A Group Theoretical Identification of Integrable Equations in the Liénard Type Equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$: Part II: Equations having Maximal Lie Point Symmetries

S. N. Pandey*

Department of Physics, Motilal Nehru National Institute of Technology, Allahabad - 211 004, India

P. S. Bindu, M. Senthilvelan and M. Lakshmanan†

Centre for Nonlinear Dynamics, School of Physics, Bharathidasan University, Tiruchirapalli -620 024, India

(Dated: July 31, 2009)

Abstract

In this second of the set of two papers on Lie symmetry analysis of a class of Liénard type equation of the form $\ddot{x} + f(x)\dot{x} + g(x) = 0$, where over dot denotes differentiation with respect to time and $f(x)$ and $g(x)$ are smooth functions of their variables, we isolate the equations which possess maximal Lie point symmetries. It is well known that any second order nonlinear ordinary differential equation which admits eight parameter Lie point symmetries is linearizable to free particle equation through point transformation. As a consequence all the identified equations turn out to be linearizable. We also show that one can get maximal Lie point symmetries for the above Liénard equation only when $f_{xx} = 0$ (subscript denotes differentiation). In addition, we discuss the linearising transformations and solutions for all the nonlinear equations identified in this paper.

I. INTRODUCTION

The present paper continues the work on the classification of Lie point symmetries described in the previous paper[1] of the second order ordinary differential equation of the Liénard type,

$$A(x,\dot{x},\ddot{x}) \equiv \ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where f and g are smooth functions of x and over-dot denotes differentiation with respect to t. In the first part[1] while solving the determining equations we have assumed the case in which one of the symmetry functions $b(x) = 0$ and identified the corresponding equations. The question now naturally arises as to what are the invariant equations when $b(x) \neq 0$. In this part we present the answer to this question.

On solving the determining equations with $b(x) \neq 0$ we find that a class of equations possess eight parameter Lie-point symmetries. The forms of f and g which lead to the maximum number of symmetry generators are as follows (in increasing order of generality):

$$\begin{aligned}
&\text{(i)} \quad f = 0, && g = \lambda_1 x + \lambda_2,\\
&\text{(ii)} \quad f = k, && g = \lambda_1 x + \lambda_2,\\
&\text{(iii)} \quad f = kx, && g = \frac{1}{9}k^2x^3 + \lambda_1 x + \lambda_2,\\
&\text{(iv)} \quad f = k_1x + k_2, && g = \frac{1}{9}k_1^2x^3 + \frac{1}{3}k_1k_2x^2 + \lambda_1 x + \lambda_2,
\end{aligned} \tag{2}$$

where k, k_1, k_2, λ_1 and λ_2 are constants. The first case includes the equation of motion of the free particle and simple harmonic oscillator. It is well known that both the equations admit eight symmetry generators[2,3,4] satisfying an $sl(3,R)$ algebra. Similarly, the case (ii) which corresponds to the damped harmonic oscillator which has also been shown to admit $sl(3,R)$ symmetry algebra[5]. From case (iii), by restricting the parameters $\lambda_1 = \lambda_2 = 0$, one can get the modified Emden type equation which is the first nonlinear ODE shown by Mahomed and Leach to admit a rich set of Lie point symmetry generators[6]. Here, we show that the general modified Emden type equation with linear term and constant external forcing also admits eight point symmetry generators. We also report the class of equations which falls under case (iv) admitting eight symmetry generators and this may be considered as the most general form of (1). It is important to note that all these equations are linearizable under point transformations. The explicit forms of the infinitesimal symmetries and their associated generators for the general cases are also reported for the first time, as far as the authors' knowledge goes.

In all the above cases, it turns out that $f_{xx} = 0$. By considering the next higher degree polynomial in the variable x for f, say quadratic, $f = k_1x^2 + k_2x + k_3$, so that $f_{xx} \neq 0$, we get

$$g = \frac{1}{9}k_1^2x^5 + \frac{5}{18}k_1k_2x^4 + \lambda_1x^3 + \lambda_2x^2 + \lambda_3x + \lambda_4,$$

which leads to one symmetry generator only. In general, we prove that for $f_{xx} \neq 0$ one obtains lesser parameter symmetry group only.

The plan of the paper is as follows. In Sec. II we recall the derivation of the determining equations for the infinitesimal symmetries and describe a procedure to solve them. In Sec. III by demanding all the four symmetry functions to be non-zero, we construct systematically the class of functions f and g which admit eight symmetry generators. We start our analysis with $f = 0, g = 0$ in Eq. (1), which lead to the equation of motion of the free particle, and derive all earlier known equations from (1) which admit eight point symmetries. Proceeding further by assuming the function f to be a constant we deduce damped harmonic oscillator equation and its variant equations. We also present the infinitesimal symmetries and their associated generators for these equations. Next, in Sec. IV, by considering the function f to be linear in x, we deduce two new family of nonlinear ODEs which admit eight symmetry generators. In Sec. V, we prove from the symmetry determining equations that when $f_{xx} \neq 0$ only lesser parameter symmetries can exist. We also show the equivalence of our results with the Lie's linearization criterion. Finally in Sec. VI we provide a summary of our results. The solutions of the determining equations encountered in the symmetry analysis of new equations are presented in Appendices A and B.

II. SYMMETRY DETERMINING EQUATION OF (1) WITH $b \neq 0$

To make this paper self contained we recall the essential determining equations very briefly here. The invariance of Eq. (1) under one parameter Lie group of infinitesimal transformations, $\tilde{t} = t + \epsilon\xi(t,x) + O(\epsilon^2)$, $\tilde{x} = x + \epsilon\eta(t,x) + O(\epsilon^2)$, $\epsilon \ll 1$, leads us to four determining equations for the functions ξ and η (vide Eqs. (8)-(11) in Paper I). The first two determining equations can be integrated straightforwardly to yield $\xi = a(t) + b(t)x$, $\eta =$

$\dot{b}x^2 - 2b\Im(x) + c(t)x + d(t)$, where $\Im_x = F(x) = \int_0^x f(x')dx'$ and $\Im_{xx} = f(x)$. The remaining two determining equations, after substituting the forms of ξ and η, read

$$(\dot{b}x^2 - 2b\Im + cx + d)f_x + (\dot{a} + \dot{b}x)f + 3bg - 4\dot{b}F + 3\ddot{b}x + 2\dot{c} - \ddot{a} = 0, \tag{3}$$

and

$$\begin{aligned}(\dot{b}x^2 - 2b\Im + cx + d)g_x - (c - 2\dot{a} - 2bF)g - 2\ddot{b}\Im \\ +(\ddot{b}x^2 - 2\dot{b}\Im + \dot{c}x + \dot{d})f + \dddot{b}x^2 + \ddot{c}x + \ddot{d} = 0.\end{aligned} \tag{4}$$

Since g is linear in (3) we can rewrite (3) to obtain

$$g = \frac{1}{3b}[-(\dot{b}x^2 - 2b\Im + cx + d)f_x - (\dot{a} + \dot{b}x)f + 4\dot{b}F - 3\ddot{b}x - 2\dot{c} + \ddot{a}], \ b \neq 0. \tag{5}$$

While deriving Eq. (5) we assumed that $b \neq 0$. On the other hand one may also consider the case $b = 0$. In fact this was the case of the Paper I. Now we solve the determining equations (3) and (4) with $b \neq 0$ and identify the functions that are invariant under eight parameter Lie point symmetries.

III. MAXIMAL LIE POINT SYMMETRIES OF LIÉNARD TYPE SYSTEMS: LINEAR ODES

To begin with we consider polynomial forms for f in the variable x with increasing degrees and derive the corresponding explicit forms of g through Eq. (5). Substituting the expressions of f and g into Eq. (4) and solving the resultant equation we obtain the infinitesimal symmetries ξ and η which in turn leads to the classification of the class of second order ODEs which admit maximum number of symmetry generators. We find that the maximal number (eight) of symmetries exists only when $f_{xx} = 0$. When $f_{xx} \neq 0$, lesser parameter Lie point symmetries only exist.

A. Linear Undamped Systems: $f = 0$

To start with we consider the simple form $f = 0$. Correspondingly from Eq. (5) we get

$$g = \lambda_1 x + \lambda_2, \tag{6}$$

where λ_1 and λ_2 are arbitrary constants. Substituting Eq. (6) into (4) and solving it consistently we can find the infinitesimal symmetries. However, to classify the results systematically we consider the following subcases, *viz.*, *(i)* $\lambda_1 = \lambda_2 = 0$, *(ii)* $\lambda_2 \neq 0,\ \lambda_1 = 0$, *(iii)* $\lambda_1 \neq 0,\ \lambda_2 = 0$ and *(iv)* $\lambda_1 \neq \lambda_2 \neq 0$.

1. Free particle motion ($\lambda_1 = \lambda_2 = 0$)

When both λ_1 and λ_2 are equal to zero, the function g also becomes zero and so Eq. (1) takes the form

$$\ddot{x} = 0 \tag{7}$$

which is obviously the equation of motion of a free particle. Substituting $f = g = 0$ into Eq. (4) we obtain the equation $\dddot{b}x^2 + \ddot{c}x + \ddot{d} = 0$. Equating the coefficients of $x^i,\ i = 0, 1, 2$, to zero separately one obtains $\dddot{b} = 0$, $\ddot{c} = 0$ and $\ddot{d} = 0$. Solving the later two equations one obtains $c = c_1 + c_2 t$ and $d = d_1 + d_2 t$, where $c_i's$ and $d_i's,\ i = 1, 2$, are arbitrary constants. Similarly from Eq. (3), we obtain $\ddot{b} = 0$ and $\ddot{a} = 2\dot{c}$. Solving these two equations one obtains $b = b_1 + b_2 t$ and $a = a_1 + a_2 t + c_2 t^2$, where $b_i's$ and $a_i's,\ i = 1, 2$, are arbitrary constants. Inserting the forms $a(t), b(t), c(t)$ and $d(t)$ in $\xi = a(t) + b(t)x$ and $\eta = \dot{b}x^2 + c(t)x + d(t)$ one gets

$$\xi = a_1 + a_2 t + c_2 t^2 + (b_1 + b_2 t)x, \quad \eta = d_1 + d_2 t + (c_1 + c_2 t)x + b_2 x^2. \tag{8}$$

The associated eight infinitesimal generators take the following form

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial t}, \quad X_4 = x\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right), \\ &X_5 = x\frac{\partial}{\partial x}, \quad X_6 = t\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right), \quad X_7 = \frac{\partial}{\partial x}, \quad X_8 = t\frac{\partial}{\partial x}, \end{aligned} \tag{9}$$

which satisfy an $sl(3, R)$ algebra. The symmetries and their generators (9) coincide exactly with the known ones given in Refs.2,3.

2. Free falling particle ($\lambda_2, \neq 0,\ \lambda_1 = 0$)

If we choose $\lambda_1 = 0$ and λ_2 as arbitrary in Eq. (6), then from Eq. (1) we get

$$\ddot{x} + \lambda_2 = 0, \tag{10}$$

which is the equation of motion of a free falling particle. Now solving the determining Eq. (4) with $f = 0$ and $g = \lambda_2$ we obtain[2]

$$\begin{aligned}
\xi &= a_1 + a_2 t + (c_2 + \frac{3}{2}\lambda_2 b_1)t^2 + \frac{\lambda_2}{2} b_2 t^3 + (b_1 + b_2 t)x, \\
\eta &= d_1 + d_2 t + \frac{\lambda_2}{2}(c_1 - 2a_2)t^2 - \lambda_2(\lambda_2 b_1 + \frac{1}{2}c_2)t^3 - \frac{\lambda_2^2}{4} b_2 t^4 + (c_1 + c_2 t)x + b_2 x^2, \quad (11)
\end{aligned}$$

where a_i, b_i, c_i and d_i, $i = 1, 2$, are arbitrary constants. Here too, we have the following eight generators[2]

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} - \lambda_2 t^2 \frac{\partial}{\partial x}, \quad X_3 = (x + \frac{3}{2}\lambda_2 t^2)\frac{\partial}{\partial t} - \lambda_2^2 t^3 \frac{\partial}{\partial x}, \\
X_4 &= t(x + \frac{\lambda_2}{2}t^2)\frac{\partial}{\partial t} + (x^2 - \frac{\lambda_2^2}{4}t^4)\frac{\partial}{\partial x}, \quad X_5 = (x + \frac{\lambda_2}{2}t^2)\frac{\partial}{\partial x}, \\
X_6 &= t^2 \frac{\partial}{\partial t} + t(x - \frac{\lambda_2}{2}t^2)\frac{\partial}{\partial x}, \quad X_7 = \frac{\partial}{\partial x}, \quad X_8 = t\frac{\partial}{\partial x}, \quad (12)
\end{aligned}$$

which also form an $sl(3, R)$ algebra.

