Modulational instability of coupled Bose Einstein condensation with three body interaction potential

R. Murali and K. Porsezian

Abstract— ***In this paper, we consider the coupled Gross-Pitaevskii (CGP) equations in a quasi-one dimensional geometry, which are governed by the dynamics of the coupled Bose Einstein condensations (BECs) in the presence of the three body interaction potential. Using the linear stability analysis (LSA), we obtain the dispersion relation for the system of coupled BECs. Analytically, we obtain a new explicit expression for the growth rate. Besides, we investigate the modulational instability (MI) criterion of constant amplitude of both uncoupled and coupled BECs for different values of the three body interaction potential. Finally, we compare our results of both uncoupled and coupled BECs in the presence and absence of three body interaction term.***

I. INTRODUCTION

In recent times, the study of solitons, MI and coherent structures in BEC has come to the forefront of experimental and theoretical efforts in soft condensed matter physics, drawing the attention of atomic and nonlinear physicists alike. In 1995, the first realization of BEC in dilute atomic gases exploited the powerful method for cooling of alkali metal atoms by using laser. At low temperature, high density BEC cannot be produced by the laser cooling method alone, it is followed by evaporative cooling state. In evaporative cooling state the more energetic atoms are removed from the trap, there by cooling the remaining atoms. Using this method, BEC was first realized experimentally in dilute alkali elements like ^{7}Li, ^{23}Na and ^{87}Rb [1], [2]. BEC in dilute atomic gases is a macroscopic quantum phenomenon with similarities to superfluidity, superconductivity and the laser [3]. The macroscopic behavior of the BEC near zero temperature is modeled very well by the Gross-Pitaevskii (GP) equation which is time dependent nonlinear Schrodingör equation with external potential [4].

The dynamical (”modulational”) instabilities of BEC have been investigated by both experimentally and theoretically [5], [6], [7]. L.D. Carr et al. have studied MI of a non-uniform initial state in presence of a harmonic potential both analytically and numerically in the context of mean-field approximation BEC [8]. The MI is a general feature of continuum as well as of discrete nonlinear wave equations and manifest in diverse fields ranging from fluid dynamics [9] and nonlinear optics [10] to plasma physics [11]. The recent experimental investigation of BEC is formed in optical lattice, and its dynamical properties are discussed [12], [13]. In addition, several papers have also reported the theoretical investigation of the linear properties of such lattices [14], [15]. Smerzi et al [16] proposed an experiment to observe a superfluid-insulator mean field transition, which is due to a discrete MI. Recently, Rapti et al [17] examined the modulational and parametric instabilities arising in a non-autonomous, discrete Nonlinear Schrodingör equation setting for deep optical lattice in the context of BEC.

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In Ref. [18], Ya Li et al, have analyzed the effect of three body recombination losses on a condensates tunnel coupled by a double-well potential based on a simplified two mode approximation. In this way, they have reported the stationary and nonstationary features of the system through analytical and numerical methods. The creation, propagation, interaction and stability of dark solitions in two component condensates have been analyzed [19].

It is well known that the MI causes an exponential growth of small perturbation of a carrier wave which is a result of the interplay between the dispersion and nonlinearity. In Ref.[20], Weiping Zhang et al have studied MI of the single BEC in the presence of the three body interaction potential through analytical and numerical methods. Recently, Kourakis et al, examined the MI for the collision of two BECs in the absence of the three body interaction potential [21]. Motivated by the above two work, we give the simple treatment of MI for the two coupled BECs in the presence of the three body interaction potential. Analytically, we obtain the dispersion relation with the help of linear stability analysis, which is appropriate for investigating the MI of constant amplitude of the coupled BECs. Finally, we obtain explicit expression for the growth rate of the instability. The aim of this paper is to investigate the MI of the quasi- 1D coupled Gross-Pitaevskii (CGP) equations for the two coupled BECs in the presence of the three body interaction potential.

