On certain new integrable second order nonlinear differential equations and their connection with two dimensional Lotka - Volterra system

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Abstract

In this paper, we consider a second order nonlinear ordinary differential equation of the form $\ddot{x}+k_1\frac{\dot{x}^2}{x}+(k_2+k_3x)\dot{x}+k_4x^3+k_5x^2+k_6x=0$, where k_i's, $i=1,2,...,6$, are arbitrary parameters. By using the modified Prelle-Singer procedure, we identify five new integrable cases in this equation besides two known integrable cases, namely (i) $k_2=0$, $k_3=0$ and (ii) $k_1=0$, $k_2=0$, $k_5=0$. Among these five, four equations admit time dependent first integrals and the remaining one admits time independent first integral. From the time independent first integral, nonstandard Hamiltonian structure is deduced thereby proving the Liouville sense of integrability. In the case of time dependent integrals, we either explicitly integrate the system or transform to a time-independent case and deduce the underlying Hamiltonian structure. We also demonstrate that the above second order ordinary differential equation is intimately related to the two-dimensional Lotka-Volterra (LV) system. From the integrable parameters of above nonlinear equation and all the known integrable cases of the latter can be deduced thereby.

I. INTRODUCTION AND STATEMENT OF RESULTS

A. Introduction

In this paper we investigate the integrability properties associated with the second order nonlinear ordinary differential equation (ODE) of the form

$$\ddot{x} + k_1\frac{\dot{x}^2}{x} + (k_2 + k_3x)\dot{x} + k_4x^3 + k_5x^2 + k_6x = 0, \tag{1}$$

where k_i's, $i = 1, 2, ..., 6$, are arbitrary parameters and the over dot denotes differentiation with respect to 't'. Equation (1) is a combination of two different classes of equations, namely Liénard type equation ($k_1 = 0$) and equation with quadratic friction ($k_2 = k_3 = 0$). Using the transformation $z = \dot{x}$ one can reduce Eq. (1) to the Abel equation of the second kind[1,2]. However, the resultant equation is not integrable in general. A set of integrable parametric choices of this equation has been listed in Refs. 1 & 2. The importance of the study of Eq. (1) arises from the fact that this equation contains many physically interesting equations such as the modified Emden equation, unforced Duffing oscillator, Helmholtz oscillator, etc and is intimately related to the much analyzed biological model, namely the two dimensional Lotka-Volterra system[3,4] (LV),

$$\dot{x} = x(a_1 + b_{11}x + b_{12}y), \tag{2a}$$

$$\dot{y} = y(a_2 + b_{21}x + b_{22}y). \tag{2b}$$

In order to study the integrable properties of Eq. (1) we employ the modified Prelle-Singer procedure (PS) which has been intensively used to identify new integrable cases for several equations[5,6]. Using this procedure we identify several new integrable parametric choices and deduce their corresponding integrals of motion. In order to prove the integrability we either associate a conservative Hamiltonian structure to the equation constructed from the time independent integrals of motion, thereby proving Liouville sense of integrability, or explicitly integrate the integrals of motion to obtain general solution, proving complete integrability.

During the past three decades or so several attempts have been made to explore the integrable cases in the LV system because of the immense importance of the problem in mathematical ecology and biology[7–12]. Integrals of motion of the LV system have been constructed for several parametric choices[9,10,12–19]. However, we find that all these parametric choices effectively reduce to one of the three general parametric choices for which integrals of motion have been reported[9]. Interestingly, we recover all these three integrable cases in addition to many subcases by comparing and expressing the parameters appearing in (1) in terms of the LV system parameters. The results exactly coincide with the reported ones in the literature. Thus, as a by-product, we recover all the known integrable cases of LV by investigating the integrability properties of Eq. (1). At this point we stress the fact that as far as integrability is concerned one can extract complete information about (2) from (1) and not vice-versa. In other words when one tries to deduce the integrable cases of (1) from (2) only partial results can be extracted. That is one will loose a major portion of the results when we proceed in the reverse way (see Sec. VI for more details).

B. Results

Our main results can be summarized as follows. Using the modified PS method we identify seven integrable parametric choices of which five seem to be new as far as our knowledge goes. The underlying forms of the equations are

$$\ddot{x}+k_1\frac{\dot{x}^2}{x}+(k_2+k_3x)\dot{x}+k_4x^3+\frac{k_2k_4(3+2k_1)}{k_3(1+k_1)}x^2+\frac{k_4k_2^2(2+k_1)}{k_3^2(1+k_1)}x=0, \tag{3}$$

$$\ddot{x}+k_1\frac{\dot{x}^2}{x}+(k_2+k_3x)\dot{x}+\frac{(1+k_1)k_3^2}{(3+2k_1)^2}x^3+\frac{k_2k_3}{(3+2k_1)}x^2+k_6x=0, \tag{4}$$

$$\ddot{x}+k_1\frac{\dot{x}^2}{x}+(k_2+k_3x)\,\dot{x}+\frac{k_3k_2(1+k_1)}{(3+2k_1)}x^2+\frac{(2+k_1)k_2^2}{(3+2k_1)^2}x=0, \tag{5}$$

$$\ddot{x}+k_1\frac{\dot{x}^2}{x}+k_2\dot{x}+k_5x^2+\frac{2(3+2k_1)k_2^2}{(5+4k_1)^2}x=0, \tag{6}$$

$$\ddot{x}+k_1\frac{\dot{x}^2}{x}+(k_2+k_3x)\dot{x}+k_4x^3+\frac{k_2k_3}{(3+2k_1)}x^2+\frac{(2+k_1)k_2^2}{(3+2k_1)^2}x=0. \tag{7}$$

For these five cases we prove integrability either in the Liouville sense by constructing the time independent Hamiltonian structure or deduce the solution by explicitly integrating the time dependent integrals of motion.

The remaining two cases which are already known in the literature are

$$\ddot{x}+k_1\frac{\dot{x}^2}{x}+k_4x^3+k_5x^2+k_6x=0 \tag{8}$$

and

$$\ddot{x}+k_3x\dot{x}+k_4x^3+k_6x=0\,. \tag{9}$$

We note that equation (8) can be reduced to the Bernoulli's equation, which can then be integrated to give the solution in terms of quadratures. Equation (9) is the modified Emden type equation with linear forcing term which has been studied in some detail in Refs. 20 & 21. The Hamiltonian structure for this equation for the parametric choices $k_3^2 \geq 8k_4$ has been given in Ref. 20. Here we present the Hamiltonian structure for the parametric choice $k_3^2 < 8k_4$ as well. In addition to these seven integrable parameteric choices we find the following equation

$$\ddot{x} + k_1 \frac{\dot{x}^2}{x} + (k_2 + k_3 x)\dot{x} + \frac{k_3(k_2 \pm \omega)}{2(2+k_1)} x^2 + k_6 x = 0, \tag{10}$$

where $\omega = \sqrt{k_2^2 - 4(1+k_1)k_6}$, for which we obtain a time-dependent integral of motion. However, we are able to prove its complete integrability only with an additional parametric restriction $k_5 = \frac{k_3 k_2 (1+k_1)}{(3+2k_1)}$, and $k_6 = \frac{(2+k_1)k_2^2}{(3+2k_1)^2}$, which reduces Eq. (10) to Eq. (5). In this parametric choice we are able to explicitly integrate the time-dependent integral of motion to find the general solution.

The plan of the paper is as follows. In the following section we briefly describe the extended Prelle-Singer procedure[5,6] applicable to second-order ODEs. In Sec. III we identify the integrable parametric choices of Eq. (1) which admit time independent integral of motion through the extended PS procedure. In Sec. IV we construct explicit conservative Hamiltonians from the time independent integral of motion. Further we transform these Hamiltonians to simpler forms using canonical transformations in order to explicitly integrate the canonical equations of motion. In Sec. V, we then identify the integrable cases of (1) which admit explicit time-dependent integrals of motion. To establish the complete integrability of these cases we transform the time-dependent integrals of motion into time-independent integrals of motion and integrate the latter and derive the general solution. In Sec. VI we present the connection between Eq. (1) and the LV equation. In Sec. VII we rewrite the results in terms of the LV parameters and point out the integrable equations. Finally, we present our conclusions in Sec. VIII.

II. EXTENDED PRELLE-SINGER (PS) PROCEDURE

Let us rewrite Eq. (1) in the form

$$\ddot{x} = -(k_1 \frac{\dot{x}^2}{x} + (k_2 + k_3 x)\dot{x} + k_4 x^3 + k_5 x^2 + k_6 x) \equiv \phi(x, \dot{x}). \tag{11}$$

Further, we assume that Eq. (11) admits a first integral $I(t,x,\dot{x}) = C$, with C constant on the solutions, so that the total differential becomes

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0, \tag{12}$$

where subscript denotes partial differentiation with respect to that variable. Rewriting Eq. (11) in the form $\phi dt - d\dot{x} = 0$ and adding a null term $S(t,x,\dot{x})\dot{x}dt - S(t,x,\dot{x})dx$ to the latter, we obtain that on the solutions the 1-form

$$(\phi + \dot{x}S)dt - Sdx - d\dot{x} = 0. \tag{13}$$

Hence, on the solutions, the 1-forms (12) and (13) must be proportional. Multiplying (13) by the function $R(t,x,\dot{x})$ which acts as the integrating factor for (13), we have on the solutions that

$$dI = R(\phi + S\dot{x})dt - RSdx - Rd\dot{x} = 0. \tag{14}$$

Comparing Eq. (12) with (14), we have the relations

$$I_t = R(\phi + \dot{x}S), \quad I_x = -RS, \quad I_{\dot{x}} = -R. \tag{15}$$

Then the compatibility conditions, $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, between the different equations of (15), provide us the relations

$$S_t + \dot{x}S_x + \phi S_{\dot{x}} = -\phi_x + \phi_{\dot{x}}S + S^2, \tag{16}$$

$$R_t + \dot{x}R_x + \phi R_{\dot{x}} = -(\phi_{\dot{x}} + S)R, \tag{17}$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \tag{18}$$

Solving Eqs. (16)-(18) one can obtain expressions for S and R. It may be noted that two sets of independent special solutions (S, R) are sufficient for our purpose. Once these forms are determined the integral of motion $I(t,x,\dot{x})$ can be deduced from the expression

$$I = r_1 - r_2 - \int \left[R + \frac{d}{d\dot{x}}(r_1 - r_2)\right] d\dot{x}, \tag{19}$$

where

$$r_1 = \int R(\phi + \dot{x}S)dt, \quad r_2 = \int (RS + \frac{d}{dx}r_1)dx.$$

Equation (19) can be derived straightforwardly by integrating Eq. (15). We solve Eq. (1) through the extended PS procedure in the following way. For the given second-order ODE (1), the first integral I should be either a time-independent or time dependent one. In the former case, it is a conservative system and we have $I_t = 0$ and in the latter case we have $I_t \neq 0$. So, let us first consider the case $I_t = 0$ and determine the null forms and the corresponding integrating factors, and from these we construct the integrals of motion and then we extend the analysis to the case $I_t \neq 0$.

