Indian J. pure appl. Math., **29**(12) : 1275-1279, December 1998

THE PRIME AND MAXIMAL IDEALS IN $R[X]$, R : A ONE-DIMENSIONAL PRÜFER DOMAIN

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(*Received* 20 *August* 1997; *after revision* 20 *February* 1998; *Accepted* 15 *July* 1998)

In this paper, the prime and maximal ideals of a polynomial ring $R[x]$ in an indeterminate x over a Prüfer domain R of Krull dimension one are explicitly determined, using elementary results of ring theory.

Key Words : Ideal; Multiplicative Ideal Theory

INTRODUCTION

In 1990, Lemmer and Naude[1] gave a simple method for the description of the already known prime and maximal ideals of the polynomial ring $R[x]$, R a principal ideal domain. Our goal here is to explicitly determine all the maximal ideals of $R[x]$, using only elementary results of ring theory, where R is a Prüfer domain of Krull dimension one. We also classify the prime ideals of $R[x]$ of height one and two. It may be remarked that the prime ideals of $R[x]$ are already known (cf. [2, Chapter 1]).

As in [3, §11], an integral domain R is called a Prüfer domain of Krull dimension one if for every non-zero prime ideal $I\!P$ of R (*i*) $R_{I\!P}$ is a valuation ring. (*ii*) $I\!P$ is a maximal ideal of R. Throughout $V_{I\!P}$ will stand for the valuation of the field of quotients of R with valuation ring $R_{I\!P}$ and residue field $R/I\!P$, $I\!P$ being a non-zero prime ideal of R. If $f(x)$ is in $R_{I\!P}[x]$, $\bar{f}(x) \in R/I\!P[x]$ will denote the polynomial obtained by replacing the coefficients of f by their respective images under the canonical homomorphism from $R_{I\!P}$ onto $R/I\!P$. We shall prove :

Theorem 1.1 — *Let R be a Prüfer domain of Krull dimension one with quotient field K. Then*

(*a*) *the maximal ideals of* $R[x]$ *are* :

(*i*) $I\!PR[x] + f(x)R[x]$ *with* $I\!P$ *a non-zero prime ideal in* R, $f(x) \in R[x]$ *a monic polynomial and* $\bar{f}(x)$ *irreducible in* $R/I\!P[x]$.

*Research was partially supported by CSIR, New Delhi vide grant no. 25/(0072)/93 /EMR-II & 9/135/(245)/93/EMR-I.

(*ii*) $(g(x)K[x]) \cap R[x]$ *with* $g(x) = \sum_i c_i x^i$ *irreducible in* $K[x]$, *and* (*) $v_{\mathbb{P}}(c_0) < v_{\mathbb{P}}(c_i)$ *for all* $i \geq 1$ *and for each non-zero prime ideal* $\mathbb{P}$ *of* R.

(*b*) *The non-maximal prime ideals of* $R[x]$ *are* :

(*iii*) $\mathbb{P}R[x]$, *with* $\mathbb{P}$ *a non-zero prime ideal in* R.

(*iv*) $(g(x)K[x]) \cap R[x]$ *with* $g(x) = \sum_i c_i x^i$ *irreducible in* $K[x]$, *and* (*) *not, satisfied.*

Remark : If R is a generalized Dedekind domain (or in particular a Dedekind domain) with infinitely many prime ideals, then (*) is never satisfied because for any $c \in K$, $v_{\mathbb{P}}(c) = 0$ for all but finitely many $\mathbb{P}$ by virtue of [3, Theorem 12.10]. So in this case, the maximal ideals of $R[x]$ are only those given by (*i*).

Keeping in view that for any valuation v of a field K, its canonical extension v^x to the simple transcendental extension $K(x)$, defined by $v^x\left(\sum_i a_i x^i\right) = \min_i \{v(a_i)\}$ is a valuation of $K(x)$, the following corollary can be easily deduced from the above theorem.

Corollary 1.2 — Let R be the valuation ring of a rank 1 valuation v of a field K with maximal ideal $\mathbb{M}$. Then

(*a*) *the maximal ideals of* $R[x]$ *are* :

(i) $\mathbb{M}R[x] + f(x)R[x]$, $f(x) \in R[x]$, $\bar{f}(x)$ irreducible in $R/\mathbb{M}[x]$.

(ii) $g(x)R[x]$ with $g(x) = \sum_i c_i x^i$ irreducible in $K[x]$, and (**) $v(c_0) = 0$, $v(c_i) > 0$ for all $i \geq 1$.

(*b*) *The non-maximal prime ideals of* $R[x]$ *are* :

(iii) $\mathbb{M}R[x]$,

(iv) $g(x)R[x]$ with $g(x) = \sum_i c_i x^i \in R[x] \setminus M[x]$ irreducible over K and (**) not satisfied.