II. BASIC FORMALISM

We recall that, in the mean field approximation, the three dimensional CGP equations for the wave functions $\psi_1(\mathbf{r},t)$ and $\psi_2(\mathbf{r},t)$ of nonlinearly interacting coupled BECs, generalized to include three body interaction is given by [18], [19], [20]

$$\imath\hbar\frac{\partial\psi_1}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_1 + V(\mathbf{r})\psi_1 + [g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2]\psi_1 + \gamma_1|\psi_1|^4\psi_1 - \mu_1\psi_1, \tag{1}$$

$$\imath\hbar\frac{\partial\psi_2}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_2 + V(\mathbf{r})\psi_2 + [g_{22}|\psi_2|^2 + g_{21}|\psi_1|^2]\psi_2 + \gamma_2|\psi_2|^4\psi_2 - \mu_2\psi_2. \tag{2}$$

We assume both the components are having same mass, where $\hbar$ is the Plank's constant, g_{jj} (j=1 for first BEC species and j=2 for second BEC species) characterizes the effective nonlinear parameter which describes the interaction between the bosons in condensates, and g_{jl} (j,l=1,2 but $j \neq l$) characterizes the nonlinear coupling parameter. γ_j represents the coefficient of three body interaction potential. μ_j are the chemical potentials. Both the components are trapped by the same potential, which is given by $V(\mathbf{r}) = V_z(z) + V_\perp(x,y) = \frac{1}{2}m[\omega^2 z^2 + \omega_\perp^2(x^2+y^2)]$.

In general, three dimensional CGP equations of coupled BECs can be approximately considered as quasi one dimensional as long as the transverse trapping potential is so tight and longitudinal trapping potential is so loose. In this description, the transverse dimension of the condensates is much smaller than the healing length. Accordingly, the results of BECs are found to be 'cigar shaped', where the bosonic interaction can be effectively described by one dimensional modes [22], [23]. In this paper, we consider 'cigar shaped' BECs because the atomic fields are tightly confined in the transverse dimension but free to move in the third direction such that motional degrees of freedom in the x-y plane are frozen. Then the model for quasi-1D CGP equations can be derived in the following way. The wave function can be expressed as $\psi_j(\mathbf{r},t) = \rho_j(x,y)\phi_j(z,t)$. Substituting the values of $\psi_j(\mathbf{r},t)$ and $V(\mathbf{r})$ in Eqs.(1) and (2), ρ_j and ϕ_j approximately satisfies

$$-\frac{\hbar^2}{2m}\nabla_\perp^2\rho_j + V_\perp\rho_j = E_\perp\rho_j, \tag{3}$$

$$-\imath\hbar\frac{\partial\phi_1}{\partial t} - \frac{\hbar^2}{2m}\frac{\partial^2\phi_1}{\partial z^2} + V_z(z)\phi_1 + [G_{11}|\phi_1|^2 + G_{12}|\phi_2|^2]\phi_1 + \eta_1|\phi_1|^4\phi_1 - \mu_1\phi_1 = -E_\perp\phi_1, \tag{4}$$

$$-\imath\hbar\frac{\partial\phi_2}{\partial t} - \frac{\hbar^2}{2m}\frac{\partial^2\phi_2}{\partial z^2} + V_z(z)\phi_2 + [G_{22}|\phi_2|^2 + G_{21}|\phi_1|^2]\phi_2 + \eta_2|\phi_2|^4\phi_2 - \mu_2\phi_2 = -E_\perp\phi_2, \tag{5}$$

Eqn.(3) can be solved analytically and its average over the first transverse mode eigen function is given by

$$\rho_j = \sqrt{\frac{m\omega_\perp}{\pi\hbar}}\exp[-(\frac{m\omega_\perp}{2\hbar})\sigma^2], \tag{6}$$

with the eigen value $E_\perp = \hbar\omega_\perp$, where $\sigma^2 = x^2 + y^2$. By integrating over the x-y plane with the transformation $\phi_j(z,t) = U_j e^{-i\omega_\perp t}$, U_j satisfies the above Eqs.(4) and (5)

$$\imath\frac{\partial U_1}{\partial t} + \frac{\hbar}{2m}\frac{\partial^2 U_1}{\partial z^2} - \frac{1}{\hbar}V_z(z)U_1 - \frac{1}{\hbar}[G_{11}|U_1|^2 + G_{12}|U_2|^2]U_1 - \frac{\eta_1}{\hbar}|U_1|^4U_1 + \frac{\mu_1}{\hbar}U_1 = 0, \tag{7}$$

