Comment on "Intermittent Synchronization in a Pair of Coupled Chaotic Pendula"

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In a recent Letter, Baker, Blackburn and Smith [1] reported that permanent synchronization of a pair of unidirectionally coupled identical chaotic pendula does not occur except as a numerical artifact arising from finite computational precisions. Baker *et al.* showed that the synchronization of a pair of unidirectionally coupled pendula is always *intermittent*, for any value of coupling coefficient c, by using their numerical and analytical tests. This was also found to be true in the case of coupled Duffing oscillators [1]. However, on careful numerical analysis using Conditional Lyapunov Exponents (CLEs), we find that there exists some specific range of c values for which persistent synchronization can occur for these systems.

The main point of this comment is to clarify the conditions under which coupled chaotic systems can exhibit permanent synchronization. According to Pecora and Carroll[2], the master and slave systems will perfectly synchronize only if the sub-Lyapunov exponents or CLEs are all negative. The two analytic criteria mentioned in Ref. [1] for asymptotic stability are, (i) the largest eigenvalue of the Jacobian matrix corresponding to the flow evaluated on the synchronization manifold must always be negative and (ii) existence of a suitable Lyapunov function. However, it is not possible, in general, to show analytically the above two criteria are uniformly obeyed for all c values for most of the chaotic systems, as shown in the case of coupled pendula (cf. Eqs. (9) and (11) of Ref. [1]). In such cases, it is always possible to compute the Lyapunov exponents of the coupled system for a range of c values and show that the CLEs take large negative values for permanent synchronization.

We have computed the Lyapunov exponents for the coupled pendula and the Duffing oscillators using the standard Wolf *et al.* algorithm. For numerical simulations we have used the same parameter values as in Ref. [1] and used 5000 drive cycles for calculations after leaving 5000 drive cycles as transient. Figure 1(a) shows the variation of CLE as a function of c for the coupled pendula and it takes negative values for $0.796 \leq c < 1.0$ only. The CLE value for $c = 0.79$ is $+0.00320$ ($\approx 10^{-3}$), for which synchronization can not occur and hence the observed intermittent synchronization (cf. Fig. 1 in Ref. [1]) is a computer artifact. However, we find that for $c = 0.932$, the CLE value becomes the lowest (-0.06825) and is a large negative value for which intermittent synchronization is absent as shown in Fig. 1(b). This has been verified upto 2×10^6 drive cycles with the addition of tiny noise levels showing that permanent synchronization does occur. The same phenomenon persists over a range of c values close to 0.932.

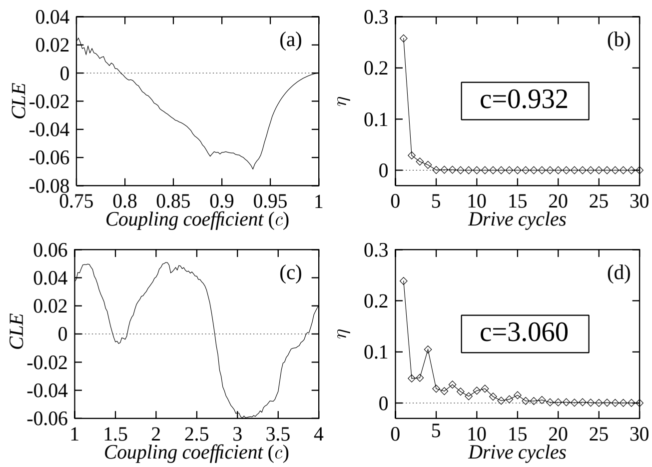

FIG. 1: (a) Variation of the Conditional Lyapunov Exponent (CLE) as a function of c, (b) synchronization error η (given by Eq. (5) in Ref. [1]) versus time in drive cycles for coupled pendula; (c) CLE versus c and (d) η versus drive cycles for coupled Duffing oscillators.

Similar arguments hold good for the coupled Duffing oscillator also and the results are shown in Figs. 1(c) and (d). The CLE takes small negative values close to zero ($\approx 10^{-3}$) for $c \in (1.48, 1.64)$ where synchronization is intermittent and numerical artifact arises in this range also. The CLE value for $c = 1.5$ is -0.00562 and hence the observed hard locking becomes intermittent with the addition of noise in the 8-th digit (cf. Fig. 4 in Ref. [1]). In addition, there exists another range of c values, $2.74 \leq c \leq 3.84$ where the CLEs are negative (cf. Fig. 1(c)). We find that for $c = 3.06$, the CLE takes the lowest value of -0.05967 (large negative value) for which persistent synchronization occurs and the intermittent synchronization is absent (cf. Fig. 1(d)). Again the same phenomenon persists over a range of c values close to 3.06.

In summary, the main aim of our comment is to emphasize that the conditional Lyapunov exponents play an important role in distinguishing between intermittent and persistent synchronization, when it is not possible to show analytically the criteria for asymptotic stability are uniformly obeyed. We find that intermittent synchronization can occur when CLEs are very small positive or negative values close to zero while persistent synchronization occurs when CLEs become sufficiently large negative values.

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[1] G. L. Baker, J. A. Blackburn and H. J. T. Smith, Phys. Rev. Lett. **81**, 554 (1998)
[2] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990)