III. TIME INDEPENDENT INTEGRALS : INTEGRABLE PARAMETRIC CHOICES

In this section, we identify a set of parametric choices of (1) for which time independent integrals exist. For this purpose we first find the null forms and integrating factors corresponding to equation (1) and using these functions we construct time independent integrals of (1) through the relation (19).

A. Null forms and integrating factors

Since $I_t = 0$, one can easily fix the null form S from the first equation in (15) as

$$S = \frac{-\phi}{\dot{x}} = \frac{(k_1 \frac{\dot{x}^2}{x} + (k_2 + k_3 x)\dot{x} + k_4 x^3 + k_5 x^2 + k_6 x)}{\dot{x}}. \tag{20}$$

Of course one can easily check that the S form given above satisfies Eq. (16).

Substituting this form of S, given in (20), into (17) we get

$$\dot{x}R_x - (k_1 \frac{\dot{x}^2}{x} + (k_2 + k_3 x)\dot{x} + k_4 x^3 + k_5 x^2 + k_6 x)R_{\dot{x}} = \left(\frac{k_1 \dot{x}}{x} - \frac{k_4 x^3 + k_5 x^2 + k_6 x}{\dot{x}}\right)R. \tag{21}$$

Since we are interested in time independent integrals we take $R_t = 0$. As we noted earlier any particular solution of the above equation along with the null form S is sufficient to construct an integral of motion. To derive a particular solution of (21) we make an ansatz for R of the form

$$R = \frac{\dot{x}}{(A(x) + B(x)\dot{x} + C(x)\dot{x}^2)^r}, \tag{22}$$

where r is a constant and $A(x)$, $B(x)$ and $C(x)$ are arbitrary functions of their argument. The reason for choosing the above form of ansatz is as follows. To deduce the time independent first integral I we assume a rational form for I, that is, $I = \frac{f(x,\dot{x})}{g(x,\dot{x})}$, where f and g are arbitrary functions of x and $\dot{x}$. Using (12) we obtain $S = I_x/I_{\dot{x}} = (f_x g - f g_x)/(f_{\dot{x}} g - f g_{\dot{x}})$. and $R = I_{\dot{x}} = (f_{\dot{x}} g - f g_{\dot{x}})/g^2$ and from these two expressions we find that the numerator of R should be the denominator of S and so we fixed the numerator of the R in Eq. (22) as $\dot{x}$ (see expression (20) for S), that is $R = \frac{\dot{x}}{h(x,\dot{x})}$, where h is an arbitrary function. However, it is difficult to proceed with this choice of h. So, we further assume that $h(x,\dot{x})$ is a polynomial in $\dot{x}$. To begin with we consider the case in which h is a quadratic function in $\dot{x}$, that is $h = A(x) + B(x)\dot{x} + C(x)\dot{x}^2$. Since R is in rational form while taking differentiation or integration the form of the denominator remains the same but the power of the denominator increases or decreases by a unit order from that of the initial one. So, instead of considering h to be of the form $h = A(x) + B(x)\dot{x} + C(x)\dot{x}^2$, we consider a more general form $h = (A(x) + B(x)\dot{x} + C(x)\dot{x}^2)^r$, where r is a constant to be determined. We note here that this form of ansatz played a crucial role in deducing the time independent integrals of certain dissipative systems, see for example Ref. 20.

Substituting (22) into (21) and solving the resultant equation, we find that the solution exists only for certain specific choices of k_i's. In the following we provide these parametric restrictions with the resultant forms of R:

Case **(i) :**

$$k_5 = k_4(3+2k_1)\rho_1,\ k_6 = k_4(1+k_1)(2+k_1)\rho_1^2,\ \ \rho_1 = \frac{k_2}{k_3(1+k_1)},\ k_1 \neq -1$$

$$R = \begin{cases} \dfrac{\dot{x}}{((2+k_1)\dot{x} + k_3 x(\rho_1(2+k_1)+x))\dot{x} + k_4 x^2(\rho_1(2+k_1)+x)^2}, & k_3^2 < 4k_4(2+k_1) \\ \dfrac{\dot{x}x^{(2-r)k_1}}{[k_3(r-1)(\rho_1(2+k_1)+x)x + (2+k_1)r\dot{x}]^r}, & k_3^2 \geq 4k_4(2+k_1) \end{cases} \quad (23)$$

where $r = \dfrac{k_3^2 \pm k_3\sqrt{k_3^2 - 4k_4(2+k_1)}}{2k_4(2+k_1)}$.

For the choice $k_1 = -1$, we find $k_4 = 0, k_6 = \dfrac{k_2 k_5}{k_3}$, $R = \dfrac{\dot{x}}{k_3 x\dot{x} + k_5 x^2}$ (24)

Case **(ii)** **:**

$$k_2 = 0,\ k_3 = 0,\ R = \dot{x}x^{2k_1} \tag{25}$$

Case **(iii)** **:**

$$k_1 = 0,\ k_2 = 0,\ k_5 = 0,$$

$$R = \begin{cases} \dfrac{\dot{x}}{2k_4\dot{x}^2 + k_3\dot{x}(k_4x^2 + k_6) + (k_4x^2 + k_6)^2}, & k_3^2 < 8k_4 \\ \dfrac{\dot{x}}{(\dot{x} + \frac{(r-1)}{2r}k_3x^2 + \frac{rk_6}{k_3})^r}, & k_3^2 \geq 8k_4,\ r = \frac{k_3^2 \pm k_3\sqrt{k_3^2 - 8k_4}}{4k_4}. \end{cases} \tag{26}$$

The question now remains as to whether the functions S and R given above satisfy the third of the determining equations (vide Eq. (18)) or not. One can easily check that all the above three sets of functions S and R do indeed satisfy the equation (18). The parametric restrictions given above fix the equation of motion (1) to the following specific forms:

Case **(i)** *Eq. (3):*

$$\ddot{x} + k_1\frac{\dot{x}^2}{x} + k_3(\rho_1(1+k_1) + x)\dot{x} + k_4x^3 + k_4\rho_1(3+2k_1)x^2 + k_4(1+k_1)(2+k_1)\rho_1^2 x = 0,$$

where $\rho_1 = \dfrac{k_2}{k_3(1+k_1)}$

Case **(ii)** *Eq. (8):*

$$\ddot{x} + k_1\frac{\dot{x}^2}{x} + k_4x^3 + k_5x^2 + k_6x = 0.$$

Case **(iii)** *Eq. (9):*

$$\ddot{x} + k_3x\dot{x} + k_4x^3 + k_6x = 0.$$

In the next section, we construct time independent integrals for the above equations. We mention here that the last equation (9) is nothing but the modified Emden equation with additional linear force term which admits a conservative Hamiltonian description for all values of k_3, k_4 and k_6. For the special choice, $k_4 = \frac{k_3^2}{9}$, this equation has been shown to exhibit unusual phenomena like amplitude independent frequency of oscillations and conservative Hamiltonian structure (for more details one may refer 21).

B. Integrals of motion

Having determined the explicit forms of S and R, one can proceed to construct the integrals of motion using the expression (19) for the above cases. Substituting the corresponding forms of S and R into the general form of the integral of motion (19) and evaluating the resultant integrals, we obtain the following time independent integrals for the above three cases:

Case **(i)**:

$$I_1 = \begin{cases} \log\Big[x^{2k_1}\{(2+k_1)\dot{x}^2 + [(2+k_1)\rho_1 + x][k_3 x\dot{x} \\ \qquad +k_4 x^2((2+k_1)\rho_1 + x)]\}\Big] - \frac{2k_3 \tan^{-1}\left[\frac{\Phi(x,\dot{x})}{\Omega(x)}\right]}{\sqrt{4k_4(2+k_1)-k_3^2}}, & k_3^2 < 4k_4(2+k_1) \\ \dfrac{x^{k_1(2-r)}[k_3 x((2+k_1)\rho_1 + x) + (2+k_1)r\dot{x}]}{\{k_3(r-1)x((2+k_1)\rho_1+x) + (2+k_1)r\dot{x}\}^{r-1}}, & k_3^2 > 4k_4(2+k_1) \\ \log\left[\left(k_3 x[(2+k_1)\rho_1 + x] + 2(2+k_1)\dot{x}\right)x^{k_1}\right] \\ \qquad -\dfrac{2(2+k_1)\dot{x}}{k_3 x((2+k_1)\rho_1 + x) + 2(2+k_1)\dot{x}}, & k_3^2 = 4k_4(2+k_1) \\ k_2\frac{\dot{x}}{x} + k_2 k_3 x + \left(k_2^2 + \frac{k_2 k_5}{k_3}\right)\log[x] - \frac{k_2 k_5}{k_3}\log\left[k_2\dot{x} + \frac{k_2 k_5}{k_3}\right], & k_1 = -1 \end{cases} \tag{27}$$

where $\Phi(x,\dot{x}) = k_3 x^2 + (2+k_1)(k_3 x\rho_1 + 2\dot{x})$,
and $\Omega(x) = \sqrt{4(2+k_1)k_4 - k_3^2}\left[(2+k_1)\rho_1 + x\right]x$.