3. **Free linear harmonic oscillator** ($\lambda_1 \neq 0, \ \lambda_2 = 0$)

We consider λ_1 as arbitrary and $\lambda_2 = 0$ so that Eq. (1) becomes

$$\ddot{x} + \lambda_1 x = 0, \quad (13)$$

which is nothing but the equation of a linear harmonic oscillator (for $\lambda_1 > 0$). Solving Eq. (4) with $f = 0$ and $g = \lambda_1 x$ we get the following well known infinitesimal symmetry transformations[2]

$$\begin{aligned}
\xi &= a_1 + a_2 \sin 2\alpha t + a_3 \cos 2\alpha t + (b_1 \sin \alpha t + b_2 \cos \alpha t)x, \quad (14) \\
\eta &= d_1 \sin \alpha t + d_2 \cos \alpha t + (c_1 + a_2\alpha \cos 2\alpha t - a_3\alpha \sin 2\alpha t)x + \alpha(b_1 \cos \alpha t - b_2 \sin \alpha t)x^2,
\end{aligned}$$

for Eq. (13) where $\alpha^2 = \lambda_1$ and a_i, b_j, d_j, $i = 1, 2, 3$, $j = 1, 2$ and c_1 are arbitrary constants. The associated generators forming the $sl(3, R)$ algebra are[2]

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \sin 2\alpha t \frac{\partial}{\partial t} + \alpha x \cos 2\alpha t \frac{\partial}{\partial x}, X_3 = \cos 2\alpha t \frac{\partial}{\partial t} - \alpha x \sin 2\alpha t \frac{\partial}{\partial x}, \\
X_4 &= x(\sin \alpha t \frac{\partial}{\partial t} + \alpha x \cos \alpha t)\frac{\partial}{\partial x}, X_5 = x(\cos \alpha t \frac{\partial}{\partial t} - \alpha x \sin \alpha t)\frac{\partial}{\partial x}, \\
X_6 &= x\frac{\partial}{\partial x}, \quad X_7 = \sin \alpha t \frac{\partial}{\partial x}, \quad X_8 = \cos \alpha t \frac{\partial}{\partial x}. \quad (15)
\end{aligned}$$

On the other hand in the case $\lambda_1 < 0$, Eq. (13) becomes the repulsive harmonic oscillator which is invariant under the following infinitesimal symmetries

$$\begin{aligned}\xi &= a_1 + a_2 e^{2\alpha t} + a_3 e^{-2\alpha t} + (b_1 e^{\alpha t} + b_2 e^{-\alpha t})x, \\ \eta &= d_1 e^{\alpha t} + d_2 e^{-\alpha t} + (c_1 + \alpha(a_2 e^{2\alpha t} - a_3 e^{-2\alpha t}))x + \alpha(b_1 e^{\alpha t} - b_2 e^{-\alpha t})x^2, \end{aligned} \tag{16}$$

The associated $sl(3, R)$ symmetry generators are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = e^{2\alpha t}(\frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x}), \quad X_3 = e^{-2\alpha t}(\frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x}), \quad X_4 = x e^{\alpha t}(\frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x}), \\ X_5 &= x e^{-\alpha t}(\frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x}), \quad X_6 = e^{\alpha t}\frac{\partial}{\partial x}, \quad X_7 = e^{-\alpha t}\frac{\partial}{\partial x}, \quad X_8 = x\frac{\partial}{\partial x}. \end{aligned} \tag{17}$$

4. Displaced linear harmonic oscillator ($\lambda_1 \neq \lambda_2 \neq 0$)

Finally, we consider both λ_1 and λ_2 to be arbitrary (not equal to zero) and we have an equation of the form

$$\ddot{x} + \lambda_1 x + \lambda_2 = 0, \tag{18}$$

which corresponds to a displaced simple harmonic motion. Eq. (18) can be transformed to (13) by a simple transformation, $x \rightarrow x' = x + (\frac{\lambda_2}{\lambda_1})$ and consequently the infinitesimal symmetries of the former can be derived from that of Eq. (14).

B. Linear Damped Systems: $f = constant = k$

In the previous subsection, III A, we considered the case $f = 0$ and derived the form of g. Now we fix f = constant = k and deduce the associated form of g. Substituting $f = k$ in Eq. (5), we again obtain g as a linear function in x, that is

$$g = \lambda_1 x + \lambda_2, \tag{19}$$

where λ_1 and λ_2 are arbitrary constants. Consequently one can solve Eq. (4) with (19) to obtain the infinitesimal symmetries. However, to relate the present results with those of the literature, we consider the following four cases separately: (i) $\lambda_1 = \lambda_2 = 0$, (ii) $\lambda_1 \neq 0,\ \lambda_2 = 0$, (iii) $\lambda_2 \neq 0,\ \lambda_1 = 0$, and (iv) $\lambda_1 \neq \lambda_2 \neq 0$.

1. **Free particle in a viscous medium** $(\lambda_1 = \lambda_2 = 0)$

Let $\lambda_1 = \lambda_2 = 0$. Then from Eq. (19) we obtain $g = 0$. Substituting $f = k$ and $g = 0$ in Eq. (1), we get

$$\ddot{x} + k\dot{x} = 0, \tag{20}$$

which corresponds to the equation of a free particle in a viscous medium. Solving Eq. (4) with $f = k$ and $g = 0$ we get

$$\begin{aligned} \xi &= a_1 + \frac{a_2}{k}e^{kt} + \frac{c_2}{k^2}e^{-kt} + (b_1 + \frac{b_2}{k}e^{kt})x, \\ \eta &= d_1 + \frac{d_2}{k}e^{-kt} + (c_1 - \frac{c_2}{k}e^{-kt})x - kb_1x^2, \end{aligned} \tag{21}$$

where a_i, b_i, c_i and d_i, $i = 1, 2$, are arbitrary constants. The corresponding generators satisfying the $sl(3, R)$ algebra are[8]

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \;\; X_2 = \frac{1}{k}e^{kt}\frac{\partial}{\partial t}, \;\; X_3 = x\left(\frac{\partial}{\partial t} - kx\frac{\partial}{\partial x}\right), \;\; X_4 = \frac{x}{k}e^{kt}\frac{\partial}{\partial t}, \\ X_5 &= x\frac{\partial}{\partial x}, \;\; X_6 = \frac{1}{k}e^{-kt}\left(\frac{1}{k}\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}\right), \;\; X_7 = \frac{\partial}{\partial x}, \;\; X_8 = \frac{1}{k}e^{-kt}\frac{\partial}{\partial x}. \end{aligned} \tag{22}$$

2. **Damped linear harmonic oscillator** $(\lambda_1 \neq 0, \;\lambda_2 = 0)$

Considering λ_1 as arbitrary and $\lambda_2 = 0$ in Eq. (19), we obtain from (1), the equation of a damped harmonic oscillator,

$$\ddot{x} + k\dot{x} + \lambda_1 x = 0. \tag{23}$$

The invariance property of this equation has been discussed in detail in Ref.5. Following our procedure we find Eq. (23) to be invariant under the infinitesimal transformation,

$$\begin{aligned} \xi &= a_1 + a_2 \sin 2\beta t - a_3 \cos 2\beta t + e^{\alpha t}(b_1 \cos \beta t + b_2 \sin \beta t)x, \\ \eta &= [b_3 + a_2(\beta \cos 2\beta t - \alpha \sin 2\beta t) + a_3(\beta \sin 2\beta t - \alpha \cos 2\beta t)]\, x \\ &\quad - \left[b_1 e^{\alpha t}(\alpha \cos \beta t + \beta \sin \beta t) + b_2 e^{\alpha t}(\alpha \sin \beta t - \beta \cos \beta t)\right] x^2 \\ &\quad + e^{-\alpha t}(d_1 \cos \beta t + d_2 \sin \beta t), \end{aligned} \tag{24}$$

where $\alpha = \frac{k}{2}$, $\beta = \frac{1}{2}(4\lambda_1 - k^2)^{\frac{1}{2}}$ and $a_i, b_i, d_j, \; i = 1, 2, 3, \; j = 1, 2$, are arbitrary constants.

The $sl(3,R)$ symmetry generators[5] are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \sin 2\beta t\frac{\partial}{\partial t} + x(\beta\cos 2\beta t - \alpha\sin 2\beta t)\frac{\partial}{\partial x}, \\
X_3 &= -\cos 2\beta t\frac{\partial}{\partial t} + x(\beta\sin 2\beta t - \alpha\cos 2\beta t)\frac{\partial}{\partial x}, \\
X_4 &= xe^{\alpha t}\left(\cos\beta t\frac{\partial}{\partial t} - x(\alpha\cos\beta t + \beta\sin\beta t)\frac{\partial}{\partial x}\right), \\
X_5 &= xe^{\alpha t}\left(\sin\beta t\frac{\partial}{\partial t} + x(\beta\cos\beta t - \alpha\sin\beta t)\frac{\partial}{\partial x}\right), \\
X_6 &= x\frac{\partial}{\partial x}, \quad X_7 = e^{-\alpha t}\cos\beta t\frac{\partial}{\partial x}, \quad X_8 = e^{-\alpha t}\sin\beta t\frac{\partial}{\partial x}. \qquad (25)
\end{aligned}$$

In the above derivation, $4\lambda_1 \leq k^2$. Then for the choice $4\lambda_1 = k^2$, we have the following infinitesimal symmetries:

$$\begin{aligned}
\xi = &\quad a_1 + a_2 t + a_3 t^2 + (b_1 + (b_2 + b_3 t)e^{-\frac{k}{2}t})x, \qquad (26)\\
\eta = &\quad -\left(\frac{3}{2}(b_2 + b_3 t) - b_3\right)x^2 e^{-\frac{k}{2}t} + \left(c_1 + (a_3 - \frac{k}{2}a_2)t - \frac{k}{2}a_3 t^2\right)x + (d_1 + d_2 t)e^{\frac{-k}{2}t}.
\end{aligned}$$

The respective infinitesimal vector fields forming an $sl(3,R)$ algebra are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} - \frac{k}{2}tx\frac{\partial}{\partial x}, \quad X_3 = t^2\frac{\partial}{\partial t} + xt(1 - \frac{k}{2}t)\frac{\partial}{\partial x}, \\
X_4 &= xe^{kt}\frac{\partial}{\partial t}, \quad X_5 = xe^{-\frac{k}{2}t}\left(\frac{\partial}{\partial t} - \frac{3}{2}kx\frac{\partial}{\partial x}\right), \quad X_6 = xe^{-\frac{k}{2}t}\left(t\frac{\partial}{\partial t} - \frac{3}{2}kx\frac{\partial}{\partial x}\right), \\
X_7 &= x\frac{\partial}{\partial x}, \quad X_8 = e^{-\frac{k}{2}t}\frac{\partial}{\partial x}. \qquad (27)
\end{aligned}$$

Finally the choice $4\lambda_1 > k^2$ leads to the following infinitesimal symmetries:

$$\begin{aligned}
\xi &= a_1 + \frac{a_2}{\beta}e^{\beta t} - \frac{a_3}{\beta}e^{-\beta t} + (b_1 e^{(\alpha+\beta)t} + b_2 e^{(\alpha-\beta)t})x, \\
\eta &= \left((\beta - \alpha)b_1 e^{(\alpha+\beta)t} - (\beta + \alpha)b_2 e^{(\alpha-\beta)t}\right)x^2 + d_1 e^{(-\alpha+\beta)t} \\
&\quad + \left(c_1 + \frac{1}{2}(1 - \frac{2\alpha}{\beta})a_2 e^{\beta t} + \frac{1}{2}(1 + \frac{2\alpha}{\beta})a_3 e^{-\beta t}\right)x + d_2 e^{-(\alpha+\beta)t}, \qquad (28)
\end{aligned}$$

where $\alpha = \frac{k}{2},\ \beta = \frac{1}{2}(k^2 - 4\lambda_1)^{\frac{1}{2}}$. The associated vector fields forming an $sl(3,R)$ algebra are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = e^{\beta t}\left(\frac{1}{\beta}\frac{\partial}{\partial t} + \frac{1}{2}(1 - \frac{2\alpha}{\beta})x\frac{\partial}{\partial x}\right), \\
X_3 &= e^{-\beta t}\left(-\frac{1}{\beta}\frac{\partial}{\partial t} + \frac{1}{2}(1 + \frac{2\alpha}{\beta})x\frac{\partial}{\partial x}\right), \quad X_4 = e^{(\alpha+\beta)t}\left(x\frac{\partial}{\partial t} + (\beta - \alpha)x^2\frac{\partial}{\partial x}\right), \qquad (29)\\
X_5 &= e^{(\alpha-\beta)t}\left(x\frac{\partial}{\partial t} - (\alpha + \beta)x^2\frac{\partial}{\partial x}\right), \quad X_6 = x\frac{\partial}{\partial x}, \quad X_7 = e^{(-\alpha+\beta)t}\frac{\partial}{\partial x}, \quad X_8 = e^{-(\alpha+\beta)t}\frac{\partial}{\partial x}.
\end{aligned}$$

It is well known that equation (23) admits a time dependent Hamiltonian of the form $H = \frac{p^2}{2m}e^{-kt} + \frac{\lambda_1}{2}x^2e^{kt}$. However, very recently we have proved that the system (23) also admits a conservative Hamiltonian description[9] for all the values of k and λ_1. The explicit form of the Hamiltonian can be deduced from equation (71) of paper I by fixing $\alpha = k$, $\beta = \lambda_1$ and $q = 1$ in it.

3. **Falling particle in a viscous medium** ($\lambda_2 \neq 0,\ \lambda_1 = 0$)

Choosing $\lambda_1 = 0$ and λ_2 to be arbitrary in Eq. (19), we obtain from Eq. (1)

$$\ddot{x} + k\dot{x} + \lambda_2 = 0, \tag{30}$$

which is the equation of a falling particle in a viscous medium. Eq. (30) can be transformed into (20) through the transformation $x \longrightarrow x' = x + (\frac{\lambda_2}{k})t$, and so its infinitesimal symmetries and generators can be derived from (21) and (22) by appropriately replacing the variable x.