$$\imath\frac{\partial U_2}{\partial t} + \frac{\hbar}{2m}\frac{\partial^2 U_2}{\partial z^2} - \frac{1}{\hbar}V_z(z)U_2 - \frac{1}{\hbar}[G_{22}|U_2|^2 + G_{21}|U_1|^2]U_2 - \frac{\eta_2}{\hbar}|U_2|^4U_2 + \frac{\mu_2}{\hbar}U_2 = 0, \tag{8}$$

where reduced interbosonic interaction coefficients are $G_{jj} = 2a_{jj}\hbar\omega_\perp$, $G_{jl} = 2a_{jl}\hbar\omega_\perp$, and $\eta_j = \frac{\gamma_j}{3}(\frac{m\omega_\perp}{\hbar})^2$. The Eqs.(7) and (8) represent the quasi-one dimensional CGP equations for the system of coupled BECs.

III. Linear Stability Analysis

In this section, we investigate the MI of constant amplitude coupled BECs in the presence of three body interaction under the linear stability analysis (LSA). We study the MI for 'cigar shaped' BEC then the longitudinal trapping potential is switched off ($V(z) = 0$), and the transverse trapping potential is fixed because the motion of the BEC is along the longitudinal dimension. According to LSA, we perturb the system slightly as follows

$$U_j = (\chi_{j0} + \varepsilon_j)e^{\imath\varphi_j t}, \tag{9}$$

where χ_{j0} is real constant amplitude and φ_j is real phase. Where $\varepsilon_j << \chi_{jo}$, which is a complex number. Substituting Eqn.(9) in Eqs.(7) and (8), we obtain the first order approximation,

$$\imath\frac{\partial\varepsilon_1}{\partial t} + \frac{\hbar}{2m}\frac{\partial^2\varepsilon_1}{\partial z^2} - \frac{1}{\hbar}[G_{11}|\chi_{10}|^2 + 2\eta_1|\chi_{10}|^4]A_1 - \frac{G_{12}}{\hbar}\chi_{10}\chi_{20}A_2 = 0, \tag{10}$$

$$\imath\frac{\partial\varepsilon_2}{\partial t} + \frac{\hbar}{2m}\frac{\partial^2\varepsilon_2}{\partial z^2} - \frac{1}{\hbar}[G_{22}|\chi_{20}|^2 + 2\eta_2|\chi_{20}|^4]A_2 - \frac{G_{21}}{\hbar}\chi_{10}\chi_{20}A_1 = 0, \tag{11}$$

where $A_j = (\varepsilon_j + \varepsilon_j^*)$. Substituting $\varepsilon_j = (\alpha_j + \imath\beta_j)$ in Eqs.(10) and (11) respectively and separating the imaginary and real parts, we get

$$\frac{\partial\alpha_j}{\partial t} + \frac{\hbar}{2m}\frac{\partial^2\beta_j}{\partial z^2} = 0, \tag{12}$$

$$-\frac{\partial\beta_1}{\partial t} + \frac{\hbar}{2m}\frac{\partial^2\alpha_1}{\partial z^2} - \frac{2}{\hbar}[G_{11}|\chi_{10}|^2 + 2\eta_1|\chi_{10}|^4]\alpha_1 - \frac{2G_{12}}{\hbar}\chi_{10}\chi_{20}\alpha_2 = 0, \tag{13}$$

$$-\frac{\partial\beta_2}{\partial t} + \frac{\hbar}{2m}\frac{\partial^2\alpha_2}{\partial z^2} - \frac{2}{\hbar}[G_{22}|\chi_{20}|^2 + 2\eta_2|\chi_{20}|^4]\alpha_2 - \frac{2G_{21}}{\hbar}\chi_{10}\chi_{20}\alpha_1 = 0. \tag{14}$$

The general solution of Eqn.(12) can be obtained analytically by assuming $\alpha_j = \alpha_{j0}\exp[i(kz - \Omega t)]$ as

$$\beta_j = -\frac{2im\Omega}{\hbar k^2}\alpha_j, \tag{15}$$

where Ω and k are the frequency and propagation constant of the modulated wave. Now substituting the values of α_j and β_j in Eqs.(13) and (14), we obtain the despersion relation.