Case **(ii)**:

$$I_1 = \left(\frac{\dot{x}^2}{x^2} + \frac{k_6}{1+k_1} + \frac{2(2+k_1)k_5 x + (3+2k_1)k_4 x^2}{(2+k_1)(2k_1+3)}\right)x^{2(1+k_1)}, \tag{28}$$

Case **(iii)**:

$$I_1 = \begin{cases} \log[2k_4\dot{x}^2 + k_3(k_4x^2+k_6)\dot{x} + (k_4x^2+k_6)^2] \\ \qquad + \dfrac{2k_3\tan^{-1}\left[\frac{k_3\dot{x}+2(k_4x^2+k_6)}{\dot{x}\sqrt{8k_4-k_3^2}}\right]}{\sqrt{8k_4-k_3^2}}, & k_3^2 < 8k_4 \\ \left(\dot{x} + \frac{(r-1)}{2r}k_3x^2 + \frac{rk_6}{k_3}\right)^{-r}\left\{\dot{x}\left(\dot{x} + \frac{k_3}{2}x^2 + \frac{r^2k_6}{(r-1)k_3}\right) \right. \\ \qquad \left. + \frac{(r-1)}{r^2}\left(\frac{k_3}{2}x^2 + \frac{r^2k_6}{(r-1)k_3}\right)^2\right\}, & k_3^2 > 8k_4 \\ \dfrac{4k_3\dot{x}}{k_3^2x^2 + 4k_3\dot{x} + 8k_6} - \log[k_3^2x^2 + 4k_3\dot{x} + 8k_6], & k_3^2 = 8k_4, \end{cases} \tag{29}$$

Note that in the above the ranges of x and $\dot{x}$ should be so restricted that no multivaluedness occurs. In a recent paper[22] we have shown that the general equation of the form

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + \alpha\frac{f(x)}{g(x)}\dot{x} + \lambda\frac{f(x)}{g(x)^2}\int f(x)dx = 0, \tag{30}$$

is related to the damped harmonic oscillator equation

$$y'' + \alpha y' + \lambda y = 0, \qquad \left(' = \frac{d}{d\tau}\right) \tag{31}$$

through the nonlocal connection

$$y = \int f(x)dx, \;\; d\tau = \frac{f(x)}{g(x)}dt. \tag{32}$$

By suitably choosing $f(x)$ and $g(x)$ one can show that the damped harmonic oscillator gets transformed to Eqs. (3), (8) and (9). By applying this transformation to the time independent integrals of motion of the damped harmonic oscillator one can also obtain the above results. However, it is not possible to obtain the time dependent integrals of motion of Eq. (1) using the above form of nonlocal transformation. We mention here that the time independent integrals for the specific equation (9) (Liénard type equation) can also be deduced by the procedure described in Ref. 23.

IV. TIME INDEPENDENT INTEGRALS OF MOTION: INTEGRABILITY & GENERAL SOLUTION

A. Hamiltonian description and integrability

In the previous subsection we showed that the equations (3), (8) and (9) admit time independent integrals of motion. Interestingly one can interpret these integrals as time independent (but nonstandard) Hamiltonians for the respective systems and they can be treated as conservative systems. In the following we deduce the underlying Hamiltonian structure for the Eqs. (3)-(9) from the first integrals (27)-(29). To do so we assume a Hamiltonian of the form

$$H(x,p) = I(x,\dot{x}) = p\dot{x} - L(x,\dot{x}), \tag{33}$$

where $L(x,\dot{x})$ is the Lagrangian and p is the canonically conjugate momentum. From (33) we get

$$\frac{\partial I}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x}, \tag{34}$$

from which we identify

$$p = \int \frac{I_{\dot{x}}}{\dot{x}} d\dot{x}. \tag{35}$$

It is clear from (35) that once I is known p can be determined in terms of $\dot{x}$ and x and inverting one can express $\dot{x}$ in terms of x and p. Substituting the expression for $\dot{x}$ in terms of p into the expression for I and from (33) one can deduce L. Once p and L are known the same expression (33) can be utilized to derive H. Using this procedure one can deduce the Hamiltonian from the first integrals for all the three Cases (i)-(iii). The Hamiltonians and the corresponding canonical conjugate momenta read as follows:

Case **(i)**:

$$H = \begin{cases} \log\left[x^{k_1}(\rho_1(2+k_1)x + x^2)\sec\left[\frac{\Omega p}{2(2+k_1)}\right]\right] & \\ \quad -\frac{k_3 p}{2(2+k_1)}(\rho_1(2+k_1)x + x^2), & k_3^2 < 4k_4(2+k_1) \\ \text{where } \Omega(x) = \sqrt{4(2+k_1)k_4 - k_3^2}\left[(2+k_1)\rho_1 + x\right]x. & \\ \frac{(r-1)}{(r-2)}\left(px^{-k_1}\right)^{\frac{(r-2)}{(r-1)}} - \frac{k_3 p(r-1)}{r(2+k_1)}(\rho_1(2+k_1)x + x^2), & k_3^2 > 4k_4(2+k_1) \\ \log\left[\frac{x^{k_1}}{p}\right] + \frac{k_3 p x}{2(2+k_1)}((2+k_1)\rho_1 + x), & k_3^2 = 4k_4(2+k_1) \end{cases}$$

where

$$p = \begin{cases} \frac{2(2+k_1)}{\Omega}\tan^{-1}\left[\frac{k_3 x((2+k_1)\rho_1 + x) + 2(2+k_1)\dot{x}}{\Omega}\right], & k_3^2 < 4k_4(2+k_1) \\ \frac{x^{(2-r)k_1}}{\left(\dot{x} + \frac{k_3(r-1)}{r(2+k_1)}((2+k_1)\rho_1 x + x^2)\right)^{r-1}}, & k_2^2 \geq 4k_4(2+k_1). \end{cases} \tag{36}$$

Case **(ii)**:

$$H = x^{2(k_1+1)}\left(\frac{p^2}{4x^{2(2k_1+1)}} + \frac{k_6}{k_1+1} + \frac{x(2(k_1+2)k_5 + (2k_1+3)k_4 x)}{2k_1^2 + 7k_1 + 6}\right), \; p = 2x^{2k_1}\dot{x}. \tag{37}$$

Case **(iii)**:

$$H = \begin{cases} \dfrac{k_3}{4}p\left(x^2+\dfrac{k_6}{k_4}\right) - \log\left[\left(x^2+\dfrac{k_6}{k_4}\right)\sec[\dfrac{\gamma p}{2}(x^2+\dfrac{k_6}{k_4})]\right], & k_3^2 < 8k_4 \\ \dfrac{(r-1)}{(r-2)}p^{\frac{(r-2)}{(r-1)}} - \dfrac{(r-1)}{2r}k_3px^2 - \dfrac{rpk_6}{k_3}, & k_3^2 > 8k_4 \\ \log[p] - \dfrac{k_3}{2}p\left(\dfrac{x^2}{2}+\dfrac{4k_6}{k_3^2}\right), & k_3^2 = 8k_4 \end{cases} \tag{38}$$

where

$$p = \begin{cases} \frac{2}{\gamma(x^2+\frac{k_6}{k_4})}\tan^{-1}\left[\frac{4\dot{x}+k_3(x^2+\frac{k_6}{k_4})}{2\gamma(x^2+\frac{k_6}{k_4})}\right], & k_3^2 < 8k_4,\ \gamma = \frac{1}{2}\sqrt{8k_4-k_3^2} \\ \dot{x} + \frac{(r-1)}{2r}k_3x^2 + \frac{rk_6}{k_3}, & k_3^2 \geq 8k_4. \end{cases} \tag{39}$$

One can check that in all the above cases the second order equivalence of the Hamilton's equation of motion coincides exactly with the associated equations (3), (8) and (9) for the appropriate parametric choices. The existence of the time independent Hamiltonian for the Eqs. (3), (8) and (9) assures us that they are Liouville integrable[4]. However, our target is to go beyond this statement and construct the solutions for these equations. On the other hand, we find that it is difficult to integrate the Hamilton's equations of motion corresponding to the above Hamiltonians except for the second case (vide Eq. (37)) whose solution can be deduced using the standard methods. In order to deduce the solutions we introduce suitable canonical transformations so that the new Hamilton's equations of motion can be integrated. However, we are able to construct the explicit solution of Eq. (3) only for the parametric choice $\rho_1 = 0$, making the resultant equation equivalent to Eq. (7) whose solution is constructed in the subsection V C. Hence we do not discuss the solution of case (i) separately here.

B. General solution

In this sub-section we discuss the method of finding the general solution for the case (ii) and case (iii).

Case (ii):

We find that one can rewrite the first integral (28) straightforwardly into the following quadrature

$$t - t_0 = \int \frac{dx}{\sqrt{I_1 x^{-2k_1} - \frac{k_4(3+2k_1)x^4 + 2k_5(2+k_1)x^3}{2k_1^2+7k_1+6} - \frac{k_6 x^2}{1+k_1}}}. \tag{40}$$

On the other hand the solution of (8) can also be obtained by transforming it to the Bernoulli equation through the transformation $\dot{x} = z(x)$, where x is the new independent variable. The solution of this reduced Bernoulli type equation is again given in terms of quadratures[2].

Case (iii) a): Parametric choice $k_3^2 < 8k_4$

In order to obtain the solution for the under damped case ($k_3^2 < 8k_4$) we use the following canonical transformation,

$$x = -\sqrt{\frac{k_6}{k_4}} \tan\left[\sqrt{\frac{k_6}{k_4}} P\right], \quad p = \frac{k_4 U}{k_6} \cos^2\left[\sqrt{\frac{k_6}{k_4}} P\right], \tag{41}$$

and transform the Hamiltonian $H = \frac{k_3}{4} p \left(x^2 + \frac{k_6}{k_4}\right) - \log\left[\left(x^2 + \frac{k_6}{k_4}\right) \sec[\frac{\gamma p}{2}(x^2 + \frac{k_6}{k_4})]\right]$ into

$$H = \log\left[\frac{k_6}{k_4} \sec^2\left[\sqrt{\frac{k_6}{k_4}} P\right] \sec\left[\frac{\gamma}{2} U\right]\right] - \frac{k_3}{4} U. \tag{42}$$

The corresponding Hamilton equations of motion are

$$\dot{U} = 2\sqrt{\frac{k_6}{k_4}} \tan\left[\sqrt{\frac{k_6}{k_4}} P\right], \quad \dot{P} = \frac{k_3}{4} - \frac{\gamma}{2} \tan\left[\frac{\gamma}{2} U\right]. \tag{43}$$

From the first equation we can express P in terms of U, that is, $P = \sqrt{\frac{k_4}{k_6}} \tan^{-1}\left[\frac{\sqrt{k_4}\dot{U}}{2\sqrt{k_6}}\right]$ and substituting this form of P into (42) we obtain

$$H = \log\left[\frac{(4k_6 + k_4 \dot{U}^2)}{4k_4} \sec\left[\frac{\gamma}{2} U\right]\right] - \frac{k_3 U}{4} \equiv E. \tag{44}$$