4. **Displaced damped harmonic oscillator** ($\lambda_1 \neq \lambda_2 \neq 0$)

Let $\lambda_1 \neq \lambda_2 \neq 0$ in Eq. (19). Then Eq. (1) becomes

$$\ddot{x} + k\dot{x} + \lambda_1 x + \lambda_2 = 0. \tag{31}$$

Here too one can transform Eq. (31) to the damped harmonic oscillator (23) through the transformation $x \longrightarrow x' = x + \frac{\lambda_2}{\lambda_1}$ and so the invariance properties can be analyzed from the infinitesimal symmetries of the damped harmonic oscillator.

IV. MAXIMAL LIE POINT SYMMETRIES OF LIÉNARD TYPE SYSTEMS: NONLINEAR ODES

A. Modified Emden Equations: $f = kx$

From the above study, we find that the choices $f = 0$ and $f =$ constant lead to *linear* ODEs only. Thus to generate nonlinear ODEs which admit maximum number of symmetry generators one needs to consider linear/higher degree polynomial/nonpolynomial forms for the function $f(x)$. To begin with we consider f to be a linear function of x, say $f = kx$, where k is a constant. As a consequence we get (from Eq. (5))

$$g = \frac{1}{9}k^2x^3 + \lambda_1 x + \lambda_2, \tag{32}$$

where λ_1 and λ_2 are arbitrary constants. Substituting the explicit forms of f and g into Eq. (4) and solving it consistently we obtain the symmetry generators of Eq. (1). For this purpose, we consider the following four cases, (i) $\lambda_1 = \lambda_2 = 0$, (ii) $\lambda_1 \neq 0,\ \lambda_2 = 0$, (iii) $\lambda_2 \neq 0,\ \lambda_1 = 0$ and (iv) $\lambda_1 \neq \lambda_2 \neq 0$, separately.

1. Modified Emden equation: $\lambda_1 = \lambda_2 = 0$

For $\lambda_1 = \lambda_2 = 0$, we obtain from Eq. (1) the following nonlinear ODE

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 = 0, \tag{33}$$

whose linearization, invariance and integrability properties have been widely discussed in the recent literature[6,10,11,12,13,14]. From our above analysis, Eq. (33) is found to be invariant under the Lie point symmetries

$$\begin{aligned}
\xi &= a_1 + a_2 t + (c_2 + \frac{k}{2}d_1)t^2 - \frac{k}{6}d_2t^3 + \Bigg[b_1 + b_2 t - \frac{k}{6}(c_1 + a_2)t^2 \\
&\quad - \frac{k}{6}(c_2 + \frac{k}{3}d_1)t^3 + \frac{k^2}{36}d_2t^4\Bigg]x, \\
\eta &= d_1 + d_2 t + (c_1 + c_2 t - \frac{k}{2}d_2t^2)x + \left[b_2 - \frac{k}{3}(c_1 + a_2)t - \frac{k}{2}(c_2 + \frac{k}{3}d_1)t^2 + \frac{k^2}{9}d_2t^3\right]x^2 \\
&\quad - \frac{k}{3}\left[b_1 + b_2 t - \frac{k}{6}(c_1 + a_2)t^2 - \frac{k}{6}(c_2 + \frac{k}{3}d_1)t^3 + \frac{k^2}{36}d_2t^4\right]x^3,
\end{aligned} \tag{34}$$

where a_i, b_i, c_i and d_i, $i = 1, 2$, are arbitrary constants. These infinitesimal symmetries lead to the symmetry generators

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\ X_2 = t(1 - \frac{k}{6}xt)\frac{\partial}{\partial t} - \frac{k}{3}x^2t(1 - \frac{k}{6}xt)\frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial t} - \frac{k}{3}x^3\frac{\partial}{\partial x}, \\
X_4 &= xt\frac{\partial}{\partial t} + x^2(1 - \frac{k}{3}xt)\frac{\partial}{\partial x},\ X_5 = -\frac{k}{6}xt^2\frac{\partial}{\partial t} + x(1 - \frac{k}{3}xt + \frac{k^2}{18}x^2t^2)\frac{\partial}{\partial x}, \\
X_6 &= t^2(1 - \frac{k}{6}xt)\frac{\partial}{\partial t} + xt(1 - \frac{k}{2}xt + \frac{k^2}{18}x^2t^2)\frac{\partial}{\partial x},
\end{aligned}$$

$$
\begin{aligned}
X_7 &= \frac{k}{2}t^2(1-\frac{k}{9}xt)\frac{\partial}{\partial t}+(1-\frac{k^2}{6}t^2x^2+\frac{k^3}{54}t^3x^3)\frac{\partial}{\partial x},\\
X_8 &= -\frac{k}{6}t^3(1-\frac{k}{6}xt)\frac{\partial}{\partial t}+t(1-\frac{k}{2}xt+\frac{k^2}{9}x^2t^2-\frac{k^3}{108}x^3t^3)\frac{\partial}{\partial x},
\end{aligned}
\tag{35}
$$

which also satisfy an $sl(3,R)$ algebra[6]. Eq. (33) has the general solution

$$
x(t)=\frac{t+I_1}{\frac{k}{6}t^2+\frac{I_1k}{3}t+I_2},
\tag{36}
$$

where I_1 and I_2 are two integrals of motion with the explicit forms

$$
I_1=-t+\frac{x}{\frac{k}{3}x^2+\dot{x}},\quad I_2=\frac{k}{6}t^2+\frac{1-\frac{k}{3}tx}{\frac{k}{3}x^2+\dot{x}}.
\tag{37}
$$

2. **Modified Emden equation with linear term** ($\lambda_1\neq 0,\ \lambda_2=0$)

Choosing λ_1 as arbitrary and $\lambda_2=0$ in Eq. (32), we obtain

$$
g=\frac{k^2}{9}x^3+\lambda_1 x.
\tag{38}
$$

As a result, we have a nonlinear ODE of the form

$$
\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x=0,\quad \lambda_1\neq 0.
\tag{39}
$$

Using the forms of f and g in Eq. (4), we obtain the following set of coupled ODEs for the four arbitrary functions a, b, c and d:

$$
\begin{aligned}
&3\ddot{b}+3\lambda_1 b+k\dot{a}+kc=0,\quad 2\dot{c}-\ddot{a}+kd=0,\quad \ddot{c}+2\lambda_1\dot{a}+k\dot{d}=0,\\
&3\dddot{b}+3\lambda_1\dot{b}+3k\dot{c}+k^2d=0,\quad \ddot{d}+\lambda_1 d=0.
\end{aligned}
\tag{40}
$$

Note that the fourth equation in (40) can be obtained from first and second equations and one has effectively only four equations for the four unknowns. Solving equations (40) we find that eight parameter Lie point symmetry groups exist for both the choices (i) $\lambda_1>0$ and (ii) $\lambda_1<0$. In the first case the infinitesimal symmetries read

$$
\begin{aligned}
\xi &= a_1-\frac{c_2}{2\alpha^2}\sin 2\alpha t-\frac{c_3}{2\alpha^2}\cos 2\alpha t-\frac{k}{3\alpha^2}d_1\sin\alpha t-\frac{k}{3\alpha^2}d_2\cos\alpha t+(b_1\sin\alpha t\\
&\quad +b_2\cos\alpha t+\frac{k}{6\alpha^2}a_3\cos 2\alpha t+\frac{k}{6\alpha^2}a_2\sin 2\alpha t-\frac{kc_1}{3\alpha^2})x,\\
\eta &= (c_1+\frac{a_3}{2}\cos 2\alpha t+\frac{a_2}{2}\sin 2\alpha t+\frac{k}{3\alpha}d_1\cos\alpha t-\frac{k}{3\alpha}d_2\sin\alpha t)x\\
&\quad +(b_1\alpha\cos\alpha t-b_2\alpha\sin\alpha t-\frac{k}{3\alpha}a_3\sin 2\alpha t+\frac{k}{3\alpha}a_2\cos 2\alpha t)x^2\\
&\quad -\frac{1}{3}(b_1\sin\alpha t+b_2\cos\alpha t+\frac{k}{6\alpha^2}a_3\cos 2\alpha t+\frac{k}{6\alpha^2}a_2\sin 2\alpha t-\frac{kc_1}{3\alpha^2})x^3\\
&\quad +d_1\sin\alpha t+d_2\cos\alpha t,
\end{aligned}
\tag{41}
$$

where $a_1, b_i, c_j, d_i,\ i=1,2,\ j=1,2,3$, are arbitrary constants and $\alpha=\sqrt{\lambda_1}$.

The corresponding symmetry generators are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\ X_2=(\sin 2\alpha t+\frac{k}{3\alpha}x\cos 2\alpha t)\frac{\partial}{\partial t}+x(\alpha\cos 2\alpha t-\frac{2k}{3}x\sin 2\alpha t\\
&\quad -\frac{k^2}{9\alpha}x^2\cos 2\alpha t)\frac{\partial}{\partial x},\\
X_3 &= (\cos 2\alpha t-\frac{k}{3\alpha}x\sin 2\alpha t)\frac{\partial}{\partial t}-x(\alpha\sin 2\alpha t+\frac{2k}{3}x\cos 2\alpha t\\
&\quad -\frac{k^2}{9\alpha}x^2\sin 2\alpha t)\frac{\partial}{\partial x},\\
X_4 &= x\sin\alpha t\frac{\partial}{\partial t}+x^2(\alpha\cos\alpha t-\frac{k}{3}x\sin\alpha t)\frac{\partial}{\partial x},\\
X_5 &= x\cos\alpha t\frac{\partial}{\partial t}-x^2(\alpha\sin\alpha t+\frac{k}{3}x\cos\alpha t)\frac{\partial}{\partial x},\\
X_6 &= x\frac{\partial}{\partial t}-(\frac{3\alpha^2 x}{k}+\frac{1}{3}x^3)\frac{\partial}{\partial x},\quad X_7=-\frac{k}{3\alpha^2}\sin\alpha t\frac{\partial}{\partial t}+(\sin\alpha t+\frac{k}{3\alpha}x\cos\alpha t)\frac{\partial}{\partial x},\\
X_8 &= -\frac{k}{3\alpha^2}\cos\alpha t\frac{\partial}{\partial t}+(\cos\alpha t-\frac{k}{3\alpha}x\sin\alpha t)\frac{\partial}{\partial x}.
\end{aligned}\tag{42}$$

The vector fields (42) can be shown to form an $sl(3,R)$ algebra. For the case (ii) the infinitesimal symmetries involve exponential functions of t, that is

$$\begin{aligned}
\xi &= a_1+a_2e^{2\alpha t}+a_3e^{-2\alpha t}+\frac{k}{3\alpha^2}d_1e^{\alpha t}+\frac{k}{3\alpha^2}d_2e^{-\alpha t}+[\frac{k}{3\alpha^2}c_1+b_1e^{\alpha t}+b_2e^{-\alpha t}\\
&\quad -\frac{k}{3\alpha}(a_2e^{2\alpha t}-a_3e^{-2\alpha t})]x, \qquad (43)\\
\eta &= -\frac{1}{3}\Big[\frac{kc_1}{3\alpha^2}+b_1e^{\alpha t}+b_2e^{-\alpha t}-\frac{k}{3\alpha}(a_2e^{2\alpha t}-a_3e^{-2\alpha t})\Big]x^3+[\alpha b_1e^{\alpha t}-\alpha b_2e^{-\alpha t}-\frac{2k}{3}(a_2e^{2\alpha t}\\
&\quad +a_3e^{-2\alpha t})]x^2+[c_1+\alpha(a_2e^{2\alpha t}-a_3e^{-2\alpha t})-\frac{k}{3\alpha}(d_1e^{\alpha t}-d_2e^{-\alpha t})]x+d_1e^{\alpha t}+d_2e^{-\alpha t},
\end{aligned}$$

where $a_i,\ c_1, b_j, d_j,\ i=1,2,3,\ j=1,2$, are arbitrary constants.

