

Scalar Product: $a.b = g(a,b) = -a_1b_1 + a_2b_2 + a_3b_3$.

Vector Product: $a \wedge b = \begin{vmatrix} -i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$,

(*Note*: In Minkowski space a vector a is defined to be a unit vector if $g(a,a) = \pm 1$. A vector a is said to be *space like* if $g(a,a) > 0$, *time like* if $g(a,a) < 0$ and *light like* or a *null vector* if $g(a,a) = 0$).

Now, let(e_1, e_2, e_3) be the Serret-Frenet frame of a unit speed (non-null) curve $\alpha(x)$ in R^3_1. Here e_1 is the unit tangent vector field , e_2 is the normal and e_3 is the binormal to $\alpha(x)$. Let g(e_1, e_1) = $\epsilon_0 = \pm 1$, g(e_2,e_2) = $\epsilon_1 = \pm 1$. Then g(e_3,e_3) = -$\epsilon_0 \epsilon_1$. Then the Serret-Frenet equations are given by [19]

$$\begin{aligned} e_{1x} &= \epsilon_1 \kappa(x) e_2, \\ e_{2x} &= -\epsilon_0 \kappa(x) e_1 - \epsilon_0 \epsilon_1 \tau(x) e_3, \\ e_{3x} &= -\epsilon_1 \tau(x) e_2. \end{aligned} \tag{1}$$

Here τ and κ denote the torsion and curvature respectively of the given space curve α.

Using the vector product relations

$$e_1 \wedge e_2 = e_3,$$

$$e_2 \wedge e_3 = -\epsilon_1 e_1,$$

$$e_3 \wedge e_1 = -\epsilon_0 e_2, \tag{2}$$

the Serret-Frenet equations (1) can be compactly written as

$$e_{ix} = D \wedge e_i, i = 1, 2, 3, \tag{3}$$

where D is the Darboux vector defined as

$$D = -\epsilon_0 \epsilon_1 \tau e_1 - \epsilon_0 \epsilon_1 \kappa e_3. \tag{4}$$

Now, let us consider the time evolution of the curve $\alpha(x, t)$. We define an angular momentum like vector $\Omega = \Sigma \omega_i e_i$, i = 1,2,3, which gives the time evolution of the Serret-Frenet system as

$$e_{it} = \Omega \wedge e_i, i = 1, 2, 3. \tag{5}$$

From (2) and (5) we obtain

$$e_{1t} = -\epsilon_0 \omega_3 e_2 - \omega_2 e_3,$$

$$e_{2t} = \epsilon_1 \omega_3 e_1 + \omega_1 e_3,$$

$$e_{3t} = -\epsilon_1 \omega_2 e_1 + \epsilon_0 \omega_1 e_2. \tag{6}$$

In order that the above two definitions are compatible we require that

$$(e_i)_{xt} = (e_i)_{tx}, i = 1, 2, 3. \tag{7}$$

From (1), (6) and (7) we obtain

$$\kappa_t = \tau\omega_2 - \epsilon_0\epsilon_1\omega_{3x},$$

$$\tau_t = \epsilon_1\kappa\omega_2 - \epsilon_0\epsilon_1\omega_{1x},$$

$$\omega_{2x} = \epsilon_1\tau\omega_3 - \epsilon_1\kappa\omega_1. \tag{8}$$

The above equations constitute the evolution of the curvature and torsion associated with an arbitrary curve moving in R_1^3.

3. SO(2,1) Heisenberg Spin Equation and Mapping to a Space Curve in $\mathbf{R}_1^3$

Consider now the SO(2,1) Heisenberg spin equation given by

$$S_t = S \times S_{xx}, \tag{9}$$

where S is a unit vector in R_1^3, that is $-S_1^2 + S_2^2 + S_3^2 = \pm 1$. We identify S with the unit tangent vector e_1 of $\alpha(x)$. Then we obtain from the spin equation

$$e_{1t} = e_1 \times e_{1xx} = e_1 \times e_{(1x)x}$$

$$= e_1 \times (\epsilon_1\kappa e_2)_x = \epsilon_1 e_1 \times (\kappa_x e_2 + \kappa e_{2x})$$

Hence from (1) and (2) and noting $\epsilon_0^2 = \epsilon_1^2 = 1$ we have

$$e_{1t} = \epsilon_1\kappa_x e_3 - \kappa\tau e_2. \tag{10}$$

Next we have $e_2 = \epsilon_1 \frac{e_{1x}}{\kappa}$ from (1). Hence $e_{2t} = \epsilon_1 \frac{e_{1xt}}{\kappa} - \frac{e_{1x}}{\kappa^2}\kappa_t$. Using (1) and (10) we obtain

$$e_{2t} = [\epsilon_1\epsilon_0\kappa^2\tau e_1 - \epsilon_1(2\kappa_x\tau + \kappa\tau_x + \epsilon_1\kappa_t)e_2 + (\kappa_{xx} + \epsilon_0\kappa\tau^2)e_3]/\kappa. \tag{11}$$

Similarly we can deduce that

$$e_{3t} = \kappa_x e_1 + [(\epsilon_0 \kappa_{xx} + \kappa \tau^2) e_2 - \epsilon_1 (2\kappa_x \tau + \kappa \tau_x + \epsilon_1 \kappa_t) e_3]/\kappa. \tag{12}$$

Comparing the above with (6) we immediately obtain the evolution equation for the curvature as

$$\kappa_t = -\epsilon_1 (2\kappa_x \tau + \kappa \tau_x). \tag{13}$$

The compatiblity condition(7) applied to e_3 yields the evolution equation for torsion as

$$\tau_t = -\epsilon_1^2 \kappa \kappa_x - \epsilon_0 \epsilon_1 (\frac{\kappa_{xx}}{\kappa} + \epsilon_0 \tau^2)_x. \tag{14}$$

The above equations define the evolution of curvature and torsion of the curve associated with an SO (2,1) continuum Heisenberg spin system in R_1^3.