PROOF OF THEOREM 1.1 :

The proof of the theorem depends upon three lemmas, the first two of which are well-known (see [2, Theorem 28, 36]. For the sake of completeness, we prove them here. In fact, our proof of the first lemma is constructive.

Lemma 2.1 — Let R be an integral domain with quotient field K and P be a non-zero prime ideal in $R[x]$ such that $P \cap R = \{0\}$ then $P = (g(x)\ K[x]) \cap R[x]$, where $g(x) \in K[x]$ is irreducible.

PROOF : Let $f(x)$ be a non-zero element of P. Since $P \cap R = \{0\}$, $f(x)$ must be a non-constant polynomial. Write

$$f(x) = f_1(x) f_2(x) \ldots f_r(x),$$

where each $f_i(x) \in K[x]$ is irreducible over K. Choose non-zero $a_i \in R$ such that $a_i f_i(x) \in R[x]$, $1 \leq i \leq r$. As P is a prime ideal of $R[x]$, $a_j f_j(x) \in P$ for some j. Set $g(x) = a_j f_j(x)$. Claim is that

$$P = (g(x)\ K[x]) \cap R[x]. \qquad \text{... (1)}$$

Let $h(x)$ be a non-zero element of P. Suppose, if possible, $h(x)$ does not belong to the right hand side of (1). Then $h(x)$ won't be divisible by the irreducible polynomial $g(x)$. So there exist $A(x)$, $B(x) \in K[x]$ satisfying

$$g(x)\ A(x) + h(x)\ B(x) = 1. \qquad \text{... (2)}$$

Let a, b be a non-zero elements of R such that the polynomials $aA(x)$ and $bB(x)$ are in $R[x]$. Keeping in view that $g(x)$, $h(x) \in P$, it immediately follows from (2) that $ab \in P$, which contradicts the hypothesis. Hence, $P \subseteq (g(x)\ K[x]) \cap R[x]$. The other inclusion can be easily verified.

Remark : Let R and K be as in the above lemma and $g(x) \in K[x]$ be irreducible, then it can be easily verified that $g(x) \cdot K[x]$ is a prime ideal of $K[x]$ and hence $(g(x)\ K[x]) \cap R[x]$ is a prime ideal of $R[x]$, which does not contain any non-zero element of R.

Lemma 2.2 — Let R be an integral domain and $\mathbb{P}$ be a maximal ideal of R, then —

(*a*) An ideal M of $R[x]$ containing $\mathbb{P}$ is maximal if and only if $M = \mathbb{P}R[x] + f(x)\ R[x]$, where $f(x) \in R[x]$ is monic and $\bar{f}(x)$ is irreducible over $R/\mathbb{P}$.

(*b*) $\mathbb{P}R[x]$ is the only non-maximal prime ideal of $R[x]$ containing $\mathbb{P}$.

PROOF : Let $\phi : R[x] \to R/\mathbb{P}[x]$ be the natural epimorphism whose kernel is $\mathbb{P}R[x]$. So there exists a 1-1 order preserving correspondence between the ideals of $R[x]$ contianing $\mathbb{P}R[x]$ and idelas of $R/\mathbb{P}[x]$. Assertion (*a*) follows immediately by virtue of the fact that an ideal I of the principal ideal domain $R/\mathbb{P}[x]$ is maximal if and only if it is generated by a monic irreducible polynomial in $R/\mathbb{P}[x]$. As regards (*b*), observe that $\{0\}$ is the only prime ideal of $R/\mathbb{P}[x]$ which is not maximal. So $\phi^{-1}\{0\} = \mathbb{P}R[x]$ is the only prime ideal of $R[x]$ containing $\mathbb{P}$, which is not maximal.

Lemma 2.3 — Let R be a Prüfer domain of Krull dimension one with quotient field K and $P = (g(x)\ K[x]) \cap R[x]$ be a prime ideal of $R[x]$ with $g(x) = \sum c_i x^i$ an irreducible element of $K[x]$. Then P is a maximal ideal of $R[x]$ if and only if $v_{\mathbb{P}}(c_0) < v_{\mathbb{P}}(c_i)$ for all $i \geq 1$ and for each non-zero prime ideal $\mathbb{P}$ of R.

PROOF : We shall denote by $v_{\mathbb{P}}^x$, the Gaussian valuation obtained on extending the valuation $v_{\mathbb{P}}$ of K to $K(x)$, defined by

$$v_{\mathbb{P}}^x \left(\sum_i a_i x^i \right) = \min_i \{v_{\mathbb{P}}(a_i)\}.$$

Suppose first that P is a maximal ideal in $R[x]$. Let $\mathbb{P}$ be a non-zero prime ideal of R. Since $P \cap R = \{0\}$ and P is maximal, it follows that $\mathbb{P}R[x] + P = R[x]$. Consequently,

$$\mathbb{P}R_{\mathbb{P}}[x] + PR_{\mathbb{P}}[x] = R_{\mathbb{P}}[x]. \qquad \text{... (3)}$$