$$(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2) - \Omega_c^4 = 0, \tag{16}$$

where $\Omega_j^2 = 2e\Gamma_j$, $\Gamma_j = [\frac{e\hbar^2}{2} + G_{jj}|\chi_{j0}|^2 + 2\eta_{jj}|\chi_{j0}|^4]$, $\Omega_{jl}^2 = 2e\Gamma_{jl}$, $\Gamma_{jl} = G_{lj}\chi_{l0}\chi_{j0}$, $e = k^2/2m$ and $\Omega_c^4 = \Omega_{12}^2\Omega_{21}^2$.

IV. Modulational Instability of Uncoupled BECs

We consider the general case where the nonlinear coupling parameter $G_{jl} \to 0$, then $\Omega_c \to 0$, the dispersion relation (16) takes the form

$$(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2) = 0, \tag{17}$$

and the general solution of the above equation

$$\Omega^2 = \Omega_j^2. \tag{18}$$

A. *Repulsive BEC* $(G_{jj} > 0)$

First, we consider the general case where the effective nonlinear parameter G_{jj} is greater than zero ($G_{jj} > 0$) describes the repulsive bosonic interaction BEC. As a result, the solution of the Eqn.(17) turns out to be positive ($\Omega^2 > 0$). Therefore both the uncoupled BECs are stable with the inclusion of three body interaction parameter η_j.

B. *Attractive BEC* $(G_{jj} < 0)$

Next, we consider the case for which G_{jj} is less than zero ($G_{jj} < 0$). It is well known that the condition $G_{jj} < 0$ represents the attractive bosonic interaction BEC. As a result, the solution of the Eqn.(17) turns out to be negative ($\Omega^2 < 0$). Therefore both the uncoupled BECs are unstable below a critical wavenumber k_c. At a critical condition $k = k_c$, we obtain the critical wave number k_c from Eqn.(18) in the form

$$k_c = \frac{2\sqrt{m}}{\hbar}[|G_{jj}||\chi_{j0}|^2 + 2|\eta_{jj}||\chi_{j0}|^4]^{\frac{1}{2}}. \qquad (19)$$

The gain of the MI is described by $G = \imath\sqrt{-\Omega_j^2}$. Fig.(1) shows the gain spectrum of MI for the uncoupled BECs.

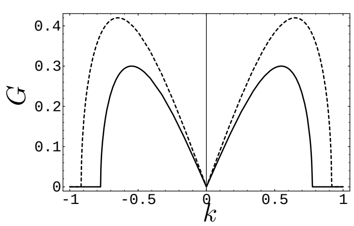


Fig. 1: MI gain spectrum of uncoupled BEC when $G_{jj} = -0.3$ and $\chi_{j0} = 1.0$ for the values of three body interaction parameter $\eta_j = -0.06$ (dotted line) and $\eta_j = 0$ (solid line) .
figure

V. Modulational Instability of coupled BECs

Now, we consider another general case for which the nonlinear coupling parameter $G_{jl} > 0$, then $\Omega_c \neq 0$. From this condition, the dispersion relation can be written as

$$\Omega^4 - \Omega^2(\Omega_1^2 + \Omega_2^2) + \Omega_1^2\Omega_2^2 - \Omega_c^4 = 0. \qquad (20)$$

the solution of Eqn.(20) is

$$\Omega_\pm^2 = \frac{1}{2}[(\Omega_1^2 + \Omega_2^2) \pm \sqrt{(\Omega_1^2 - \Omega_2^2)^2 - 4\Omega_c^4}]. \qquad (21)$$

In Eqn.(21), we note that the decrement quantity $D = (\Omega_1^2 - \Omega_2^2)^2 - 4\Omega_c^4$ is positive/negative, then the right-hand side of Eqn.(21) is real/complex.

A. *Coupling between two repulsive BECs*

First, we consider the effective nonlinear parameters G_{jj} are greater than zero (i.e. $G_{jj} > 0$) and nonlinear coupling parameters G_{jl} are greater than zero (i.e. $G_{jl} > 0$) of both BECs, which describe the coupling between the two repulsive interaction BECs. Hence the solutions of the coupled BECs $\Omega_\pm$ is greater than zero (i.e. $\Omega_\pm^2 > 0$). By this one can infer that the result of coupled BECs is stable.