4. Mapping onto the Defocusing NLSE

Let us first consider the case where $\epsilon_0 = -1$ and $\epsilon_1 = 1$. Then equations (11) and (12) reduce to

$$\kappa_t = -2\kappa_x \tau - \kappa \tau_x \tag{15}$$

and

$$\tau_t = -\kappa \kappa_x + (\frac{\kappa_{xx}}{\kappa} - \tau^2)_x. \tag{16}$$

We now make the complex transformation

$$u = \frac{\kappa}{2} e^{i \int_{-\infty}^{x} \tau dx}. \tag{17}$$

Then using (17) equations (15) and (16) are transformed into

$$iu_t + u_{xx} - 2|u|^2 u = 0, \tag{18}$$

which is nothing but the defocusing nonlinear Schrödinger equation. Now we assume the energy and current densities of the spin system to be related to the curvature and torsion respectively as

$$\epsilon(x,t) = \frac{1}{2}\frac{\partial S}{\partial x}.\frac{\partial S}{\partial x} = \frac{1}{2}\kappa^2, \tag{19}$$

$$I(x,t) = S.S_x \wedge S_{xx} \tag{20}$$

so that the continuity given by $\epsilon_t - I_x = 0$ is satisfied. This continuity equation can be easily shown to be compatible with (15). Finally we also observe that the case of $\epsilon_0 = 1$ and $\epsilon_1 = -1$ yields the solution of focusing NLSE under the condition that the curve has a constant torsion. The other two cases of $\epsilon_0 = -1$, $\epsilon_1 = -1$ and $\epsilon_0 = 1$, $\epsilon_1 = 1$ do not reduce either to the defocusing or to the focusing NLSE for the transformation given by (17).

5. Reduction to AKNS Eigenvalue Problem

Corresponding to the Serret-Frenet frame given by (1) (for $\epsilon_0 = -1$ and $\epsilon_1 = 1$) we define a new scalar variable $z_l = \frac{e_{2l}+ie_{3l}}{1-ie_{1l}}$, l=1,2,3, following [20, 21], from which we obtain

$$z_{lx} = -i\tau z_l + \frac{i\kappa}{2}(1+z_l^2). \tag{21}$$

Now differentiating z_l with respect to t and using (5), and after some detailed calculations, we arrive at

$$z_{lt} = -i\omega_1 z_l + \frac{\omega_2 + i\omega_3}{2} - \frac{(\omega_2 - i\omega_3)z_l^2}{2}. \tag{22}$$

Equations (21) and (22) are nothing but the Riccati equations. Again the compatibility of (21) and (22) , that is $(z_l)_{xt} = (z_l)_{tx}$, leads to the correct equations for $\kappa(x,t)$ and $\tau(x,t)$ as in (15) and (16). Defining $z_l = \frac{v_2}{v_1}$, equation (21) can be written as

$$v_{1x} = \frac{i\tau}{2}v_1 - \frac{i\kappa}{2}v_2,$$
$$v_{2x} = \frac{i\kappa}{2}v_1 - \frac{i\tau}{2}v_2. \tag{23}$$

Similarly, from Eq.(22) we obtain

$$v_{1t} = \frac{i\omega_1}{2}v_1 - \frac{i}{2}(\omega_3 + i\omega_2)v_2,$$
$$v_{2t} = -\frac{i\omega_1}{2}v_2 + \frac{i}{2}(\omega_3 - i\omega_2)v_1. \tag{24}$$

Using the compatibility conditions $(v_{ix})_t = (v_{it})_x, i = 1, 2$, from (23), (24) above we once again get back easily the original equations for $\kappa(x,t)$ and $\tau(x,t)$. Now introducing a suitable Galilean transformation and a gauge transformation in to (23), (24) we obtain the linear eigenvalue problem

$$\psi_{1x} = u\psi_2 - i\lambda\psi_1,$$
$$\psi_{2x} = u^*\psi_1 + i\lambda\psi_2, \tag{25}$$

and the time evolution of the eigenfunction as

$$\psi_{1t} = A\psi_1 + B\psi_2,$$
$$\psi_{2t} = C\psi_1 + D\psi_2, \tag{26}$$

where

$$A = -2i\lambda^2 - iuu^*,$$

$$B = 2u\lambda + iu_x,$$

$$C = 2u^*\lambda - iu_x^*. \tag{27}$$

and u is as defined in Eq. (17).

6. Conclusions

In this paper, we have shown how the dynamics of moving curves in three dimensional Minkowski space R_1^3 can be related to the dynamics of SO(2,1) spin equations and soliton equations of defocusing NLS type. It is possible that more number of such connections with soliton equations not related to moving curves in Euclidean space R^3 may be related to present type of formulation. The analysis can also be extended higher dimensional spaces. Analysis along these lines is in progress.

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