The corresponding symmetry generators, which also satisfy an $sl(3,R)$ algebra, are

$$\begin{aligned}
&X_1=\frac{\partial}{\partial t},\quad X_2=e^{2\alpha t}\Big[\Big(1-\frac{k}{3\alpha}x\Big)\frac{\partial}{\partial t}+\Big[\Big(\frac{k}{9\alpha}x^3-\frac{2k}{3}x^2+\alpha x\Big)\frac{\partial}{\partial x}\Big],\\
&X_3=e^{-2\alpha t}\Big[\Big(1+\frac{k}{3\alpha}x\Big)\frac{\partial}{\partial t}-\Big(\frac{k}{9\alpha}x^3+\frac{2k}{3}x^2+\alpha x\Big)\frac{\partial}{\partial x}\Big],\\
&X_4=x\frac{\partial}{\partial t}-\Big(\frac{k}{3}x^3-\frac{3\alpha^2}{k}x\Big)\frac{\partial}{\partial x},\qquad X_5=e^{\alpha t}\Big[x\frac{\partial}{\partial t}-\Big(\frac{k}{3}x^3-\alpha x^2\Big)\frac{\partial}{\partial x}\Big],\\
&X_6=e^{-\alpha t}\Big[x\frac{\partial}{\partial t}-\Big(\frac{k}{3}x^3+\alpha x^2\Big)\frac{\partial}{\partial x}\Big],\qquad X_7=e^{\alpha t}\Big[\frac{\partial}{\partial t}-\Big(\alpha x-\frac{3\alpha^2}{k}\Big)\frac{\partial}{\partial x}\Big],\\
&X_8=e^{-\alpha t}\Big[\frac{\partial}{\partial t}+\Big(\alpha x+\frac{3\alpha^2}{k}\Big)\frac{\partial}{\partial x}\Big].
\end{aligned}\tag{44}$$

The nonlinear oscillator equation (39) admits several unusual dynamical properties. In fact the dissipative system (39) admits a time independent first integral and a conservative Hamiltonian description[15] for all values of k and λ, that is

$$H = \frac{9\lambda^2}{2k^2}\left(2 - 2(1 - \frac{2kp}{3\lambda})^{\frac{1}{2}} + \frac{k^2x^2}{9\lambda} - \frac{2kp}{3\lambda} - \frac{2k^3x^2p}{27\lambda^2}\right), \tag{45}$$

where

$$p = \frac{\partial L}{\partial \dot{x}} = -\frac{27\lambda^3}{2k}\left(\frac{1}{(k\dot{x} + \frac{k^2}{3}x^2 + 3\lambda)^2}\right) + \frac{3\lambda}{2k}. \tag{46}$$

For $\lambda > 0$, the system (39) admits explicit amplitude independent sinusoidal periodic solution of the form

$$x(t) = \frac{A\sin(\omega t + \delta)}{1 - \frac{k}{3\omega}A\cos(\omega t + \delta)}, \quad 0 \leq A \leq \frac{\omega}{k}, \quad \omega = \sqrt{\lambda} \tag{47}$$

and when $\lambda \leq 0$, the solution turn out to be a front like one[15], that is

$$x(t) = \frac{3\beta(I_1 e^{2\beta t} - 1)}{kI_1I_2e^{\beta t} + k(1 + I_1e^{\beta t})}, \tag{48}$$

where $\beta = \sqrt{|\lambda|}$ and A, δ, I_1, I_2 are constants.

3. **Modified Emden equation with constant external forcing** ($\lambda_2 \neq 0,\ \lambda_1 = 0$)

In Eq. (32) let us put $\lambda_1 = 0$ and $\lambda_2 =$ arbitrary so that Eq. (1) becomes

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_2 = 0. \tag{49}$$

Substituting $f = kx$ and $g = \frac{k^2}{9}x^3 + \lambda_2$ in (4), we obtain the following determining equations for the arbitrary functions a, b, c and d:

$$\begin{aligned}
&3\ddot{b} + k\dot{a} + kc = 0, \quad 2\dot{c} - \ddot{a} + 3\lambda_2 b + kd = 0, \quad \ddot{c} + k\dot{d} = 0, \\
&\ddot{d} - \lambda_2 c + 2\lambda_2\dot{a} = 0.
\end{aligned} \tag{50}$$

Solving them consistently one obtains

$$\begin{aligned}
a(t) &= a_1 - k(m_1b_2e^{p_1t} + m_2b_3e^{p_2t} + m_2b_4e^{p_3t} + m_1b_5e^{p_4t} + m_1b_6e^{p_5t} + m_2b_7e^{p_6t}), \\
b(t) &= b_1 + \frac{b_2}{p_1}e^{p_1t} + \frac{b_3}{p_2}e^{p_2t} + \frac{b_4}{p_3}e^{p_3t} + \frac{b_5}{p_4}e^{p_4t} + \frac{b_6}{p_5}e^{p_5t} + \frac{b_7}{p_6}e^{p_6t}, \\
c(t) &= -k(m_2p_1b_2e^{p_1t} + m_1p_2b_3e^{p_2t} + m_1p_3b_4e^{p_3t} + m_2p_4b_5e^{p_4t} + m_2p_5b_6e^{p_5t} + m_1p_6b_7e^{p_6t}), \\
d(t) &= -\frac{3\lambda_2}{k}b_1 + r_1b_2e^{p_1t} + r_2b_3e^{p_2t} + r_3b_4e^{p_3t} + r_4b_5e^{p_4t} + r_5b_6e^{p_5t} + r_6b_7e^{p_6t}.
\end{aligned} \tag{51}$$

The above derivation is presented in Appendix A. Here a_1 and b_i, $i = 1, \ldots, 7$, are integration constants which are also the eight symmetry parameters. The constants p_i and γ_i, where $i = 1, \ldots, 6$, are introduced for simplicity which depend on the system parameters and can be fixed from the relations (A15), and (A20). Finally, the constants m_1 and m_2 are defined in the expression (A19).

Eq. (51) provides us the infinitesimal symmetries of the form

$$\begin{aligned} \xi &= a_1 - k(m_1 b_2 e^{p_1 t} + m_2 b_3 e^{p_2 t} + m_2 b_4 e^{p_3 t} + m_1 b_5 e^{p_4 t} + m_1 b_6 e^{p_5 t} + m_2 b_7 e^{p_6 t}) \\ &\quad + \left(b_1 + \frac{b_2}{p_1} e^{p_1 t} + \frac{b_3}{p_2} e^{p_2 t} + \frac{b_4}{p_3} e^{p_3 t} + \frac{b_5}{p_4} e^{p_4 t} + \frac{b_6}{p_5} e^{p_5 t} + \frac{b_7}{p_6} e^{p_6 t} \right) x, \\ \eta &= (b_2 e^{p_1 t} + b_3 e^{p_2 t} + b_4 e^{p_3 t} + b_5 e^{p_4 t} + b_6 e^{p_5 t} + b_7 e^{p_6 t}) x^2 - \frac{k}{3} \left(b_1 + \frac{b_2}{p_1} e^{p_1 t} \right. \\ &\quad \left. + \frac{b_3}{p_2} e^{p_2 t} + \frac{b_4}{p_3} e^{p_3 t} + \frac{b_5}{p_4} e^{p_4 t} + \frac{b_6}{p_5} e^{p_5 t} + \frac{b_7}{p_6} e^{p_6 t} \right) x^3 - \left(m_2 p_1 b_2 e^{p_1 t} \right. \\ &\quad \left. + m_1 p_2 b_3 e^{p_2 t} + m_1 p_3 b_4 e^{p_3 t} + m_2 p_4 b_5 e^{p_4 t} + m_2 p_5 b_6 e^{p_5 t} + m_1 p_6 b_7 e^{p_6 t} \right) x \\ &\quad - \frac{3\lambda_2}{k} b_1 + r_1 b_2 e^{p_1 t} + r_2 b_3 e^{p_2 t} + r_3 b_4 e^{p_3 t} + r_4 b_5 e^{p_4 t} + r_5 b_6 e^{p_5 t} + r_6 b_7 e^{p_6 t}. \end{aligned} \tag{52}$$

The associated infinitesimal generators are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = x \frac{\partial}{\partial t} - \left(\frac{3}{k} \lambda_2 + \frac{k}{3} x^3 \right) \frac{\partial}{\partial x}, \\ X_3 &= e^{p_1 t} \left[\left(-k m_1 + \frac{x}{p_1} \right) \frac{\partial}{\partial t} + \left(r_1 - m_2 p_1 x + x^2 - \frac{k}{3p_1} x^3 \right) \frac{\partial}{\partial x} \right], \\ X_4 &= e^{p_2 t} \left[\left(-k m_2 + \frac{x}{p_2} \right) \frac{\partial}{\partial t} + \left(r_2 - m_1 p_2 x + x^2 - \frac{k}{3p_2} x^3 \right) \frac{\partial}{\partial x} \right], \\ X_5 &= e^{p_3 t} \left[\left(-k m_2 + \frac{x}{p_3} \right) \frac{\partial}{\partial t} + \left(r_3 - m_1 p_3 x + x^2 - \frac{k}{3p_3} x^3 \right) \frac{\partial}{\partial x} \right], \\ X_6 &= e^{p_4 t} \left[\left(-k m_1 + \frac{x}{p_4} \right) \frac{\partial}{\partial t} + \left(r_4 - m_2 p_4 x + x^2 - \frac{k}{3p_4} x^3 \right) \frac{\partial}{\partial x} \right], \\ X_7 &= e^{p_5 t} \left[\left(-k m_1 + \frac{x}{p_5} \right) \frac{\partial}{\partial t} + \left(r_5 - m_2 p_5 x + x^2 - \frac{k}{3p_5} x^3 \right) \frac{\partial}{\partial x} \right], \\ X_8 &= e^{p_6 t} \left[\left(-k m_2 + \frac{x}{p_6} \right) \frac{\partial}{\partial t} + \left(r_6 - m_1 p_6 x + x^2 - \frac{k}{3p_6} x^3 \right) \frac{\partial}{\partial x} \right]. \end{aligned} \tag{53}$$

The general solution of Eq. (49) can be deduced explicitly following Ref. 16 as

$$x(t) = \frac{-3\alpha}{k} + \frac{1}{3k} \left(\frac{18\alpha^2 (1 - I_1 e^{\pm i\alpha\sqrt{3}t})}{3\alpha(1 - I_1 e^{\pm i\alpha\sqrt{3}t}) \mp 2i\alpha\sqrt{3} I_1 I_2 e^{\frac{3\alpha + i\alpha\sqrt{3}}{2} t} \pm i\alpha\sqrt{3}(1 + I_1 e^{\pm i\alpha\sqrt{3}t})} \right). \tag{54}$$

where α is solution of the equation $\alpha^3+\lambda_1\alpha-\frac{k}{3}\lambda_2=0$ and I_1 and I_2 are two integrals of motion whose explicit forms have also been reported in Ref.16.

4. **Modified Emden equation with linear term and constant external forcing** ($\lambda_1 \neq 0,\ \lambda_2 \neq 0$)

Finally, we choose $\lambda_1 \neq \lambda_2 \neq 0$ in Eq. (32), so that

$$g=\frac{k^2}{9}x^3+\lambda_1 x+\lambda_2.$$

For this case Eq. (1) becomes

$$\ddot{x}+kx\dot{x}+\frac{1}{9}k^2x^3+\lambda_1 x+\lambda_2=0. \tag{55}$$

The determining equations for the infinitesimal symmetries become

$$\begin{aligned}
&3\ddot{b}+k\dot{a}+3\lambda_1 b+kc=0, \quad 2\dot{c}-\ddot{a}+3\lambda_2 b+kd=0,\\
&\ddot{c}+2\lambda_1\dot{a}+k\dot{d}=0,, \quad \ddot{d}-\lambda_2 c+2\lambda_2\dot{a}+\lambda_1 d=0.
\end{aligned} \tag{56}$$

Even though Eq. (56) looks similar to (50), the appearance of additional terms makes it tedious to solve. We solve Eq. (56) with the assumption that $k,\lambda_1,\lambda_2 \neq 0$ to obtain

$$\begin{aligned}
a(t) &= a_1-\frac{2\lambda_1^2}{k^2\lambda_2}b_1+\beta_1 b_2 e^{\alpha_1 t}+\beta_2 b_3 e^{-\alpha_1 t}+b_4 e^{\alpha_2 t}(\beta_3\cos\alpha_3 t+\beta_4\sin\alpha_3 t)\\
&\quad +b_5 e^{\alpha_2 t}(\beta_3\sin\alpha_3 t-\beta_4\cos\alpha_3 t)+b_6 e^{-\alpha_2 t}(\beta_5\cos\alpha_3 t+\beta_4\sin\alpha_3 t)\\
&\quad +b_7 e^{-\alpha_2 t}(\beta_4\cos\alpha_3 t-\beta_5\sin\alpha_3 t),\\
b(t) &= b_1+\frac{b_2}{\alpha_1}e^{\alpha_1 t}-\frac{b_3}{\alpha_1}e^{-\alpha_1 t}+\frac{b_4 e^{\alpha_2 t}}{(\alpha_2^2+\alpha_3^2)}(\alpha_2\cos\alpha_3 t+\alpha_3\sin\alpha_3 t)\\
&\quad +\frac{b_5 e^{\alpha_2 t}}{(\alpha_2^2+\alpha_3^2)}(\alpha_2\sin\alpha_3 t-\alpha_3\cos\alpha_3 t)+\frac{b_6 e^{-\alpha_2 t}}{(\alpha_2^2+\alpha_3^2)}(\alpha_3\sin\alpha_3 t-\alpha_2\cos\alpha_3 t)\\
&\quad +\frac{b_7 e^{-\alpha_2 t}}{(\alpha_2^2+\alpha_3^2)}(\alpha_2\sin\alpha_3 t+\alpha_3\cos\alpha_3 t),\\
c(t) &= -\frac{3\lambda_1}{k}b_1+\gamma_1 b_2 e^{\alpha_1 t}+\gamma_2 b_3 e^{-\alpha_1 t}+b_4 e^{\alpha_2 t}(\gamma_3\cos\alpha_3 t+\gamma_4\sin\alpha_3 t)\\
&\quad +b_5 e^{\alpha_2 t}(\gamma_3\sin\alpha_3 t-\gamma_4\cos\alpha_3 t)+b_6 e^{-\alpha_2 t}(\gamma_5\cos\alpha_3 t+\gamma_6\sin\alpha_3 t)\\
&\quad +b_7 e^{-\alpha_2 t}(\gamma_6\cos\alpha_3 t-\gamma_5\sin\alpha_3 t),\\
d(t) &= -\frac{3\lambda_2}{k}b_1+\delta_1 b_2 e^{\alpha_1 t}+\delta_2 b_3 e^{-\alpha_1 t}+b_4 e^{\alpha_2 t}(\delta_3\cos\alpha_3 t+\delta_4\sin\alpha_3 t)\\
&\quad +b_5 e^{\alpha_2 t}(\delta_3\sin\alpha_3 t-\delta_4\cos\alpha_3 t)+b_6 e^{-\alpha_2 t}(\delta_5\cos\alpha_3 t+\delta_6\sin\alpha_3 t)\\
&\quad +b_7 e^{-\alpha_2 t}(\delta_6\cos\alpha_3 t-\delta_5\sin\alpha_3 t).
\end{aligned} \tag{57}$$

Here a_1 and b_i, $i = 1, \ldots, 7$, are the arbitrary constants arising from the integration of Eq. (56) which fix the eight symmetry generators. The other constants are included for simplicity sake. A detailed analysis on the above derivation and the relation between these constants and system parameters are given in Appendix B.