Rewriting the last equation (44) for $\dot{U}$ as

$$\dot{U} = 2\sqrt{\exp\left[E + \frac{k_3}{4} U\right] \cos\left[\frac{\gamma}{2} U\right] - \frac{k_6}{k_4}} \tag{45}$$

and integrating we arrive at the following quadrature

$$t - t_0 = \int \frac{dU}{2\sqrt{\exp\left[E + \frac{k_3}{4} U\right] \cos\left[\frac{\gamma}{2} U\right] - \frac{k_6}{k_4}}}. \tag{46}$$

Case (iii) b): Parametric choice $k_3^2 > 8k_4$

In order to obtain the solution for the over damped case ($k_3^2 > 8k_4$), we use the following canonical transformation $x = \frac{P}{U}, \quad p = \frac{U^2}{2}$ and obtain a new Hamiltonian of the form

$$H = \sigma_1 U^{\frac{2}{r_{12}}} - \left(\frac{k_3(r-1)P^2}{4r} + \frac{k_6 r U^2}{2k_3}\right), \tag{47}$$

where $\sigma_1 = \frac{r_{12}}{2^{\frac{1}{r_{12}}}}$. The canonical equations in the new variable assume the form

$$\dot{U} = \frac{k_3 P(1-r)}{2r}, \quad \dot{P} = \frac{k_6 r U}{k_3} - \frac{2U^{\frac{2}{r_{12}}}}{r_{12}U}. \tag{48}$$

From the first relation one can express P in terms of $\dot{U}$, that is $P = \frac{2r\dot{U}}{k_3(1-r)}$. Substituting the latter in the Hamiltonian we get

$$H \equiv E = \sigma_1 U^{\frac{2}{r_{12}}} - \frac{k_6 r}{2k_3}U^2 + \frac{r\dot{U}^2}{k_3(1-r)} \tag{49}$$

which in turn leads us to the quadrature of the form

$$t - t_0 = \left(\frac{k_3(1-r)}{r}\right)^{\frac{1}{2}} \int \frac{dU}{\sqrt{E - \sigma_1 U^{\frac{2}{r_{12}}} + \frac{k_6 r U^2}{2k_3}}} \tag{50}$$

Case (iii) c): Parametric choice $k_3^2 = 8k_4$

Finally, now we focus our attention on the critically damped case ($k_3^2 = 8k_4$). Using the same canonical transformation $x = \frac{P}{U}$, $p = \frac{U^2}{2}$ we rewrite the underlying Hamiltonian (38) as

$$H = 2\log[U] - \frac{k_3}{4}\left(\frac{P^2}{2} + k_6 U^2\right). \tag{51}$$

The corresponding canonical equations turn out to be

$$\dot{U} = \frac{-k_3 P}{4}, \quad \dot{P} = \frac{k_3 k_6}{2} - \frac{2}{U}. \tag{52}$$

Substituting $P = -\frac{4\dot{U}}{k_3}$ in the Hamiltonian, we get

$$H \equiv E = 2\log[U] - \frac{1}{4}k_3 k_4 U^2 - \frac{2}{k_3}\dot{U}^2 \tag{53}$$

Rearranging we get,

$$\dot{U} = \left(\frac{8k_3\log[U] - k_3^2 k_6 U^2 - 4Ek_3}{2}\right)^{\frac{1}{2}}, \tag{54}$$

which upon integrating reduces to the following form of quadrature

$$t - t_0 = \sqrt{2} \int \frac{dU}{\sqrt{8k_3 \log[U] - k_3^2 k_6 U^2 - 4Ek_3}}. \tag{55}$$

Summarizing the results, we find that the nonlinear equations (3), (8) and (9) admit time independent Hamiltonians for all values of the system parameters and can be classified as integrable ones in the Liouville sense. While constructing the solution for these three equations we find that the nonlinear system (8) can be transformed into Abel equation which in turn can be integrated into a quadrature. Solution of Eq. (9) is given in terms of quadrature by applying suitable canonical transformation to its Hamiltonian. By using again canonical transformations the Hamiltonian corresponding to Eq. (3) can be transformed and the canonical equations corresponding to this transformed Hamiltonian can be integrated and the solution is given in terms of quadratures for the parametric choice $\rho_1 = 0$ (see Sec. V C).

V. TIME DEPENDENT INTEGRALS ($I_t \neq 0$)

In this section, we explore the parametric choices for which (1) admits time dependent integrals. The underlying procedure is same as that of the time independent integral case but involves somewhat lengthy calculations. As a first step in this process we derive the null forms and integrating factors corresponding to Eq. (1) by solving the determining Eqs. (16)-(18).

A. Null forms and integrating factors

In the previous section we considered the case $I_t = 0$. As a consequence S turns out to be $\frac{-\phi}{\dot{x}}$. However, in the case $I_t \neq 0$, the function S has to be determined from Eq. (16), that is

$$\begin{aligned} S_t + \dot{x}S_x - \left(\frac{k_1\dot{x}^2}{x} + (k_2 + k_3 x)\dot{x} + k_4 x^3 + k_5 x^2 + k_6 x\right)S_{\dot{x}} = -\frac{k_1\dot{x}^2}{x^2} \\ +k_3\dot{x} + 3k_4x^2 + 2k_5 x + k_6 - S\left(\frac{2k_1\dot{x}}{x} + (k_2 + k_3 x)\right) + S^2. \end{aligned} \tag{56}$$

Since it is too difficult to solve Eq. (56) for its general solution, we seek a particular solution for S, which is sufficient for our purpose. The time independent integral case clearly indicates that S should be in a rational form. To begin with one may consider $S = \frac{f(t,x,\dot{x})}{g(t,x,\dot{x})}$, where f

and g are arbitrary functions. However, it is difficult to solve Eq. (56) with this form of S. So one may assume that f and g are simple polynomials in $\dot{x}$ with coefficients which are arbitrary functions of t and x. We seek a simple rational expression for S in the form

$$S = \frac{a(x,t) + b(x,t)\dot{x} + c(x,t)\dot{x}^2}{d(x,t) + e(x,t)\dot{x} + f(x,t)\dot{x}^2}, \tag{57}$$

where a, b, c, d, e and f are arbitrary functions of x and t which are to be determined. One may also consider a cubic polynomial in $\dot{x}$ both in the numerator and in the denominator. However, the resultant analysis did not yield any new result. So we confine our presentation here to the form (57) only. Substituting (57) into (56) and equating the coefficients of different powers of $\dot{x}$ to zero, we get

$$\begin{aligned}
&-a^2 + a_t d - ad_t + adk_2 - d^2k_6 + adk_3x - 2d^2k_5x - bdk_6x + aek_6x - 3d^2k_4x^2 \\
&\qquad -bdk_5x^2 + aek_5x^2 - bdk_4x^3 + aek_4x^3 = 0, \\
&a_xd - 2ab - ad_x + b_td - bd_t + a_te - ae_t + 2aek_2 - d^2k_3 - 2dek_6 + \frac{2adk_1}{x} + 2aek_3x \\
&\qquad -4dek_5x - 2cdk_6x - 6dek_4x^2 - 2cdk_5x^2 - 2cdk_4x^3 = 0, \\
&b_xd - b^2 - 2ac - bd_x + c_td - cd_t + a_xe - ae_x + b_te - be_t - cdk_2 + bek_2 - 2dek_3 - e^2k_6 \\
&\qquad +\frac{d^2k_1}{x^2} + \frac{bdk_1}{x} + \frac{3aek_1}{x} - cdk_3x + bek_3x - 2e^2k_5x - cek_6x \\
&\qquad -3e^2k_4x^2 - cek_5x^2 - cek_4x^3 = 0, \\
&c_xd - cd_x - 2bc + b_xe - be_x + c_te - ce_t - e^2k_3 + \frac{2dek_1}{x^2} + \frac{2bek_1}{x} = 0, \\
&c_xe - c^2 - ce_x + \frac{e^2k_1}{x^2} + \frac{cek_1}{x} = 0.
\end{aligned} \tag{58}$$

We obtain four parametric choices for which we are able to find nontrivial particular solutions of the above set of coupled partial differential equations. We present the parametric choices and the null forms S in Table-I. We find that the null forms S obtained are incidently independent of time.

Next, the above identified forms of S along with their corresponding parametric restrictions are substituted in the determining equation (17) for R. To solve the resultant equation for R we make use of the ansatz

$$R = \frac{F(t)S_d}{(A(x) + B(x)\dot{x} + C(x)\dot{x}^2)^r}, \tag{59}$$

where S_d is the denominator of S. We demand the above form due to the following reason. To deduce the first integral I we assume a rational form for I, that is, $I = \frac{f(t,x,\dot{x})}{g(t,x,\dot{x})}$, where f and g are arbitrary functions of t, x and $\dot{x}$. From (15), we know that $S = I_x/I_{\dot{x}} = (f_x g - f g_x)/(f_{\dot{x}} g - f g_{\dot{x}})$ and as we find the deduced S forms are independent of time, the numerator and denominator of S should share a common factor which is a function of time alone, that is, the numerator and denominator of S should be of the form $(f_x g - f g_x) = F(t) f_1(x,\dot{x})$ and $(f_{\dot{x}} g - f g_{\dot{x}}) = F(t) g_1(x,\dot{x})$, respectively. Moreover, from the relation $R = I_{\dot{x}} = (f_{\dot{x}} g - f g_{\dot{x}})/g^2$, we find that the numerator of R should be the denominator of S.

On solving the resultant equation, we obtain the integrating factor R. We present the forms of R along with the S forms in Table 1. Once S and R are determined then one has to verify the compatibility of this set (S, R) with the extra constraint Eq. (18). Having verified the compatibility of (S, R), we substitute S and R into Eq. (19) and construct the associated integral of motion. In this way we identify four sets of integrable parametric choices. We present the explicit form of these integrals of motion in Sec V B.