Let c_j be a coefficient of $g(x)$ such that $v^x_{\mathbb{P}}(g(x)) = v_{\mathbb{P}}(c_j)$. Set $g_1(x) = g(x)/c_j$, then

$$v^x_{\mathbb{P}}\ (g_1(x)) = 0. \qquad \text{... (4)}$$

Keeping in view that $P \subseteq g(x)\, K[x] = g_1(x)\, K[x]$, it is immediate from (3) that there exists $h(x) \in K[x]$ such that

$$1 - g_1(x)\, h(x) \in \mathbb{P}R_{\mathbb{P}}[x], \text{ i.e.,}$$

$$v^x_{\mathbb{P}}\ (1 - g_1(x)\, h(x)) > 0. \qquad \text{... (5)}$$

It follows from (4) and (5) that

$$v^x_{\mathbb{P}}\ (h(x)) = 0 \quad \text{and} \quad \bar{g}_1(x)\, \bar{h}(x) = \bar{1}.$$

In particular, $\bar{g}_1(x) \in R/\mathbb{P}[x]$ is a constant polynomial which is possible only if $v_{\mathbb{P}}\ (c_0) < v_{\mathbb{P}}\ (c_i)$ for all $i \geq 1$.

Conversely, suppose that P is not a maximal ideal in $R[x]$. So there exists a maximal ideal M of $R[x]$ such that $P \subset M$ choose $u(x) \in M \setminus P$, then u(x) won't be divisible by the irreducible polynomial $g(x)$ over K. So there exists $A(x)$, $B(x) \in K[x]$ such that

$$g(x)\ A(x) + u(x)\ B(x) = 1. \qquad \text{... (6)}$$

If a, b, $c \in R$ are such that the polynomials $aA(x)$, $bB(x)$ and $cg(x)$ belong to $R[x]$, then multiplying (6) by abc, we see that $abc \in M$. Thus $M \cap R \neq \{0\}$.

Since $M \cap R$ (= $\mathbb{P}$ say), is a non-zero prime ideal of the ring R having Krull dimension one, it is a maximal ideal of R. By Lemma 2.2, there exists $f(x) \in R[x]$ with $\bar{f}(x)$ irreducible over $R/\mathbb{P}$ such that $M = \mathbb{P}R[x] + f(x)\ R[x]$ which implies that

$$MR_{\mathbb{P}}\ [x] = \mathbb{P}R_{\mathbb{P}}\ [x] + f(x)\ R_{\mathbb{P}}\ [x]. \qquad \text{... (7)}$$

Let $g_1(x)$ be a constant multiple of $g(x)$ satisfying (4). Claim is $g_1(x) \in PR_{\mathbb{P}}\ [x]$. Since $g_1(x) \in R_{\mathbb{P}}\ [x]$, we can write

$$g_1(x) = \sum (a_i/b_i)\, x^i,\ a_i, b_i \in R,\ b_i \notin \mathbb{P}.$$

Then clearly $(\Pi b_i)\, g_1(x) \in P$ and hence $g_1(x) \in PR_{\mathbb{P}}\ [x] \subseteq MR_{\mathbb{P}}\ [x]$, which proves the claim. It now follows from (7) that there exists $t(x) \in R_{\mathbb{P}}\ [x]$ such that

$$g_1(x) - f(x)\, t(x) \in \mathbb{P}R_{\mathbb{P}}\ [x].$$

Consequently, $\bar{g}_1(x) = \bar{f}(x)\ \bar{t}(x)$, but $\bar{f}(x)$ being irreducible is a non-constant polynomial. Hence, so must be $\bar{g}_1(x)$, i.e., $v_{\mathbb{P}}\ (c_0) \geq v_{\mathbb{P}}\ (c_i)$ for some $i \geq 1$. The proof of lemma is now complete.

The theorem follows immediately from the above three lemmas.

Remark : As in [2, Theorem 37], one cam deduce from Lemma 2.1 that for any ring R, there can't exist in $R[x]$, a chain of three distinct prime ideals with the same contraction in R. Therefore, when R is a one-dimensional Prüfer domain, it follows quickly from Theorem 1.1 that dim $R[x]$ is 2 and that all the height two prime ideals of $R[x]$ are given by Theorem 1.1(i). If $M = \mathbb{P}R[x] + f(x)\ R[x]$ is as in Theorem 1.1(i), then keeping in view that R is integrally closed in its quotient field K and that $f(x)$ is a monic polynomial which is irreducible over R, one can easily see that $f(x)$ is irreducible over$K$; for any factorization of $f(x)$ over K leads to its factorization over R. Consequently $f(x)\ R[x]$ is a prime ideal of $R[x]$. It follows from the above discussion that every height two prime ideal of $R[x]$ contains a non-zero principal prime ideal. A similar result has been proved in [4] for height two ideals of $R[x]$, where R is a Krull domain having infinitely many height one prime ideals.

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