The infinitesimal symmetries turn out to be

$$\begin{aligned}
\xi \;=\; & a_1 - \frac{2\lambda_1^2}{k^2\lambda_2} b_1 + \beta_1 b_2 e^{\alpha_1 t} + \beta_2 b_3 e^{-\alpha_1 t} + b_4 e^{\alpha_2 t}(\beta_3 \cos\alpha_3 t + \beta_4 \sin\alpha_3 t) + b_5 e^{\alpha_2 t}(\beta_3 \sin\alpha_3 t \\
& - \beta_4 \cos\alpha_3 t) + b_6 e^{-\alpha_2 t}(\beta_5 \cos\alpha_3 t + \beta_4 \sin\alpha_3 t) + b_7 e^{-\alpha_2 t}(\beta_4 \cos\alpha_3 t - \beta_5 \sin\alpha_3 t) \\
& + b_7 e^{-\alpha_2 t}(\beta_4 \cos\alpha_3 t - \beta_5 \sin\alpha_3 t) + \Bigg[b_1 + \frac{b_2}{\alpha_1} e^{\alpha_1 t} - \frac{b_3}{\alpha_1} e^{-\alpha_1 t} + \frac{b_4 e^{\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_2 cos\alpha_3 t \\
& + \alpha_3 sin\alpha_3 t) + \frac{b_5 e^{\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_2 \sin\alpha_3 t - \alpha_3 \cos\alpha_3 t) + \frac{b_6 e^{-\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_3 \sin\alpha_3 t - \alpha_2 \cos\alpha_3 t) \\
& + \frac{b_7 e^{-\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_2 \sin\alpha_3 t + \alpha_3 \cos\alpha_3 t) \Bigg] x,
\end{aligned}$$

$$\begin{aligned}
\eta \;=\; & \frac{k}{3} \Bigg[b_1 + \frac{b_2}{\alpha_1} e^{\alpha_1 t} - \frac{b_3}{\alpha_1} e^{-\alpha_1 t} + \frac{b_4 e^{\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_2 \cos\alpha_3 t + \alpha_3 \sin\alpha_3 t) + \frac{b_5 e^{\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_2 \sin\alpha_3 t \\
& - \alpha_3 \cos\alpha_3 t) + \frac{b_6 e^{-\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_3 \sin\alpha_3 t - \alpha_2 \cos\alpha_3 t) + \frac{b_7 e^{-\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_2 \sin\alpha_3 t \\
& + \alpha_3 \cos\alpha_3 t)] \, x^3 + \Bigg[b_2 e^{\alpha_1 t} + b_3 e^{-\alpha_1 t} + e^{\alpha_2 t}(b_4 \cos\alpha_3 t + b_5 \sin\alpha_3 t) + e^{-\alpha_2 t}(b_6 \cos\alpha_3 t \\
& - b_7 \sin\alpha_3 t) \Bigg] x^2 + \Bigg[- \frac{3\lambda_1}{k} b_1 + \gamma_1 b_2 e^{\alpha_1 t} + \gamma_2 b_3 e^{-\alpha_1 t} + b_4 e^{\alpha_2 t}(\gamma_3 \cos\alpha_3 t + \gamma_4 \sin\alpha_3 t) \\
& + b_5 e^{\alpha_2 t}(\gamma_3 \sin\alpha_3 t - \gamma_4 \cos\alpha_3 t) + b_6 e^{-\alpha_2 t}(\gamma_5 \cos\alpha_3 t + \gamma_6 \sin\alpha_3 t) + b_7 e^{-\alpha_2 t}(\gamma_6 \cos\alpha_3 t \\
& - \gamma_5 \sin\alpha_3 t) \Bigg] x - \frac{3\lambda_2}{k} b_1 + \delta_1 b_2 e^{\alpha_1 t} + \delta_2 b_3 e^{-\alpha_1 t} + b_4 e^{\alpha_2 t}(\delta_3 \cos\alpha_3 t + \delta_4 \sin\alpha_3 t) \\
& + b_5 e^{\alpha_2 t}(\delta_3 \sin\alpha_3 t - \delta_4 \cos\alpha_3 t) + b_6 e^{-\alpha_2 t}(\delta_5 \cos\alpha_3 t + \delta_6 \sin\alpha_3 t) \\
& + b_7 e^{-\alpha_2 t}(\delta_6 \cos\alpha_3 t - \delta_5 \sin\alpha_3 t). \qquad (58)
\end{aligned}$$

The associated generators constituting a $sl(3, R)$ algebra are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \; X_2 = \left(x - \frac{2\lambda_1^2}{k^2\lambda_2^2} \right) \frac{\partial}{\partial t} + \left(-\frac{3\lambda_2}{k} - \frac{3\lambda_1}{k} x + \frac{k}{3} x^3 \right) \frac{\partial}{\partial x}, \\
X_3 &= \left[\left(\beta_1 + \frac{x}{\alpha_1} \right) \frac{\partial}{\partial t} + \left(\delta_1 + \gamma_1 x + x^2 + \frac{k}{3\alpha_1} x^3 \right) \frac{\partial}{\partial x} \right] e^{\alpha_1 t}, \\
X_4 &= \left[\left(\beta_2 - \frac{x}{\alpha_1} \right) \frac{\partial}{\partial t} + \left(\delta_2 + \gamma_2 x + x^2 - \frac{k}{3\alpha_1} x^3 \right) \frac{\partial}{\partial x} \right] e^{-\alpha_1 t},
\end{aligned}$$

$$
\begin{aligned}
X_5 &= \Bigg[\left(\beta_3 \cos\alpha_3 t + \beta_4 \sin\alpha_3 t + \frac{1}{(\alpha_2^2+\alpha_3^3)}(\alpha_2\cos\alpha_3 t + \alpha_3 \sin\alpha_3 t)x\right)\frac{\partial}{\partial t} \\
&+ \big(\delta_3 \cos\alpha_3 t + \delta_4 \sin\alpha_3 t + (\gamma_3 \cos\alpha_3 t + \gamma_4 \sin\alpha_3 t)x + \cos\alpha_3 t x^2 \\
&+ \frac{1}{(\alpha_2^2+\alpha_3^2)}(\alpha_2\cos\alpha_3 t + \alpha_2 \sin\alpha_3 t)x^3\big)\frac{\partial}{\partial x}\Bigg] e^{\alpha_2 t}, \\
X_6 &= \Bigg[\left(\beta_3 \sin\alpha_3 t - \beta_4 \cos\alpha_3 t + \frac{1}{(\alpha_2^2+\alpha_3^3)}(\alpha_2\sin\alpha_3 t - \alpha_3 \cos\alpha_3 t)x\right)\frac{\partial}{\partial t} \\
&+ \Bigg(-\delta_4 \cos\alpha_3 t + \delta_3 \sin\alpha_3 t + (\gamma_3 \sin\alpha_3 t - \gamma_4 \cos\alpha_3 t)x + \sin\alpha_3 t x^2 \\
&+ \frac{1}{(\alpha_2^2+\alpha_3^2)}(\alpha_2\sin\alpha_3 t - \alpha_3 \cos\alpha_3 t)x^3\Bigg)\frac{\partial}{\partial x}\Bigg] e^{\alpha_2 t}, \\
X_7 &= \Bigg[\left(\beta_5 \cos\alpha_3 t + \beta_4 \sin\alpha_3 t + \frac{1}{(\alpha_2^2+\alpha_3^3)}(\alpha_3\sin\alpha_3 t - \alpha_2 \cos\alpha_3 t)x\right)\frac{\partial}{\partial t} \\
&+ \Bigg(\delta_5 \cos\alpha_3 t + \delta_6 \sin\alpha_3 t + (\gamma_5 \cos\alpha_3 t + \gamma_6 \sin\alpha_3 t)x + \cos\alpha_3 t x^2 \\
&+ \frac{1}{(\alpha_2^2+\alpha_3^2)}(\alpha_3\sin\alpha_3 t - \alpha_2 \cos\alpha_3 t)x^3\Bigg)\frac{\partial}{\partial x}\Bigg] e^{-\alpha_2 t}, \\
X_8 &= \Bigg[\left(\beta_4 \cos\alpha_3 t - \beta_5 \sin\alpha_3 t + \frac{1}{(\alpha_2^2+\alpha_3^3)}(\alpha_2\sin\alpha_3 t + \alpha_3 \cos\alpha_3 t)x\right)\frac{\partial}{\partial t} \\
&+ \Bigg(\delta_6 \cos\alpha_3 t - \delta_5 \sin\alpha_3 t + (\gamma_6 \cos\alpha_3 t - \gamma_5 \sin\alpha_3 t)x - \sin\alpha_3 t x^2 \\
&+ \frac{1}{(\alpha_2^2+\alpha_3^2)}(\alpha_2\sin\alpha_3 t + \alpha_3 \cos\alpha_3 t)x^3\Bigg)\frac{\partial}{\partial x}\Bigg] e^{-\alpha_2 t}.
\end{aligned} \tag{59}
$$

As far the authors knowledge goes *the above symmetry generators are being reported for the first time.* Eq. (55) is also linearizable one as it admits eight parameter Lie point symmetries. The general solution[17] of Eq. (55) reads as

$$
x(t) = \frac{-3\alpha}{k} + \frac{6}{k}\left(\frac{(3\alpha^2+\lambda_1)(1-I_1 e^{\pm i\hat{\alpha}t})}{3(1-I_1e^{\pm\hat{\alpha}t}) - 2(3\alpha^2+\lambda_1)I_2 e^{(-3\alpha\pm\frac{\hat{\alpha}}{2})t} \pm \hat{\alpha}(1-I_1e^{\pm\hat{\alpha}t})}\right) \tag{60}
$$

where

$$
\alpha^3 + \alpha\lambda_1 - k\lambda_2 = 0, \quad \textit{and} \quad \hat{\alpha} = \sqrt{-3\alpha^2 - 4\lambda_1}. \tag{61}
$$

For more details about the derivation of the solution (60) one may refer to Ref.17.

B. Generalized Modified Emden Equations: $f = k_1x + k_2$

Next we consider the case where f as the most general linear function of x, that is $f = k_1x + k_2$, where k_1 and k_2 are arbitrary constants. As a consequence we arrive at

$$g = \frac{k_1^2}{9}x^3 + \frac{k_1k_2}{3}x^2 + \lambda_1x + \lambda_2, \tag{62}$$

where λ_1 and λ_2 are arbitrary constants.

Proceeding in a similar fashion as earlier, we consider the following four cases: (i) $\lambda_1 = \lambda_2 = 0$, (ii) $\lambda_1 \neq 0,\ \lambda_2 = 0$ (iii) $\lambda_2 \neq 0,\ \lambda_1 = 0$, and (iv) $\lambda_1 \neq \lambda_2 \neq 0$, separately. The corresponding equations of motion, respectively, are

$$\ddot{x} + (k_1x + k_2)\dot{x} + \frac{k_1k_2}{3}x^2 + \frac{k_1^2}{9}x^3 = 0. \tag{63}$$

$$\ddot{x} + (k_1x + k_2)\dot{x} + \frac{k_1k_2}{3}x^2 + \frac{k_1^2}{9}x^3 + \lambda_1x = 0. \tag{64}$$

$$\ddot{x} + (k_1x + k_2)\dot{x} + \frac{k_1k_2}{3}x^2 + \frac{k_1^2}{9}x^3 + \lambda_2 = 0. \tag{65}$$

$$\ddot{x} + (k_1x + k_2)\dot{x} + \frac{k_1k_2}{3}x^2 + \frac{k_1^2}{9}x^3 + \lambda_1x + \lambda_2 = 0. \tag{66}$$

However, all the above equations can be transformed straightforwardly to the equations discussed in the previous subsections. For example with the subtitution, $X = x + \frac{k_2}{k_1}$, Eq. (63) can be transformed to the form

$$\ddot{X} + k_1X\dot{X} + \frac{k_1^2}{9}X^3 - \frac{k_2^2}{3}X + \frac{2}{9}\frac{k_2^3}{k_1} = 0. \tag{67}$$

Equation (67) coincides exactly with (55) by restricting the constants $\lambda_1 = -\frac{k_2^2}{3}$ and $\lambda_2 = \frac{2}{9}\frac{k_2^3}{k_1}$ in the latter. As a result the infinitesimal symmetries and their associated symmetry generators can also be derived from the Eqs. (58) and (59).