B. Integrals of motion and general solution

Case (i) k_1, k_2, k_3, k_6 : arbitrary, $k_4 = \dfrac{(1+k_1)k_3^2}{(3+2k_1)^2}$, $k_5 = \dfrac{k_2 k_3}{3+2k_1}$

The parametric restriction given above fixes the equation of motion (1) as

$$\ddot{x} + k_1 \frac{\dot{x}^2}{x} + (k_2 + k_3 x)\dot{x} + \frac{k_3^2(1+k_1)}{(3+2k_1)^2} x^3 + \frac{k_2 k_3}{(3+2k_1)} x^2 + k_6 x = 0. \qquad (4)$$

Equation (4) reduces to the generalized modified Emden equation for the parametric choice $k_1 = 0$ whose integrability has been studied in detail in Ref. 5. The integral of motion associated with Eq. (4) for $k_1 \neq 0$ turns out to be

$$\textbf{(ia)} \quad I_1 = e^{\pm\omega t}\left(\frac{\dot{x} + \frac{(k_2 \mp \omega)}{2(1+k_1)} x + \rho_2 x^2}{\dot{x} + \frac{(k_2 \pm \omega)}{2(1+k_1)} x + \rho_2 x^2}\right), \qquad k_1 \neq -1, \omega \neq 0 \qquad (60)$$

$$\textbf{(ib)} \quad I_1 = e^{k_2 t}\left(\frac{k_2(\dot{x} + k_3 x^2) + k_6 x}{x}\right), \qquad k_1 = -1 \qquad (61)$$

$$\textbf{(ic)} \quad I_1 = t - \frac{2(3+2k_1)x}{(3+2k_1)k_2 x + 2(1+k_1)(k_3 x^2 + (3+2k_1)\dot{x})}, \quad \omega = 0 \qquad (62)$$

where $\rho_2 = \frac{k_3}{3+2k_1}$, $\omega = \sqrt{k_2^2 - 4(1+k_1)k_6}$.

TABLE I: Parametric restrictions, null forms S and integrating factors R of $\ddot{x} + k_1\frac{\dot{x}^2}{x} + (k_2 + k_3 x)\dot{x} + k_4 x^3 + k_5 x^2 + k_6 x = 0$

Case	Parametric restriction	Form of S	Form of R
(i)	$k_4 = \frac{(1+k_1)k_3^2}{(3+2k_1)^2}$, $k_5 = \frac{k_2 k_3}{3+2k_1}$. k_1, k_2, k_3 arbitrary	$\frac{k_3 x}{(3+2k_1)} - \frac{\dot{x}}{x}$	$\frac{\pm e^{\pm\omega t} x}{\left(\dot{x}+\frac{1}{2}\frac{k_2\pm\omega}{1+k_1}x+\frac{k_3}{3+2k_1}x^2\right)^2}$ $\omega = \sqrt{k_2^2 - 4(1+k_1)k_6}$
(ii)	$k_5 = \frac{k_3(k_2\pm\omega)}{2(2+k_1)}$, $k_4 = 0$. k_1, k_2, k_3 arbitrary	$\frac{1}{2}(k_2 \mp \omega + 2k_3 x) + k_1\frac{\dot{x}}{x}$	$e^{\frac{1}{2}(k_2\pm\omega)t} x^{k_1}$
(iii)	$k_6 = \frac{2(3+2k_1)k_2^2}{(5+4k_1)^2}$, $k_3 = 0, k_4 = 0$. k_1, k_2 arbitrary	$\frac{\frac{4(1+k_1)k_2^2}{5+4k_1}x^2+(5+4k_1)k_5 x^3+2(1+2k_1)k_2 x\dot{x}+k_1(5+4k_1)\dot{x}^2)}{2k_2 x^2+(5+4k_1)x\dot{x}}$	$e^{\frac{2(3+2k_1)k_2 t}{5+4k_1}} x^{2k_1}\left(2k_2 x + (5+4k_1)\dot{x}\right)$

Case	Parametric restriction	Form of S	Form of R
(iva)	$k_5 = \frac{k_2 k_3}{3+2k_1}$, $k_6 = \frac{(2+k_1)k_2^2}{(3+2k_1)^2}$. k_1, k_2, k_3, k_4 arbitrary	$k_1 \frac{\dot{x}}{x} + k_3 x + \frac{(3+2k_1)k_4 x^3}{(3+2k_1)\dot{x}+k_2 x} + \frac{(1+k_1)k_2}{3+2k_1}$	$\left(k_2 x + (3+2k_1)\dot{x}\right)\Big[(2+k_1)k_2^2 x^2 + (3+2k_1)k_2 x(k_3 x^2 + 2(2+k_1)\dot{x}) +(3+2k_1)^2(k_4 x^4 + \dot{x}(k_3 x^2 + (2+k_1)\dot{x}))\Big]^{-1}$
(ivb)	$k_5 = \frac{k_2 k_3}{3+2k_1}$, $k_6 = \frac{(2+k_1)k_2^2}{(3+2k_1)^2}$, $k_4 = \frac{(r-1)k_3^2}{(2+k_1)r^2}$. k_1, k_2, k_3 arbitrary	$k_1 \frac{\dot{x}}{x} + k_3 x + \frac{(3+2k_1)k_4 x^3}{(3+2k_1)\dot{x}+k_2 x}$	$\frac{e^{\frac{(2-r)(2+k_1)k_2 t}{3+2k_1}}(k_2 x+(3+2k_1)\dot{x})x^{k_1(2-r)}}{\left((2+k_1)r\left(\frac{k_2 x}{3+2k_1}+\dot{x}\right)+k_3(r-1)x^2\right)^r}$

Rewriting (60) - (62) as first order ODEs we find that the resultant equations are of Bernoulli type which can be solved using the standard method[2]. The general solution to (4) in each of these cases turns out to be

$$\textbf{(ia)}\ x(t) = \left\{ c_2 e^{\frac{c_2 t}{2}} \left[\frac{I_1 - e^{\pm\omega t}}{I_1}\right]^{\frac{\pm c_2}{2\omega}} (e^{\pm\omega t} - I_1)^{\frac{\pm c_1}{2\omega}} I_2 - 2\rho_2 \left[\frac{I_1 - e^{\pm\omega t}}{I_1}\right]^{\frac{\pm c_1}{2\omega}} (e^{\pm\omega t} - I_1)^{\frac{\pm c_2}{2\omega}} \right.$$
$$\left. \times F\left[\frac{\pm(c_1 - c_2)}{2\omega}, \frac{\mp c_2}{2\omega}, 1 \mp \frac{c_2}{2\omega}, \frac{e^{\pm\omega t}}{I_1}\right]\right\}^{-1} c_2 \left\{\frac{-(I_1 - e^{\pm\omega t})^2}{I_1}\right\}^{\frac{\pm c_2}{2\omega}}, k_1 \neq -1 \quad (63)$$

$$\textbf{(ib)}\ x(t) = (I_2 e^{\Phi} + k_3 e^{\Phi} \int e^{-\Phi} dt)^{-1}, \qquad k_1 = -1 \qquad (64)$$

$$\textbf{(ic)}\ x(t) = \frac{k_2(3 + 2k_1)\,(-c_3)^{\frac{1}{1+k_1}} \exp\left[\frac{c_3(I_1 - t)}{2}\right]}{(I_1 - t)^{\frac{1}{1+k_1}}} \left(2^{\frac{2+k_1}{1+k_1}} (1 + k_1) k_3 \Gamma\left[\frac{2 + k_1}{1 + k_1}, \frac{c_3(t - I_1)}{2}\right]\right.$$
$$\left. - I_2(3 + 2k_1) k_2 (-c_3)^{\frac{1}{1+k_1}}\right)^{-1}, \ \omega = 0 \qquad (65)$$

where F is the hypergeometric function, Γ is the gamma function, $c_1 = \frac{k_2 \mp \omega}{1+k_1}$, $c_2 = \frac{k_2 \pm \omega}{1+k_1}$, $c_3 = \frac{k_2}{1+k_1}$, $\Phi = \frac{k_2 k_6 t + e^{-k_2 t} I_1}{k_2^2}$, I_1 and I_2 are the integration constants.

Case (ii) k_1**,** k_2**,** k_3 : arbitrary, $k_5 = \frac{k_3(k_2 \pm \omega)}{2(2+k_1)}$, $k_4 = 0$, $\omega = \sqrt{k_2^2 - 4(1 + k_1)k_6}$

The parametric restriction given above fixes the equation of motion (1) as

$$\ddot{x} + k_1 \frac{\dot{x}^2}{x} + (k_2 + k_3 x)\dot{x} + \frac{k_3(k_2 \pm \omega)}{2(2 + k_1)} x^2 + k_6 x = 0. \qquad (10)$$

The system (10) possesses a first integral of the form

$$I_1 = e^{\frac{1}{2}(k_2 \pm \omega)t} x^{k_1} \left(\dot{x} + \frac{k_2 \mp \omega}{2(1 + k_1)} x + \rho_3 x^2\right), \ k_1 \neq -1, \qquad (66)$$

where $\rho_3 = \frac{k_3}{(2+k_1)}$.

For the parametric choice $k_1 = -1$, we obtain the same first integral as we have obtained in Case (ib) above. On the other hand for the choice $k_1 = -2$, Eq. (66) gets reduced to the Riccati equation which in turn can be integrated by the standard methods and the solution can be obtained of the form[2]

$$x(t) = \frac{2c_1 e^{c_1 t}}{I_1 + 2I_2 c_1 e^{2c_1 t} - 2\rho_3 e^{c_1 t}}, \qquad (67)$$

where $c_1 = \frac{1}{2}(k_2 \pm \omega)$, I_1 and I_2 are the integration constants.

For other choices of k_1 one is able to integrate the first integral (66) only by imposing additional parametric restriction. For example, choosing $k_6 = \dfrac{(2 + k_1)k_2^2}{(3 + 2k_1)^2}$, and $k_5 = \dfrac{k_3 k_2(1 + k_1)}{(3 + 2k_1)}$, one is able to transform the time dependent integral of motion (66) into the following time independent integral of motion

$$I_1 = w' + \frac{k_3(1+k_1)}{2+k_1} w^{\frac{2+k_1}{1+k_1}}, \quad ' = \frac{d}{dz}, \tag{68}$$

where w and z are the new dependent and independent variables, respectively and are given by

$$w = e^{\frac{(1+k_1)k_2 t}{3+2k_1}} x^{1+k_1}, \qquad z = -\frac{(3+2k_1)e^{-\frac{k_2 t}{3+2k_1}}}{k_2}. \tag{69}$$

The additional parametric restrictions fix Eq. (10) to the specific form

$$\ddot{x} + k_1 \frac{\dot{x}^2}{x} + (k_2 + k_3 x)\,\dot{x} + \frac{k_3 k_2 (1+k_1)}{(3+2k_1)} x^2 + \frac{(2+k_1)k_2^2}{(3+2k_1)^2} x = 0, \tag{5}$$

Integrating (68) we get,

$$\begin{aligned} z - z_0 &= \int \frac{dw}{I_1 - \frac{k_3(1+k_1)}{2+k_1} w^{\frac{2+k_1}{1+k_1}}}, \\ &= wF\left[\frac{1+k_1}{2+k_1}, 1, \frac{3+2k_1}{2+k_1}, \frac{\hat{k} w^{\frac{2+k_1}{1+k_1}}}{I_1}\right] I_1^{-1}, \end{aligned} \tag{70}$$

where F is the hypergeometric function[24] and $\hat{k} = \frac{k_3(1+k_1)}{2+k_1}$. We also mention here that Eq. (5) contains several sub-cases which are already known to be integrable, see for example Ref. 1.