Eq. (64) can be cast into the form,

$$\ddot{X} + k_1X\dot{X} + \frac{k_1^2}{9}X^3 + \frac{(3\lambda_1 - k_2^2)}{3}X + \frac{(2k_2^3 - 9k_2\lambda_1)}{9k_1} = 0, \tag{68}$$

through the transformation $X = x + \frac{k_2}{k_1}$. Eq. (68) is of the form (55). Therefore, the infinitesimal symmetries and the associated symmetry generators of equation (68) can also be derived from the Eqs. (58) and (59) by appropriately fixing the constants.

In a similar way, one can transform Eqs. (65) and (66) to the following form:

$$\ddot{X} + k_1X\dot{X} + \frac{k_1^3}{9}X^3 - \frac{k_2^2}{3}X + \frac{2k_2^3}{9k_1} + \frac{k_1\lambda_2}{\lambda_1} = 0. \tag{69}$$

$$\ddot{X} + k_1X\dot{X} + \frac{k_1^3}{9}X^3 + (\lambda_1 - \frac{k_2^2}{3})X + \frac{(2k_2^3 + 9k_1\lambda_2 - 9k_2\lambda_1)}{9k_1^2} = 0. \tag{70}$$

respectively. Eqs. (69) and (70) are also of the form (55) and hence we can derive the infinitesimal symmetries directly from the Eq. (58).

V. NON-MAXIMAL LIE POINT SYMMETRIES FOR $f_{xx} \neq 0$

1. General forms of $\mathbf{f}(\mathbf{x})$

One can consider higher degree polynomials or other forms for f such that ($f_{xx} \neq 0$) and classify the equations and their underlying Lie point symmetries. However, our analysis shows that one can get only lesser number of Lie point symmetries than eight whenever $f_{xx} \neq 0$. This can be proved in a rather general way as given below.

Now we assume $f = k_1 x + k_2 + l(x)$, where $l(x)$ is an arbitrary function and investigate the outcome when $l(x) \neq 0$. With this choice, g takes the form (vide Eq. (5))

$$g = \frac{1}{3b}\Big[-(\dot{b}x^2 - 2bf_2 + cx + d)f_x - (\dot{a} + \dot{b}x) + 4\dot{b}f_1 - 3\ddot{b}x - 2\dot{c} + \ddot{a}\Big]. \tag{71}$$

In the above we introduced the notation $f_1 = \int f(x)dx$ and $f_2{=}\int f_1(x)dx{=}\int(\int f(x)dx)dx$. Substituting g and its derivative into Eq. (4) and expanding the latter we arrive at an expression exclusively in terms of $l(x)$, that is

$$\begin{aligned}
&\frac{4}{3}bl\Big(l_4 + \frac{k_2}{2}x^2 + \frac{k_1}{6}x^3\Big)^2 - \frac{c^2lx^2}{3b} - \frac{2cdlx}{3b} - \frac{ld^2}{3b} + \frac{4}{3}l\Big(cx+d\Big)\Big(\frac{k_2}{2}x^2 + \frac{k_1}{6}x^3 + l_4\Big) \\
&\quad + \frac{c\dot{a}}{3b}\Big(k_2 + k_1x + l_2\Big) - \frac{2\dot{a}}{3}\Big(k_2x + \frac{k_1}{2}x^2 + l_3\Big)\Big(k_2 + k_1x + l_2\Big) - \frac{\dot{a}}{b}\Big(k_1 + l_1\Big)\Big(cx + d\Big) \\
&\quad + 2\Big(\frac{k_2}{2}x^2 + \frac{k_1}{6}x^3 + l_4\Big)\Big(k_1 + l_1\Big)\dot{a} - \frac{2\dot{a}}{3b}\Big(k_2 + k_1x + l_2\Big) - \frac{4\dot{b}c}{3b}\Big(k_2x + \frac{k_1}{2}x^2 + l_3\Big) - \frac{2\dot{a}\ddot{b}x}{b} \\
&\quad + \frac{8}{3}\dot{b}\Big(k_2x + \frac{k_1}{2}x^2 + l_3\Big)^2 + \frac{\dot{b}}{3b}\Big(k_2 + k_1x + l_2\Big)\Big(4cx + 3d\Big) - \frac{\dot{b}cx^2}{b}\Big(k_1 + l_1\Big) - \frac{\dot{b}dx}{b}\Big(k_1 + l_1\Big) \\
&\quad - 4\Big(\frac{k_2}{2}x^2 + \frac{k_1}{6}x^3 + l_4\Big)\Big(k_2 + k_1x + l_2\Big)\dot{b} - \frac{2}{3}x\Big(k_2x + \frac{k_1}{2}x^2 + l_3\Big)\Big(k_2 + k_1x + l_2\Big)\dot{b} - \frac{4\dot{a}\dot{c}}{3b} \\
&\quad + 2x\Big(\frac{k_2}{2}x^2 + \frac{k_1}{6}x^3 + l_4\Big)\Big(k_1 + l_1\Big)\dot{b} - \frac{2}{3}\dot{b}x^2l\Big(cx + d\Big) + \frac{4}{3}x^2l\dot{b}\Big(\frac{k_2}{2}x^2 + \frac{k_1}{6}x^3 + l_4\Big) - \frac{x^2\dot{b}\ddot{b}}{b} \\
&\quad + \frac{8\dot{a}\dot{b}}{3b}\Big(k_2x + \frac{k_1}{2}x^2 + l_3\Big) - \frac{2\dot{a}\dot{b}}{3b}\Big(k_2 + k_1x + l_2\Big)x - \frac{\dot{a}\dot{b}}{b}\Big(k_1 + l_1\Big)x^2 + \frac{\dot{b}^2}{b}\Big(k_2 + k_1x + l_2\Big)x^2 \\
&\quad - \frac{\dot{b}^2x^3}{b}\Big(k_1 + l_1\Big) - \frac{l\dot{b}^2x^4}{3b} + \frac{2c\dot{c}}{3b} - \frac{4}{3}\dot{c}\Big(k_2x + \frac{k_1}{2}x^2 + l_3\Big) + \dot{c}x\Big(k_2 + k_1x + l_2\Big) \\
&\quad + \Big(k_2 + k_1x + l_2\Big)\dot{d} - \frac{c\ddot{a}}{3b} + \frac{2}{3}\Big(k_2x + \frac{k_1}{2}x^2 + l_3\Big)\ddot{a} + \frac{2\dot{a}\ddot{a}}{3b} - \frac{d\ddot{b}}{b} - 2\ddot{b}\Big(k_2x + k_1x^2 + l_3\Big)x \\
&\quad + x^2\Big(k_2 + k_1x + l_2\Big)\ddot{b} + \ddot{c}x + \ddot{d} + \dddot{b}x^2 = 0.
\end{aligned} \tag{72}$$

Again in (72), for the sake of simplicity, we have used the notation $l_1(x) = \int l(x)dx$, $l_2(x) = \int [\int l(x)dx]dx$ and so on. It is clear from the above expression that the first term on the left hand side of Eq. (72), namely $\frac{4}{3}bl\Big(l_4 + \frac{k_2}{2}x^2 + \frac{k_1}{6}x^3\Big)^2$ cannot be balanced by no other term. This in turn leads us to

$$\frac{4}{3}bl\Big(l_4 + \frac{k_2}{2}x^2 + \frac{k_1}{6}x^3\Big)^2 \Longrightarrow b = 0. \tag{73}$$

When $l(x) \neq 0$ the above expression clearly indicates that the symmetry function $b(t)$ should be zero. Since one of the symmetry functions becomes zero one can get ultimately lesser Lie point symmetries only. This has in fact been analyzed in detail in Sec. III of paper I.

On the other hand restricting $l(x) = 0$, that is $f = k_1x + k_2$, then one obtains maximal Lie point symmetries as shown in the earlier sections.

2. Relationship between the symmetries and the forms of $\mathbf{f(x)}$ and $\mathbf{g(x)}$

In our discussions so far, we find that all the examples dealt with in Secs. III and IV admit eight symmetry generators, and each one of the sets form $sl(3, R)$ algebra. It is a well known fact that second order ODEs with eight symmetry generators are linearizable by point transformation[18,19], which was pointed out by Lie himself[20]. In fact the theorem that provides criteria for testing the linearization of a scalar second order ODE $\ddot{x} = f(t, x, \dot{x})$ via a point transformation has the cubic in derivative form[18,19]

$$\ddot{x} + A(t, x)\dot{x}^3 + B(t, x)\dot{x}^2 + C(t, x)\dot{x} + D(t, x) = 0, \tag{74}$$

with the coefficients A, B, C and D satisfying the following two invariant conditions,

$$3A_{tt} + 3A_tC - 3A_xD + 3AC_t + C_{xx} - 6AD_x + BC_x - 2BB_t - 2B_{tx} = 0, \tag{75}$$

$$6A_tD - 3B_xD + 3AD_t + B_{tt} - 2C_{tx} - 3BD_x + 3D_{xx} + 2CC_x - CB_t = 0, \tag{76}$$

where suffixes refer to partial derivatives. On comparing Eq. (1) with Eq. (74), we find that the functions $A(t, x) = B(t, x) = 0$ and $C(t, x)$ and $D(t, x)$ are $f(x)$ and $g(x)$ respectively which are independent of time. Substituting these forms of $C(t, x)$ and $D(t, x)$ along with $A(t, x) = B(t, x) = 0$ in Eqs. (75) and (76), we obtain

$$f_{xx} = 0, \quad 3g_{xx} - 2ff_x = 0. \tag{77}$$

Integrating Eq. (77), we obtain

$$f = k_1 x + k_2, \quad g = \frac{1}{9} k_1 k_2 x^3 + \frac{1}{3} k_2^2 x^2 + \lambda_1 x + \lambda_2, \tag{78}$$

where λ_1, λ_2 are integration constants. Thus for the function $f_{xx} \neq 0$, one obtains lesser parameter symmetries only . However, our analysis leads to a more general form from which one can determine systems exhibiting lesser parameter symmetry groups. Thus this analysis broadens our knowledge on both integrable and linearizable equations belonging to Eq. (1).

Note that the above relation (78) is the same as the one obtained from the analysis of the determining equations, Eq. (21) for the Lie symmetries in Sec. II A of paper I for the case $f_{xx} = 0$.

VI. DISCUSSION AND CONCLUSIONS

In this paper, we have studied in some detail the symmetry properties of the Liénard type equation (1), where none of the symmetry functions is equal to zero. We have also isolated the class of equations which possess eight Lie point symmetries. For the sake of completeness we have also given the solutions for all the nonlinear ODEs which we have identified as linearizable in this procedure. We have also indicated the explicit Hamiltonian forms for these equations.

A question now naturally arises is about the Noether symmetries of these equations. As it is well known, Noether's theorem is based upon the invariance of action integral and essentially one needs a Lagrangian to start the analysis. Since our recent studies show that even dissipative systems can have time independent Lagrangian[21] and so it is worth exploring Noether symmetries of these systems as well. Finally, it is also of importance to study the detailed symmetry structure when the functions f and g in (1) are also time dependent. These are being pursued currently.

Acknowledgments

One of us (SNP) is grateful to the Centre for Nonlinear Dynamics, Bharathidasan University, Tiruchirappalli, for warm hospitality. The work of SNP forms part of a Department of Science and Technology, Government of India sponsored research project. The work of

MS forms part of a research project sponsored by the National Board for Higher Mathematics, Government of India. The work of ML forms part of a Department of Science and Technology (DST), Ramanna Fellowship and is also supported by a DST-IRHPA research project.

APPENDIX A: METHOD OF SOLVING THE DETERMINING EQS. (50)

In this section we solve the determining equations (50) to obtain the general form of ξ and η which leaves Eq. (49) invariant. The determining equations to be solved are

$$3\ddot{b} + k\dot{a} + kc = 0, \tag{A1}$$

$$2\dot{c} - \ddot{a} + 3\lambda_2 b + kd = 0, \tag{A2}$$

$$\ddot{c} + k\dot{d} = 0, \tag{A3}$$

$$\ddot{d} - \lambda_2 c + 2\lambda_2 \dot{a} = 0. \tag{A4}$$

In the following, we solve eqs. (A1)-(A4) with the assumption that the parameters $k, \lambda_2 \neq 0$ and deduce the infinitesimal symmetries.