Case (iii) $k_1,\, k_2,\, k_5$: arbitrary, $k_3 = 0,\, k_4 = 0,\, k_6 = \frac{2(3+2k_1)k_2^2}{(5+4k_1)^2}$

The parametric restriction given above fix the equation of motion (1) as

$$\ddot{x} + k_1 \frac{\dot{x}^2}{x} + \rho_5 (5+4k_1)\dot{x} + k_5 x^2 + 2(3+2k_1)\rho_5^2 x = 0, \tag{6}$$

where $\rho_5 = \frac{k_2}{(5+4k_1)}$. The first integral for this equation is

$$I_1 = x^{2k_1}\left(\dot{x}^2 + 4\rho_5^2 x^2 + \frac{2k_5 x^3}{3+2k_1} + 4\rho_5 x\dot{x}\right) e^{2(3+2k_1)\rho_5 t}. \tag{71}$$

This time independent integral can be transformed into a time independent integral by introducing a transformation of the form

$$w = \frac{1}{\sqrt{2}} x^{1+k_1} e^{2(1+k_1)\rho_5 t},\; z = -\frac{e^{-\rho_5 t}}{\rho_5}. \tag{72}$$

where w and z are new dependent and independent variables, respectively. Rewriting the integral of motion in terms of the new variables, we obtain

$$I_1 = w'^2 + 2^{\left(\frac{2k_1+3}{2(1+k_1)}\right)} \frac{k_5(1+k_1)^2}{(3+2k_1)} w^{\frac{3+2k_1}{(1+k_1)}}. \tag{73}$$

Eq. (73) can be integrated further and one can obtain the general solution as

$$z - z_0 = w\sqrt{\frac{I_1}{I_1 - \hat{k}w^{\frac{3+2k_1}{1+k_1}}}} F\left[\frac{1+k_1}{3+2k_1}, \frac{1}{2}, \frac{4+3k_1}{3+2k_1}, \frac{\hat{k}w^{\frac{3+2k_1}{1+k_1}}}{I_1}\right] \tag{74}$$

where F is the hypergeometric function[24] and $\hat{k} = 2^{\frac{(3+2k_1)}{2(1+k_1)}} \frac{k_5(1+k_1)^2}{(3+2k_1)}$.

Case (iv) k_1, k_2, k_3 : arbitrary, $k_5 = \frac{k_2k_3}{3+2k_1}$, $k_6 = \frac{(2+k_1)k_2^2}{(3+2k_1)^2}$.

The equation of motion in this case turns out to be

$$\ddot{x} + k_1\frac{\dot{x}^2}{x} + (k_2 + k_3x)\dot{x} + k_4x^3 + \frac{k_2k_3}{(3+2k_1)}x^2 + \frac{k_2^2(2+k_1)}{(3+2k_1)^2}x = 0. \tag{7}$$

The first integral reads

(iva) $$I_1 = \log\left[x^{2k_1}\left((2+k_1)\left(\dot{x}+\rho_6x\right)^2 + k_3x^2(\dot{x}+\rho_6x) + k_4x^4\right)\right] - \frac{2k_3\tan^{-1}\left(\frac{2(2+k_1)(\dot{x}+\rho_6x)+k_3x^2}{x^2\sqrt{4(2+k_1)k_4-k_3^2}}\right)}{\sqrt{4(2+k_1)k_4-k_3^2}} + 2(2+k_1)\rho_6t, \quad 4k_4(2+k_1) > k_3^2$$

(ivb) $$I_1 = \frac{x^{k_1(2-r)}\left(k_3x^2 + (2+k_1)r(\dot{x}+x\rho_6)\right)e^{(2+k_1)(2-r)\rho_6t}}{\left(k_3(r-1)x^2 + (2+k_1)r(\dot{x}+x\rho_6)\right)^{r-1}}, \qquad 4k_4(2+k_1) < k_3^2$$

(ivc) $$I_1 = (2+k_1)\rho_6t + \log\left[k_3x^{2+k_1} + 2(2+k_1)x^{k_1}(\dot{x}+\rho_6x)\right] - \frac{2(2+k_1)(\dot{x}+\rho_6x)}{k_3x^2 + 2(2+k_1)(\dot{x}+x\rho_6)}, \qquad 4k_4(2+k_1) = k_3^2,$$

where $\rho_6 = \frac{k_2}{(3+2k_1)}$, $r = \frac{k_3^2 \pm k_3\sqrt{k_3^2-4k_4(2+k_1)}}{2k_4(2+k_1)}$.

In the above forms of I_1, we now introduce the following transformation,

$$w = xe^{\rho_6t}, \quad z = -\frac{e^{-\rho_6t}}{\rho_6}, \tag{75}$$

so that in the new variables the integrals of motion read,

(iva) $$I_1 = \log\left[w^{2k_1}\{(2+k_1)w'^2 + [k_3w^2w' + k_4w^4]\}\right] - \frac{2k_3\tan^{-1}\left[\frac{k_3w^2+2(2+k_1)w'}{w^2\sqrt{4(2+k_1)k_4-k_3^2}}\right]}{\sqrt{4k_4(2+k_1)-k_3^2}}, \tag{76}$$

(ivb) $$I_1 = \frac{w^{k_1(2-r)}[k_3w^2 + (2+k_1)rw']}{\{k_3(r-1)w^2 + (2+k_1)rw'\}^{r-1}}, \tag{77}$$

(ivc) $$I_1 = \log\left[\left(k_3w^2 + 2(2+k_1)w'\right)w^{k_1}\right] - \frac{2(2+k_1)w'}{k_3w^2 + 2(2+k_1)w'}. \tag{78}$$

In terms of the new variables w and z Eq. (7) reduces to the form

$$w'' + k_1 \frac{w'^2}{w} + k_3 w w' + k_4 w^3 = 0 \tag{79}$$

which is equivalent to Eq. (3) with $k_2 = 0$. We find that it is very difficult to integrate the resultant integrals even after removing the time dependent factors. To establish the integrability of Eq. (7) we transform these time independent integrals into time independent Hamiltonian and thereby establish the Liouville integrability of Eq. (7).

By following the procedure given in Sec. IV A we obtain the following Hamiltonian for the above cases, namely

$$\textbf{(iva)}\ H = \log\left[w^{2+k_1}\sec\left[\frac{w^2\sqrt{4(2+k_1)k_4 - k_3^2}p}{2(2+k_1)}\right]\right] - \frac{k_3 p w^2}{2(2+k_1)},\ 4k_4(2+k_1) > k_3^2 \tag{80}$$

$$\textbf{(ivb)}\ H = \frac{(r-1)}{(r-2)}\left(pw^{-k_1}\right)^{\frac{(r-2)}{(r-1)}} - \frac{k_3 p(r-1)}{r(2+k_1)}w^2, \qquad 4k_4(2+k_1) < k_3^2 \tag{81}$$

$$\textbf{(ivc)}\ H = \log\left[\frac{w^{k_1}}{p}\right] + \frac{k_3 p w^2}{2(2+k_1)}, \qquad 4k_4(2+k_1) = k_3^2 \tag{82}$$

where p is the canonical momenta defined by

$$\textbf{(iva)}\ p = \frac{2(2+k_1)}{w^2\sqrt{4(2+k_1)k_4 - k_3^2}}\tan^{-1}\left[\frac{k_3 w^2 + 2(2+k_1)w'}{w^2\sqrt{4(2+k_1)k_4 - k_3^2}}\right], \qquad 4k_4(2+k_1) > k_3^2 \tag{83}$$

$$\textbf{(ivb,c)}\ p = \frac{w^{(2-r)k_1}}{\left(w' + \frac{k_3(r-1)w^2}{r(2+k_1)}\right)^{r-1}}, \qquad 4k_4(2+k_1) \leq k_3^2 \tag{84}$$

The existence of the time independent Hamiltonian confirms that the system (7) is an integrable one. However, in the following we briefly point out the method of integrating the underlying Hamilton equations of motion associated with the Hamiltonians (80) - (82)

C. General solution

To derive the general solution, we use the following canonical transformations.

Case (iv a): Parametric choice $k_3^2 < 4k_4(2+k_1)$

By introducing the canonical transformation, $w = \frac{U}{P}$ and $p = \frac{P^2}{2}$, where U and P are new canonical variables, we transform the Hamiltonian (80) (for the choice $k_3^2 < 4k_4(2+k_1)$) to the form

$$H \equiv E = 4(2+k_1)\log\left[\left(\frac{U}{P}\right)^{2+k_1}\sec\left[\frac{U^2\sqrt{4(2+k_1)k_4 - k_3^2}}{4(2+k_1)}\right]\right] - k_3 U^2. \tag{85}$$

The underlying canonical equations of motion then become

$$U' = \frac{-4(2+k_1)^2}{P}, \qquad \left(' = \frac{d}{dz}\right) \tag{86a}$$

$$P' = 2k_3U - \frac{4(2+k_1)^2}{U} - 2U\left(\sqrt{4(2+k_1)k_4 - k_3^2}\right)\tan\left[\frac{U^2\sqrt{4(2+k_1)k_4-k_3^2}}{4(2+k_1)}\right]. \tag{86b}$$

Eq. (86) can be solved in the following way. Expressing P in terms of $\dot{U}$, by using the Eq. (86a), and substituting it in (85) we obtain

$$E = 4(2+k_1)\log\left[\mu_1(UU')^{2+k_1}\sec\left[\frac{U^2\sqrt{4(2+k_1)k_4-k_3^2}}{4(2+k_1)}\right]\right] - k_3U^2, \tag{87}$$

where $\mu_1 = \frac{4^{-(2+k_1)}(-1)^{k_1}}{(2+k_1)^{2(2+k_1)}}$ and E is an arbitrary constant.

By splitting U' and U in (87),

$$U' = \frac{\exp\left[\frac{E+k_3U^2}{4(2+k_1)^2}\right]}{U\left(\mu_1\sec\left[\frac{U^2\sqrt{4(2+k_1)k_4-k_3^2}}{4(2+k_1)}\right]\right)^{\frac{1}{(2+k_1)}}}, \tag{88}$$

and integrating the above expression, we get

$$\begin{aligned} z - z_0 = \quad & F\left[\frac{1}{2+k_1}, \frac{4a_1+ik_3}{8a_1(2+k_1)}, \frac{4a_1(5+2k_1)+ik_3}{8a_1(2+k_1)}, -e^{2ia_1U^2}\right] \\ & \times \frac{(2+k_1)2^{\frac{3+k_1}{2+k_1}}e^{\frac{-(E+k_3U^2)+2ia_1U^2}{2(2+k_1)}}}{(k_3-4ia_1)\mu_1^{\frac{1}{2+k_1}}} \end{aligned} \tag{89}$$

where $a_1 = \frac{\sqrt{4(2+k_1)k_4-k_3^2}}{4(2+k_1)}$ and F is the hypergeometric function[2].

Case (iv b): Parametric choice $k_3^2 > 4k_4(2+k_1)$

The Hamiltonian (81) (for the choice $k_3^2 > 4k_4(2+k_1)$) can be rewritten in terms of the new canonical variables, $w = \frac{U}{P}$ and $p = \frac{P^2}{2}$, as

$$H = 2(2+k_1)rr_{12}\left(P^{2+k_1}U^{-k_1}\right)^{r_{12}} - 2^{r_{12}}k_3(r-1)U^2, \tag{90}$$

where we have defined $r_{12} = \frac{(r-1)}{(r-2)}$. The Hamilton equations of motion corresponding to the above Hamiltonian are

$$U' = \frac{2(2+k_1)^2rr_{12}^2}{P}\left(\frac{P^{2+k_1}}{U^{k_1}}\right)^{r_{12}}, \tag{91a}$$

$$P' = 2^{r_{12}}k_3(r-1)2U + \frac{2k_1(2+k_1)rr_{12}^2}{U}\left(\frac{P^{(2+k_1)}}{U^{k_1}}\right)^{r_{12}}. \tag{91b}$$

Now we integrate Eq. (91a) by following the same analogy described in the previous subcase. First we rewrite Eq. (91a) for P and obtain

$$P = \left(\frac{U^{k_1 r_{12}} U'}{2(2+k_1)^2 r r_{12}^2}\right)^{\frac{1}{(2+k_1)r_{12}-1}}. \tag{92}$$

$$H = \mu_2 U^{m_1} U'^{m_2} + \mu_3 U^2 \equiv E, \tag{93}$$

where

$$\begin{aligned} \mu_2 &= \frac{2(2+k_1) r r_{12}}{(2(2+k_1)^2 r r_{12}^2)^{m_3 (2+k_1) r_{12}}}, \quad \mu_3 = -2^{r_{12}} k_3 (r-1), \\ m_1 &= k_1 \left((2+k_1) m_3 - r_{12}\right), \qquad m_2 = r_{12}(2+k_1) m_3. \end{aligned}$$

From (93) one can express

$$U' = \left(\frac{E - \mu_3 U^2}{\mu_2 U^{m_1}}\right)^{\frac{1}{m_2}}. \tag{94}$$

Integrating (94) we get

$$z - z_0 = \frac{m_2 U}{m_1 + m_2} \left(\frac{\mu_2 U^{m_1}}{E}\right)^{\frac{1}{m_2}} F\left[\frac{m_1+m_2}{2m_2}, \frac{1}{m_2}, \frac{m_1+3m_2}{2m_2}, \frac{\mu_3 U^2}{E}\right], \tag{95}$$

where F is the hypergeometric function[2] and t_0 is an integration constant.

Case (iv c): Parametric choice $k_3^2 = 4k_4(2+k_1)$

We use the same canonical transformation $w = \frac{U}{P}$, $p = \frac{P^2}{2}$ and rewrite the Hamiltonian (82) with the parametric choice $k_3^2 = 4(2+k_1)$ as

$$H = \log\left[\frac{U^{k_1}}{P^{k_1}}\right] - \log[P^2] + \frac{k_3}{4(2+k_1)} U^2. \tag{96}$$

The associated canonical equations of motion now become

$$U' = -\frac{1}{P}(2+k_1), \tag{97a}$$

$$P' = -\left(\frac{k_1}{U} + \frac{2k_3 U}{4(2+k_1)}\right). \tag{97b}$$

Rewriting (97a) for $P = \frac{(2+k_1)}{-U'}$ and substituting the latter into (96) we get

$$H = \log[U^{k_1}(-4U')^{2+k_1}] + \frac{k_3 U^2}{4(2+k_1)} \equiv E \tag{98}$$

which in turn can be brought to the form

$$U' = -\frac{1}{4}\left(U^{-k_1}\exp[E - \frac{k_3}{4(2+k_1)}U^2]\right)^{\frac{1}{(2+k_1)}}. \tag{99}$$

Now integrating the above equation (99) we get

$$z - z_0 = \frac{\tilde{E}U^{\frac{2(1+k_1)}{2+k_1}}}{\left(\frac{-k_3U^2}{(2+k_1)^2}\right)^{\frac{(1+k_1)}{2+k_1}}}\Gamma\left[\frac{1+k_1}{2+k_1}, \frac{-k_3U^2}{4(2+k_1)^2}\right], \tag{100}$$

where $\tilde{E} = -2^{\frac{4+4k_1}{2+k_1}}\exp\left(\frac{-E}{2+k_1}\right)$ and Γ is the gamma function[24].

We summarize the results obtained in this section. Solving the determining equations (16)-(18) we find that the system (1) admits time dependent integrals for four parametric choices and their respective equations are (4),(5), (6) and (7). For (4) explicit solution is deduced by integrating the corresponding time dependent integral of motion. Using suitable variable transformations, solutions of Eq. (5), Eq. (6) and Eq. (7) are found in an implicit form.

VI. CONNECTION WITH 2D LOTKA-VOLTERRA SYSTEM (LV)

The detailed study made on the integrability of the second order ODE (1) in the previous sections helps one to identify the dynamics of 2D-LV system which we find to be related to the second order ODE (1) under appropriate choice of k_is. The 2D-LV system

$$\begin{aligned} \dot{x} &= x(a_1 + b_{11}x + b_{12}y), \\ \dot{y} &= y(a_2 + b_{21}x + b_{22}y), \end{aligned} \tag{2}$$

models the the population dynamics of two interacting species[3]. Interestingly Lotka-Volterra systems arises in other branches of physics also such as the coupling of waves in laser physics[25] and the evolution of electrons, ions and neutral species in plasma physics. In hydrodynamics they model the convective instability in the Benard problem[26]. Similarly, they appear in the interaction of gases in a background host medium[27]. In the theory of partial differential equation they can be obtained as a discretized form of the Korteweg-de Vries equation[28].

Since the 2D-LV system is a planar dynamical system it is also being thoroughly investigated from a mathematical point of view. Due to the multi-faceted importance of the

2D-LV system, several in-depth and independent studies have been made to classify the integrable cases[7−19]. Integrals of motion of the 2D-LV system (2) have been studied for several parametric choices all of which reduces to any one of the following 3 parametric choices or subcases thereof :

$$a_2 = \frac{a_1 b_{22}(b_{11} - b_{21})}{b_{11}(b_{12} - b_{22})}, \tag{102}$$

$$b_{21} = \frac{b_{11} b_{22}}{b_{12}}, \tag{103}$$

$$a_1 = a_2. \tag{104}$$

for which integrals of motion have been explicitly deduced[9]. Interestingly, we find that all these three cases (see LV 1, LV 8, LV 15 below), in addition to several subcases of the above parametric choices, and their associated integrals of motion can be deduced from the results of Eq. (1) straightforwardly.

In the following we show that Eq. (2) can be transformed to the form (1) and thus the integrable cases of (1) can be correlated with the integrable cases of (2).

A. Transformation

To transform the system (2) to the form of Eq. (1), first we rewrite Eq. (2a) for the variable y as

$$y = \frac{1}{b_{12}}\left(\frac{\dot{x}}{x} - b_{11}x - a_1\right), \; b_{12} \neq 0 \tag{105}$$

Then we substitute the latter into Eq. (2b) and obtain the following equation,

$$\begin{aligned} \ddot{x} - \left(1 + \frac{b_{22}}{b_{12}}\right)\frac{\dot{x}^2}{x} + \left((2b_{11}\frac{b_{22}}{b_{12}} - b_{11} - b_{21})x + (2a_1\frac{b_{22}}{b_{12}} - a_2)\right)\dot{x} + \left(b_{21}b_{11} - \frac{b_{22}}{b_{12}}b_{11}^2\right)x^3 \\ + \left(b_{11}a_2 + b_{21}a_1 - 2a_1 b_{11}\frac{b_{22}}{b_{12}}\right)x^2 + \left(a_1 a_2 - \frac{b_{22}}{b_{12}}a_1^2\right)x = 0. \end{aligned} \tag{106}$$

which is of the same form as (1). Now comparing (106) with Eq. (1) we find the parameters are connected in the following way

$$\begin{aligned} &k_1 = -(1 + \frac{b_{22}}{b_{12}}), \quad k_2 = (2a_1\frac{b_{22}}{b_{12}} - a_2), \quad k_3 = 2b_{11}\frac{b_{22}}{b_{12}} - b_{11} - b_{21}, \\ &k_4 = (b_{21}b_{11} - \frac{b_{22}}{b_{12}}b_{11}^2), \quad k_5 = (b_{11}a_2 + b_{21}a_1 - 2a_1 b_{11}\frac{b_{22}}{b_{12}}), \quad k_6 = (a_1 a_2 - \frac{b_{22}}{b_{12}}a_1^2). \end{aligned} \tag{107}$$

One may note that both the Eqs. (1) and (106) contain six parameters and thus one ends up with six relations connecting them. Here we emphasize that one can obtain the results of the LV equation from the results of Eq. (1) straightforwardly, while it is difficult to deduce all the results pertaining to Eq. (1) from the known results of LV equation. This is illustrated with an example. One of the integrable parametric choices of the LV equation (2) is $a_2 = \frac{a_1 b_{22}(b_{11}-b_{21})}{b_{11}(b_{12}-b_{22})}$. Upon substituting the above parametric choice in the relations (107) and solving for k_i's we get, $k_5 = \frac{k_2 k_4(3+2k_1)}{k_3(1+k_1)}$, $k_6 = \frac{k_4 k_2^2(2+k_1)}{k_3^2(1+k_1)}$ and $b_{11} = \frac{-k_3 \pm \sqrt{k_3^2-4(2+k_1)k_4}}{2(2+k_1)}$ where b_{11} is a real and arbitrary constant. We find that this b_{11} is equivalent to $r = \frac{-k_3 \pm \sqrt{k_3^2-4(2+k_1)k_4}}{2(2+k_1)}$ of the parametric choice $k_3^2 > 4(2+k_1)$. However, we have obtained integrals of motion for all the three parametric choices $k_3^2 \geq 4(2+k_1)$ and $k_3^2 < 4(2+k_1)$ which cannot be obtained going back from the results of LV.