Rewriting (A1) we get

$$c = -\frac{3}{k}\ddot{b} - \dot{a}. \tag{A5}$$

Substituting (A5) into (A2) and rearranging, we obtain an expression for the function d of the form

$$d = \frac{6}{k^2}\dddot{b} + \frac{3}{k}\ddot{a} - \frac{3\lambda_2}{k}b. \tag{A6}$$

Now substituting the forms of c and d and their derivatives into (A3) and (A4) and simplifying, we get two equations for the unknowns a and b:

$$3\frac{d^4 b}{dt^4} + 2k\frac{d^3 a}{dt^3} - 3k\lambda_2\frac{db}{dt} = 0, \tag{A7}$$

$$6\frac{d^5 b}{dt^5} + 3k\frac{d^4 a}{dt^4} + 3k^2\lambda_2\frac{da}{dt} = 0. \tag{A8}$$

From Eq. (A7) we have

$$\frac{d^3 a}{dt^3} = -\frac{3}{2k}\frac{d^4 b}{dt^4} + \frac{3}{2}\lambda_2\frac{db}{dt}. \tag{A9}$$

Substituting (A9) into (A8) we get

$$\frac{d^5b}{dt^5}+3k\lambda_2\frac{d^2b}{dt^2}+2k^2\lambda_2\frac{da}{dt}=0. \tag{A10}$$

Rearranging the above we obtain

$$\frac{da}{dt}=-\frac{1}{2k^2\lambda_2}\frac{d^5b}{dt^5}-\frac{3}{2k}\frac{d^2b}{dt^2}. \tag{A11}$$

On integration (A11) yields

$$a(t)=a_1-\frac{1}{2k^2\lambda_2}\frac{d^4b}{dt^4}-\frac{3}{2k}\frac{db}{dt}, \tag{A12}$$

where a_1 is an integration constant. Using (A11), Eq. (A7) can be written as

$$\frac{d^7b}{dt^7}+3k^2\lambda_2^2\frac{db}{dt}=0. \tag{A13}$$

A general solution for (A13) takes the form

$$b(t)=b_1+\frac{b_2}{p_1}e^{p_1t}+\frac{b_3}{p_2}e^{p_2t}+\frac{b_4}{p_3}e^{p_3t}+\frac{b_5}{p_4}e^{p_4t}+\frac{b_6}{p_5}e^{p_5t}+\frac{b_7}{p_6}e^{p_6t}, \tag{A14}$$

where $b_i's,\ i=1,\ldots,7$, are integration constants and $p_j's,\ j=1,\ldots,6$, are given by

$$\begin{aligned} p_1 &= (-3)^{\frac{1}{6}}(k\lambda_2)^{\frac{1}{3}},\ \ p_2=-(-3)^{\frac{1}{6}}(k\lambda_2)^{\frac{1}{3}},\ \ p_3=(-1)^{\frac{1}{3}}(-3)^{\frac{1}{6}}(k\lambda_2)^{\frac{1}{3}}, \\ p_4 &= -(-1)^{\frac{1}{3}}(-3)^{\frac{1}{6}}(k\lambda_2)^{\frac{1}{3}},\ \ p_5=(-1)^{\frac{2}{3}}(-3)^{\frac{1}{6}}(k\lambda_2)^{\frac{1}{3}},\ \ p_6=-(-1)^{\frac{2}{3}}(-3)^{\frac{1}{6}}(k\lambda_2)^{\frac{1}{3}}. \end{aligned} \tag{A15}$$

Substituting the form of b given by (A14), and its derivatives into (A12), we get

$$\begin{aligned} a(t) &= a_1-\frac{(3+\sqrt{-3})}{2k}b_2e^{p_1t}-\frac{(3-\sqrt{-3})}{2k}b_3e^{p_2t}-\frac{(3-\sqrt{-3})}{2k}b_4e^{p_3t} \\ &\quad -\frac{(3+\sqrt{-3})}{2k}b_5e^{p_4t}-\frac{(3+\sqrt{-3})}{2k}b_6e^{p_5t}-\frac{(3-\sqrt{-3})}{2k}b_7e^{p_6t}. \end{aligned} \tag{A16}$$

Using (A14) and (A16), an explicit form for c can be derived from the relation (A5):

$$\begin{aligned} c(t) &= \frac{(-3+\sqrt{-3})}{2k}p_1b_2e^{p_1t}-\frac{(3+\sqrt{-3})}{2k}p_2b_3e^{p_2t}-\frac{(3+\sqrt{-3})}{2k}p_3b_4e^{p_3t} \\ &\quad +\frac{(-3+\sqrt{-3})}{2k}p_4b_5e^{p_4t}+\frac{(-3+\sqrt{-3})}{2k}p_5b_6e^{p_5t}-\frac{(3+\sqrt{-3})}{2k}p_6b_7e^{p_6t}. \end{aligned} \tag{A17}$$

Now with the known expressions for the functions a, b and c one can deduce the form of $d(t)$ through the relation (A6), namely,

$$\begin{aligned} d(t) &= -\frac{3\lambda_2}{k}b_1+\frac{m_2}{2k_1^2}p_1^2b_2e^{p_1t}+\frac{m_1}{2k_1^2}p_2^2b_3e^{p_2t}+\frac{m_1}{2k_1^2}p_3^2b_4e^{p_3t}+\frac{m_2}{2k_1^2}p_4^2b_5e^{p_4t} \\ &\quad +\frac{m_2}{2k_1^2}p_5^2b_6e^{p_5t}+\frac{m_1}{2k_1^2}p_6^2b_7e^{p_6t}, \end{aligned} \tag{A18}$$

where

$$m_1 = \frac{(3+\sqrt{-3})}{2k^2}, m_2 = \frac{(3-\sqrt{-3})}{2k^2}. \qquad (A19)$$

For further analysis we redefine the constants

$$\begin{aligned} r_1 &= \frac{m_2}{2k_1^2}p_1^2, \quad r_2 = \frac{m_1}{2k_1^2}p_2^2, \quad r_3 = \frac{m_1}{2k_1^2}p_3^2 \\ r_4 &= \frac{m_2}{2k_1^2}p_4^2, \quad r_5 = \frac{m_2}{2k_1^2}p_5^2, \quad r_6 = \frac{m_1}{2k_1^2}p_6^2. \end{aligned} \qquad (A20)$$

and write the function $d(t)$ in the form

$$d(t) = -\frac{3\lambda_2}{k}b_1 + r_1 b_2 e^{p_1 t} + r_2 b_3 e^{p_2 t} + r_3 b_4 e^{p_3 t} + r_4 b_5 e^{p_4 t} + r_5 b_6 e^{p_5 t} + r_6 b_7 e^{p_6 t}. \qquad (A21)$$

APPENDIX B: METHOD OF SOLVING THE DETERMINING EQ. (56)

Here we solve the equations

$$3\ddot{b} + k\dot{a} + 3\lambda_1 b + kc = 0, \qquad (B1)$$

$$2\dot{c} - \ddot{a} + 3\lambda_2 b + kd = 0, \qquad (B2)$$

$$\ddot{c} + 2\lambda_1\dot{a} + k\dot{d} = 0, \qquad (B3)$$

$$\ddot{d} - \lambda_2 c + 2\lambda_2\dot{a} + \lambda_1 d = 0 \qquad (B4)$$

$$\dddot{b} + k\lambda_2 b + \lambda_1\dot{b} + k\dot{c} + \frac{k^2}{3}d = 0, \qquad (B5)$$

and deduce the general form of $a(t),\ b(t),\ c(t)$ and $d(t)$.

From (B1) and (B2) we get

$$\begin{aligned} c &= -\frac{3}{k}\ddot{b} - \frac{3}{k}\lambda_1 b - \dot{a}, \qquad (B6) \\ d &= \frac{6}{k^2}\dddot{b} + \frac{6}{k^2}\lambda_1\dot{b} + \frac{3}{k}\ddot{a} - \frac{3\lambda_2}{k}b. \qquad (B7) \end{aligned}$$

By substituting (B6) and (B7) and their derivatives into (B3) and (B4) we obtain

$$\begin{aligned} &3\frac{d^4b}{dt^4} + 3\lambda_1\frac{d^2b}{dt^2} - 3k\lambda_2\frac{db}{dt} + 2k\frac{d^3a}{dt^3} + 2k\lambda_1\frac{da}{dt} = 0, \qquad (B8) \\ &2\frac{d^5b}{dt^5} + 4\lambda_1\frac{d^3b}{dt^3} + 2\lambda_1^2\frac{db}{dt} + k\frac{d^4a}{dt^4} + k\lambda_1\frac{d^2a}{dt^2} + \lambda_2 k^2\frac{da}{dt} = 0. \qquad (B9) \end{aligned}$$

From (B8) we have

$$\frac{d^3a}{dt^3} + \lambda_1\frac{da}{dt} = -\frac{3}{2k_1}\frac{d^4b}{dt^4} - \frac{3\lambda_1}{2k}\frac{d^2b}{dt^2} + \frac{3\lambda_2}{2}\frac{db}{dt}. \qquad (B10)$$

Substitute (B10) into (B9) and rearranging we get

$$\frac{da}{dt} = -\frac{1}{2k^2\lambda_2}\frac{d^5b}{dt^5} - \frac{5\lambda_1}{2k^2\lambda_2}\frac{d^3b}{dt^3} - \frac{3}{2k}\frac{d^2b}{dt^2} - \frac{2\lambda_1^2}{k^2\lambda_2}\frac{db}{dt}. \tag{B11}$$

Integration of (B11) leads to

$$a(t) = a_1 - \frac{1}{2k^2\lambda_2}\frac{d^4b}{dt^4} - \frac{5\lambda_1}{2k^2\lambda_2}\frac{d^2b}{dt^2} - \frac{3}{2k}\frac{db}{dt} - \frac{2\lambda_1^2}{k^2\lambda_2}b, \tag{B12}$$

where a_1 is an integration constant. Substituting (B11) and its derivatives into (B8) we arrive at

$$\frac{d^7b}{dt^7} + 6\lambda_1\frac{d^5b}{dt^5} + 9\lambda_1^2\frac{d^3b}{dt^3} + (4\lambda_1^3 + 3k^2\lambda_2^2)\frac{db}{dt} = 0. \tag{B13}$$

Introducing $\frac{db}{dt} = P$, where P is the new dependent variable, we can reduce an order of the Eq. (B13), that is,

$$\frac{d^6P}{dt^6} + 6\lambda_1\frac{d^4P}{dt^4} + 9\lambda_1^2\frac{d^2P}{dt^2} + (4\lambda_1^3 + 3k^2\lambda_2^2)P = 0. \tag{B14}$$

Eq. (B14) is a sixth order linear ODE with constant coefficients whose solution can be found in the following way. Since Eq. (B14) contains constant coefficients a general solution of this equation can be written of the form

$$P = P_1e^{m_1t} + P_2e^{m_2t} + P_3e^{m_3t} + P_4e^{m_4t} + P_5e^{m_5t} + P_6e^{m_6t}, \tag{B15}$$

where P_i, $i = 1, \ldots, 6$, are integration constants and $m_i's$, $i = 1, \ldots, 6$, are roots of the characteristic equation

$$m^6 + 6\lambda_1m^4 + 9\lambda_1^2m^2 + (4\lambda_1^3 + 3k^2\lambda_2^2) = 0. \tag{B16}$$

Now choosing $m^2 = X$ we can rewrite the sixth power polynomial equation (B16) as a cubic polynomial equation in X, namely,

$$X^3 + 6\lambda_1X^2 + 9\lambda_1^2X + (4\lambda_1^3 + 3k^2\lambda_2^2) = 0. \tag{B17}$$

Let us choose

$$a = 6\lambda_1, \quad b = 9\lambda_1^2, \quad c = 4\lambda_1^3 + 3k^2\lambda_2^2, \tag{B18}$$

and introduce a transformation[22]

$$X = Y - \frac{a}{3} \tag{B19}$$

in (B17) so that it becomes

$$Y^3 + pY + q = 0, \tag{B20}$$

where the new constants, p and q are related to the old constants, a, b and c, by the following relations

$$p = b - \frac{a^2}{3}, \quad q = \frac{2}{27}a^3 - \frac{ab}{3} + c. \tag{B21}$$

The cubic equation (B20) has one real root and two complex conjugate roots[22], namely,

$$Y_1 \ = \ A + B, \ \ Y_{2,3} = -\frac{(A+B)}{2} \pm i\frac{\sqrt{3}(A-B)}{2}, \tag{B22}$$

where

$$A = \left(-\frac{q}{2} + \sqrt{Q}\right)^{\frac{1}{3}}, \quad B = \left(-\frac{q}{2} - \sqrt{Q}\right)^{\frac{1}{3}}, \quad Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2, \tag{B23}$$

so that the roots of Eq. (B17) can now be expressed through the relation (B19), that is,

$$\begin{aligned} X_1 \ &= \ A + B - \frac{a}{3}, \quad X_2 = -\frac{(A+B)}{2} - \frac{a}{3} + i\frac{\sqrt{3}(A-B)}{2}, \\ X_3 \ &= \ -\frac{(A+B)}{2} - \frac{a}{3} - i\frac{\sqrt{3}(A-B)}{2}. \end{aligned} \tag{B24}$$

From the identity $m^2 = X$, we find the roots of the characteristic equation (B16):

$$\begin{aligned} &m_{1,2} = \pm\sqrt{X_1} = \pm\sqrt{A + B - \frac{a}{3}}, \quad m_{3,4} = \pm\sqrt{X_2} = \pm\sqrt{C + iD}, \\ &m_{5,6} = \pm\sqrt{X_3} = \pm\sqrt{C - iD}, \end{aligned} \tag{B25}$$

where again for simplicity we have introduced the constants C and D which can be fixed from the relation

$$C = -\frac{(A+B)}{2} - \frac{a}{3}, \quad D = \frac{\sqrt{3}(A-B)}{2}. \tag{B26}$$

Eq. (B25) can be put in the following convenient way

$$\begin{aligned} m_1 \ &= \ +\sqrt{A + B - \frac{a}{3}}, \quad m_2 = -\sqrt{A + B - \frac{a}{3}}, \\ m_3 \ &= \ (C^2 + D^2)^{\frac{1}{4}} \left[\cos\phi + i \sin\phi\right], \quad m_4 = -(C^2 + D^2)^{\frac{1}{4}} \left[\cos\phi + i \sin\phi\right], \\ m_5 \ &= \ (C^2 + D^2)^{\frac{1}{4}} \left[\cos\phi - i \sin\phi\right], \quad m_6 = -(C^2 + D^2)^{\frac{1}{4}} \left[\cos\phi - i \sin\phi\right], \end{aligned}$$

where

$$\phi = \left(\frac{\tan^{-1}(D/C)}{2}\right). \tag{B27}$$

With the help of these $m_i's, \ i = 1, \ldots, 6$, a general solution for (B14) can be written in the form

$$P = b_2 e^{\alpha_1 t} + b_3 e^{-\alpha_1 t} + e^{\alpha_2 t}(b_4 \cos\alpha_3 t + b_5 \sin\alpha_3 t) + e^{-\alpha_2 t}(b_6 \cos\alpha_3 t - b_7 \sin\alpha_3 t), \tag{B28}$$

where $b_i's, \ i = 2, \ldots, 7$, are integration constants and α_2, α_3 are real and imaginary parts of m_3, m_4, m_5 and m_6 respectively, where $\alpha_2 = \sqrt{C^2 + D^2}\cos\phi, \ \alpha_3 = \sqrt{C^2 + D^2}\sin\phi$.