VII. RESULTS IN TERMS OF LOTKA-VOLTERRA EQUATION PARAMETERS

Now we rewrite the results obtained in Secs. III - V in terms of the LV parameters using the relation (107). In total we obtain 16 parametric choices in terms of the LV parameters (designated as LV 1 - LV 16) and the results are tabulated in Tables II and III, corresponding to time independent and time dependent integrals, respectively. It may be noted that out of the 16 parametric choices, 15 of them reduce to any one of the three parametric choices LV 1, LV 8 and LV 15 (given in the last columns of Tables II and III), while the remaining one (LV 6) is the uncoupled case. Also we note that no new parametric choice is obtained other than the ones already reported in the literature [9,12,13,15,16] as far as the Lotka-Volterra system (2) is concerned.

To deduce the integrable choices LV 1 - LV 16 from the results of Eq. (1) one can derive the corresponding integrals of motion for each one of the cases from the results of the second order equation and the relation (107). As we pointed out earlier, the integrability of the parametric choices obtained in this analysis have already been established in Ref. [9,12,13,15,16]. In the following we illustrate the procedure to deduce the time independent integral from the results of the second order equation (1) for the parametric choices LV 1 and LV 2. Integrals of motion for the remaining cases can be deduced similarly and the procedure is straightforward. Therefore we do not present the details here.

TABLE II: Parametric cases of Eq. (1) possessing time independent integrals of motion in terms of the LV parameters

Case	Parametric restrictions in Eq. (1)	Parametric restrictions in terms of LV parameters	Integrable choices of the LV equation
(i)	$k_5 = \frac{k_2 k_4(3+2k_1)}{k_3(1+k_1)}$	(i) $(a_2 b_{11}(b_{12}-b_{22}) + a_1(b_{21}-b_{11})b_{22}) = 0,$	**LV 1** $a_2 = \frac{a_1 b_{22}(b_{11}-b_{21})}{b_{11}(b_{12}-b_{22})}$
	$k_6 = \frac{k_4 k_2^2(2+k_1)}{k_3^2(1+k_1)}.$	(ii) $a_1 = a_2,\ b_{21} = 0$	**LV 2** $a_1 = a_2,\ b_{21} = 0$
(ii)	$k_2 = 0,\ k_3 = 0$	$a_2 - \frac{2a_1 b_{22}}{b_{12}} = 0,\ b_{11} + b_{21} - \frac{2b_{11}b_{22}}{b_{12}} = 0$	**LV 3** $a_2 = \frac{2a_1 b_{22}}{b_{12}},\ b_{21} = b_{11}\left(\frac{2b_{22}}{b_{12}} - 1\right)$
(iii)	$k_1 = 0,\ k_2 = 0,$	$\frac{b_{22}}{b_{12}} + 1 = 0,\ 2a_1\frac{b_{22}}{b_{12}} - a_2 = 0,$	**LV 4** $b_{12} = -b_{22},\ a_1 = -\frac{a_2}{2},\ b_{21} = 0$
	$k_5 = 0$	$b_{11}a_2 + b_{21}a_1 - 2a_1 b_{11}\frac{b_{22}}{b_{12}} = 0$	**LV 5** $b_{12} = -b_{22},\ a_1 = a_2 = 0$

TABLE III: Parametric cases of Eq. (1) possessing time dependent integrals of motion in terms of the LV parameters

Case	Parametric restrictions in Eq. (1)	Parametric restrictions in terms of LV parameters	Integrable choices of the LV equation
(i)	$k_4=\frac{(1+k_1)k_3^2}{(3+2k_1)^2}$,	$\frac{b_{21}(b_{11}(b_{12}-2b_{22})+b_{21}b_{22})}{b_{12}-2b_{22}}=0$,	**LV 6** $b_{21}=0,\ b_{12}\neq 0$
	$k_5=\frac{k_2k_3}{3+2k_1}$	$\frac{b_{21}(a_1-a_2)}{b_{12}-2b_{22}}=0$	**LV 7** $a_1=a_2,\ b_{11}=-\frac{b_{21}b_{22}}{b_{12}-2b_{22}}$
(ii)	$k_4=0$,	$b_{11}b_{21}-\frac{b_{11}^2b_{22}}{b_{12}}=0$,	**LV 8** $b_{21}=\frac{b_{11}b_{22}}{b_{12}}$
	$k_5=\frac{k_3(k_2\pm\omega)}{2(2+k_1)}$	$\frac{a_2b_{12}(b_{11}-b_{21})+2a_1(b_{12}b_{21}-b_{11}b_{22})\pm a_2(b_{11}b_{12}+b_{12}b_{21}-2b_{11}b_{22})}{b_{12}-b_{22}}=0$	**LV 9** $a_1=a_2,\ b_{11}=0$, **LV 10** $a_1=0,\ b_{11}=0$
(iii)	$k_3=0,\ k_4=0$,	$\frac{2b_{11}b_{22}}{b_{12}}-b_{21}-b_{11}=0,\ b_{11}b_{21}-\frac{b_{11}^2b_{22}}{b_{12}}=0$,	**LV 11** $a_1=2a_2,\ b_{11}=0,\ b_{21}=0$
	$k_6=\frac{2(3+2k_1)k_2^2}{(5+4k_1)^2}$	$\frac{(a_1-2a_2)(a_1b_{22}+2a_2b_{22}-a_2b_{12})}{b_{12}-4b_{22}}=0$	**LV 12** $a_1=\frac{a_2(b_{12}-2b_{22})}{b_{22}},\ b_{11}=0,\ b_{21}=0$,
			LV 13 $a_1=2a_2,\ b_{11}=b_{21},\ b_{12}=b_{22}$
			LV 14 $a_1=-a_2,\ b_{11}=b_{21},\ b_{12}=b_{22}$
(iv)	$k_5=\frac{k_2k_3}{3+2k_1}$,	$\frac{b_{21}(a_1-a_2)}{b_{12}-2b_{22}}=0$,	**LV 15** $a_1=a_2$
	$k_6=\frac{(2+k_1)k_2^2}{(3+2k_1)^2}$	$\frac{(a_1-a_2)(a_1b_{22}+a_2(b_{22}-b_{12}))}{b_{12}-2b_{22}}=0$	**LV 16** $a_1=\frac{a_2(b_{12}-b_{22})}{b_{22}},\ b_{21}=0$

Considering the integrals of motion of the second order system (1) we note that case (i) have three types of integrals (vide Eqs. (27)) depending on the values of the parameters. To rewrite these integrals for the first order LV system (2) first we check whether the LV parametric relations LV 1 and LV 2 are consistent with these conditions. While verifying this we find that both the LV systems are subcases of the overdamped parametric choices $k_3^2 > 4k_4(2+k_1)$, see Eq. (27).

Once the respective integral has been identified then one can replace the variable $\dot{x}$ in terms x and y (vide eq (2)) in (27). The integrals of motion for the above two LV cases turn out to be, respectively,

$$\textbf{LV 1}\ I_1 = yx^{\frac{b_{22}(b_{21}-b_{11})}{b_{11}(b_{12}-b_{22})}}(a_1(b_{11}-b_{21})+b_{11}(b_{11}-b_{21})x+b_{11}(b_{12}-b_{22})y)^{\frac{b_{22}b_{11}-b_{12}b_{21}}{b_{11}(b_{12}-b_{22})}} \quad (108)$$

$$\textbf{LV 2}\ I_1 = (a_1+b_{22}y)(b_{11}x+(b_{12}-b_{22})y)^{\frac{b_{22}}{b_{12}-b_{22}}}x^{\frac{-b_{22}}{b_{12}-b_{22}}} \quad (109)$$

We also note that the well known LV equation

$$\dot{x} = x(a_1 - b_{21}y), \quad (110a)$$

$$\dot{y} = y(a_2 + b_{21}x), \quad (110b)$$

is a subcase of LV 1. The integral of motion corresponding to (110) is deduced from the integral (27) (with $k_1 = -1$) as

$$I_1 = b_{21}(x+y) + a_2\log[x] - a_1\log[y]). \quad (111)$$

For the remaining cases LV 3- LV 16, similar analysis can be performed straightforwardly.

VIII. CONCLUSION

In this paper, we have investigated the integrability properties of Eq. (1) and shown that it admits a set of integrable parametric choices. To identify them we have divided our analysis into two categories, that is systems which admit time independent first integrals and equations which possess time dependent integrals. After carrying out the detailed analysis we found that there exists a new equation which admits time independent integral. To interpret this integral as a Hamiltonian we first deduce the corresponding Lagrangian and then construct the Hamiltonian using the Legendre transformation. Since we have identified a conservative Hamiltonian description for a dissipative system, we expect the study can be extended to the quantum case as well in future. The other two systems which admit time independent integrals are already known in the literature. However we have also given the Hamiltonian description for both of them.

We then moved on to identify the systems which admit time dependent integrals. Our results show that there exist four integrable cases in (1) that admit time dependent integrals. We have also reported the explicit forms of these integrals. For the first three equations we have also found the general solution from these integrals. Since the integral of the fourth equation turned out to be a very complicated one it became difficult to integrate it straightforwardly. So first we have transformed the time dependent integral into a time independent one. Then from the latter we identified a Hamiltonian. We then introduced a canonical transformation to this Hamiltonian and transformed the latter into a relatively simpler Hamiltonian. This Hamiltonian has been integrated to obtain the general solution.

We have transformed the identified integrable choices of the second order equation to the LV system. Out of the 16 LV parametric choices obtained, 15 reduces to any one of the three parametric choice LV 1, LV 8, LV 15. The 16^{th} one, namely LV 6, is an uncoupled case. Interestingly our results reproduce all the known integrable cases of the LV system in the literature.

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