From the identity $\frac{db}{dt} = P$, the function $b(t)$ upon integration can be deduced as

$$\begin{aligned} b(t) \ &= \ b_1 + \frac{b_2}{\alpha_1} e^{\alpha_1 t} - \frac{b_3}{\alpha_1} e^{-\alpha_1 t} + \frac{b_4 e^{\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_2 \cos\alpha_3 t + \alpha_3 \sin\alpha_3 t) \\ &+ \frac{b_5 e^{\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_2 \sin\alpha_3 t - \alpha_3 \cos\alpha_3 t) + \frac{b_6 e^{-\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_3 \sin\alpha_3 t \\ &- \alpha_2 \cos\alpha_3 t) + \frac{b_7 e^{-\alpha_2 t}}{(\alpha_2^2 + \alpha_3^2)}(\alpha_2 \sin\alpha_3 t + \alpha_3 \cos\alpha_3 t), \end{aligned} \tag{B29}$$

where b_1 is an integration constant. Once $b(t)$ is known the function $a(t)$ can be fixed from the relation (B12),

$$\begin{aligned} a(t) \ &= \ a_1 - \frac{2\lambda_1^2}{k^2\lambda_2} b_1 + \beta_1 b_2 e^{\alpha_1 t} + \beta_2 b_3 e^{-\alpha_1 t} + b_4 e^{\alpha_2 t}(\beta_3 \cos\alpha_3 t + \beta_4 \sin\alpha_3 t) \\ &+ b_5 e^{\alpha_2 t}(\beta_3 \sin\alpha_3 t - \beta_4 \cos\alpha_3 t) + b_6 e^{-\alpha_2 t}(\beta_5 \cos\alpha_3 t + \beta_4 \sin\alpha_3 t) \\ &+ b_7 e^{-\alpha_2 t}(\beta_4 \cos\alpha_3 t - \beta_5 \sin\alpha_3 t), \end{aligned} \tag{B30}$$

where a_1 is an integration constant. For simplicity in the expression $a(t)$ we have introduced the constants $\beta_i, \ \ i = 1, \ldots, 5$. These new constants are related to the old parameters through the following relations

$$\begin{aligned} \beta_1 \ &= \ -\left(\frac{\alpha_1^3}{2k^2\lambda_2} + \frac{5\lambda_1\alpha_1}{2k^2\lambda_2} + \frac{3}{2k} + \frac{2\lambda_1^2}{k^2\lambda_2\alpha_1}\right), \ \ \beta_2 = \left(\frac{\alpha_1^3}{2k^2\lambda_2} + \frac{5\lambda_1\alpha_1}{2k^2\lambda_2} - \frac{3}{2k} + \frac{2\lambda_1^2}{k^2\lambda_2\alpha_1}\right), \\ \beta_3 \ &= \ -\left(\frac{\alpha_2(\alpha_2^2 - 3\alpha_3^2)}{2k^2\lambda_2} + \frac{5\lambda_1\alpha_2}{2k^2\lambda_2} + \frac{3}{2k} + \frac{2\lambda_1^2\alpha_2}{k^2\lambda_2(\alpha_2^2 + \alpha_3^2)}\right), \\ \beta_4 \ &= \ -\left(\frac{\alpha_3(\alpha_3^2 - 3\alpha_2^2)}{2k^2\lambda_2} - \frac{5\lambda_1\alpha_3}{2k^2\lambda_2} + \frac{2\lambda_1^2\alpha_3}{k^2\lambda_2(\alpha_2^2 + \alpha_3^2)}\right), \\ \beta_5 \ &= \ \left(\frac{\alpha_2(\alpha_2^2 - 3\alpha_3^2)}{2k^2\lambda_2} + \frac{5\lambda_1\alpha_2}{2k^2\lambda_2} - \frac{3}{2k} + \frac{2\lambda_1^2\alpha_2}{k^2\lambda_2(\alpha_2^2 + \alpha_3^2)}\right). \end{aligned} \tag{B31}$$

With the forms of $a(t)$ and $b(t)$ the function $c(t)$ can be deduced using the relation (B6) in the form,

$$\begin{aligned} c(t) &= -\frac{3\lambda_1}{k}b_1 + \gamma_1 b_2 e^{\alpha_1 t} + \gamma_2 b_3 e^{-\alpha_1 t} + b_4 e^{\alpha_2 t}(\gamma_3 \cos\alpha_3 t + \gamma_4 \sin\alpha_3 t) \\ &+ b_5 e^{\alpha_2 t}(\gamma_3 \sin\alpha_3 t - \gamma_4 \cos\alpha_3 t) + b_6 e^{-\alpha_2 t}(\gamma_5 \cos\alpha_3 t + \gamma_6 \sin\alpha_3 t) \\ &+ b_7 e^{-\alpha_2 t}(\gamma_6 \cos\alpha_3 t - \gamma_5 \sin\alpha_3 t), \end{aligned} \tag{B32}$$

where again we have defined new constants $\gamma_i' s,\ i = 1, \ldots, 5$, for simplicity sake. Here too one can fix their forms using the following relations,

$$\begin{aligned} \gamma_1 &= -\left(\frac{3\alpha_1}{k} + \frac{3\lambda_1}{k\alpha_1} + \alpha_1\beta_1\right), \quad \gamma_2 = \left(\frac{3\alpha_1}{k} + \frac{3\lambda_1}{k\alpha_1} + \alpha_1\beta_2\right), \qquad \text{(B33)} \\ \gamma_3 &= -\left(\frac{3\alpha_2}{k} + \frac{3\lambda_1\alpha_2}{k(\alpha_2^2+\alpha_3^2)} + \alpha_2\beta_3 + \alpha_3\beta_4\right), \quad \gamma_4 = \left(\frac{3\alpha_3}{k} - \frac{3\lambda_1\alpha_3}{k(\alpha_2^2+\alpha_3^2)} + \alpha_3\beta_3 - \alpha_2\beta_4\right), \\ \gamma_5 &= \left(\frac{3\alpha_2}{k} + \frac{3\lambda_1\alpha_2}{k(\alpha_2^2+\alpha_3^2)} + \alpha_2\beta_5 - \alpha_3\beta_4\right), \quad \gamma_6 = \left(\frac{3\alpha_3}{k} - \frac{3\lambda_1\alpha_3}{k(\alpha_2^2+\alpha_3^2)} + \alpha_2\beta_4 + \alpha_3\beta_5\right). \end{aligned}$$

Finally, the function $d(t)$ can be derived from the relation (B7) by simply substituting the forms of $a(t)$ and $b(t)$ and their derivatives into it, that is,

$$\begin{aligned} d(t) &= -\frac{3\lambda_2}{k}b_1 + \delta_1 b_2 e^{\alpha_1 t} + \delta_2 b_3 e^{-\alpha_1 t} + b_4 e^{\alpha_2 t}(\delta_3 \cos\alpha_3 t + \delta_4 \sin\alpha_3 t) \\ &+ b_5 e^{\alpha_2 t}(\delta_3 \sin\alpha_3 t - \delta_4 \cos\alpha_3 t) + b_6 e^{-\alpha_2 t}(\delta_5 \cos\alpha_3 t + \delta_6 \sin\alpha_3 t) \\ &+ b_7 e^{-\alpha_2 t}(\delta_6 \cos\alpha_3 t - \delta_5 \sin\alpha_3 t). \end{aligned} \tag{B34}$$

Again for simplicity we have defined the constants $\delta_i' s,\ i = 1, \ldots, 6$, which can be determined from the following relations,

$$\begin{aligned} \delta_1 &= \frac{1}{k}\left(\frac{6\alpha_1^2}{k} + \frac{6\lambda_1}{k} + 3\alpha_1^2\beta_1 - \frac{3\lambda_2}{\alpha_1}\right), \quad \delta_2 = \frac{1}{k}\left(\frac{6\alpha_1^2}{k} + \frac{6\lambda_1}{k} + 3\alpha_1^2\beta_1 + \frac{3\lambda_2}{k}\right), \\ \delta_3 &= \frac{1}{k}\left(3(\alpha_2^2-\alpha_3^2)(\beta_3 + \frac{2}{k}) + \frac{6\lambda_1}{k} + 6\alpha_2\alpha_3\beta_4 - \frac{3\alpha_2\lambda_2}{(\alpha_2^2+\alpha_3^2)}\right), \\ \delta_4 &= -\frac{1}{k}\left(6\alpha_2\alpha_3(\beta_3 + \frac{2}{k}) - 3\beta_4(\alpha_2^2-\alpha_3^2) + \frac{3\alpha_3\lambda_2}{(\alpha_2^2+\alpha_3^2)}\right), \\ \delta_5 &= \frac{1}{k}\left(3(\alpha_2^2-\alpha_3^2)(\beta_5 + \frac{6}{k}) + \frac{6\lambda_1}{k} - 6\alpha_2\alpha_3\beta_4 + \frac{3\alpha_2\lambda_2}{(\alpha_2^2+\alpha_3^2)}\right), \\ \delta_6 &= \frac{1}{k}\left(6\alpha_2\alpha_3(\beta_5 + \frac{2}{k}) + 3\beta_4(\alpha_2^2-\alpha_3^2) - \frac{3\lambda_2\alpha_3}{(\alpha_2^2+\alpha_3^2)}\right). \end{aligned} \tag{B35}$$

---

* Electronic address: snp@mnnit.ac.in (S. N. Pandey)

[†] Electronic address: lakshman@cnld.bdu.ac.in (M. Lakshmanan)

[1] S. N. Pandey, P. S. Bindu, M. Senthilvelan and M. Lakshmanan, *A Group Theoretical Identification of Integrable Cases of the Liénard Type Equation* $\ddot{x}+f(x)\dot{x}+g(x)=0$ *: Part I: Equations having Non-maximal Number of Lie point Symmetries*, J. Math. Phys. (submitted)

[2] M. Aguirre and J. Krause, J. Math. Phys. **25**, 210 (1984)

[3] P. Hydon, *Symmetry Methods for Differential Equations: A Beginner's Guide* (Cambridge University Press, Cambridge, 2000)

[4] C. E. Wulfman and B. G. Wybourne, J. Phys. A: Math. Gen. **9**, 507 (1976)

[5] J. M. Cervero and J. Villarroel, J. Phys. A: Math. Gen. **17**, 1777 (1984)

[6] F. M. Mahomed and P. G. L. Leach, Quaestiones Math. **8**, 241 (1985); **12**, 121 (1989)

[7] M. Lutzky, J. Phys. A: Math. Gen. **11**, 249 (1978)

[8] G. E. Prince, P. G. L. Leach and R. M. Santilli, Hadronic J. **3**, 390 (1979)

[9] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, J. Math. Phys. **48**, 032701 (2007)

[10] W. Sarlet, F. M. Mahomed and P. G. L. Leach, J. Phys. A: Math. Gen. **20**, 277 (1987)

[11] L. G. S. Duarte, S. E. S. Durarte and I. Moreira, J. Phys. A: Math. Gen. **20**, L701 (1987)

[12] P. G. L. Leach, M. R. Feix and S. Bouquet, J. Math. Phys. **29** 2563 (1988)

[13] R.L.Lemmer and P.G.L.Leach, J. Phys. A: Math. Gen. **26**, 5017 (1993)

[14] V. K. Chandrasekar, A. Kundu, M. Senthilvelan and M. Lakshmanan, J. Phys. A: Math. Gen. **39**, 9743 (2006); **39**, 10945 (2006)

[15] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, Phys. Rev. E **72**, 066203 (2005); *A nonlinear oscillator with unusual dynamical properties* in Proceedings of the Third National Conference on Nonlinear Systems and Dynamics (NCNSD-2006) pp.1-4 (2006)

[16] V. K. Chandrasekar, S.N.Pandey, M. Senthilvelan and M. Lakshmanan, Chaos, Solitons and Fractals **26**, 1399 (2005)

[17] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, Proc. R. Soc. London **A 461**, 2451 (2005)

[18] N. H. Ibragimov, *Elementary Lie group Analysis and Ordinary Differential Equations* (John Wiley & Sons, New York, 1999)

[19] F. M. Mahomed, Math. Meth. Appl. Sci. **30**, 1995 (2007)

[20] S. Lie, Arch. Math. Naturv. **9**, 371 (1883)

[21] R. Gladwin Pradeep, V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, J. Math. Phys.

50, 052901 (2009)

[22] F. A. Korn, T. M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw-Hill, New York 1968)