B. *Coupling between two different BECs*

We consider the case for which the coupling between two different BECs i.e the effective nonlinear parameter for the first species BEC G_{11} is less than zero ($G_{11} < 0$) and the second species BEC G_{22} is greater than zero ($G_{22} > 0$). The solution of coupled system Ω_-^2 is less than zero ($\Omega_-^2 < 0$). Such type of coupled BECs is unstable because in the presence of a one attractive component BEC should be de-stabilizing its counterpart. The gain of MI is described by $G = i\sqrt{-\Omega_-^2}$. The gain spectra of MI for the coupling of two different BECs with the various values of physical parameters are shown in Fig.(2).

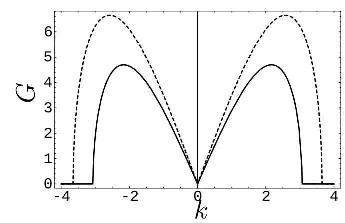


Fig. 2: MI gain spectrum of coupled of two different BECs when $G_{11} = -0.6$, $G_{22} = 0.6$, $|\chi_{20}|2 = 3.0$ and $G_{12} = -0.2$, $G_{21} = 0.2$ for the values of three body interaction potential coefficients $\eta_1 = \eta_2 = -0.06$ (dotted line) and $\eta_1 = \eta_2 = 0$ (Solid line).
figure

C. *Coupling between two attractive BECs*

Now, we consider the effective nonlinear parameters G_{jj} are less than zero ($G_{jj} < 0$) and nonlinear coupling parameters G_{jl} are greater than zero ($G_{jl} > 0$) of both BECs, which describe the coupling between the two attractive interaction BECs. Hence the solutions of coupled BECs $\Omega_\pm^2$ is less than zero ($\Omega_\pm^2 < 0$). Then this type of coupled BECs is unstable. The gain spectra of MI for the coupling of two attractive BECs with the various values of physical parameters are shown in Fig.(3).

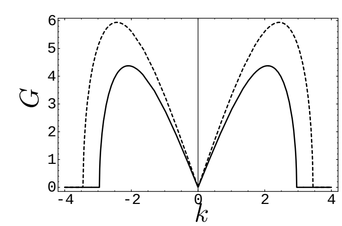
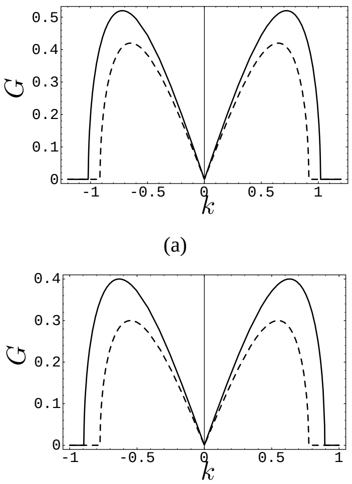


Fig. 3: MI gain spectrum of coupled of two attractive BECs when $G_{11} = G_{22} = -0.6$, $|\chi_{20}|^2 = 3.0$ and $G_{12} = G_{21} = -0.2$ for the values of three body interaction potential coefficients $\eta_1 = \eta_2 = -0.06$ (dotted line) and $\eta_1 = \eta_2 = 0$ (Solid line).
figure

Finally we compare our results of both uncoupled and coupled BECs in the presence and absence of three body interaction potential η_j. Fig.(4) shows the growth rate of both uncoupled (dashed line) and coupled (solid line) in the presence and absence of η_j. When comparing both the Figs.(4a) and (4b), one can infer that the maximum gain is seen only in the case of coupled BECs for both presence and absence of three body interaction potential. However, the maximum gain of MI is noticed with the case of three body interaction potential.



(a)

(b)

Fig. 4: Comparison of MI gain spectrum between coupled when $\chi_{10} = \chi_{20} = 1.0$, $G_{11} = G_{22} = -0.3$, $G_{12} = G_{21} = -0.1$ (solid line) and uncoupled when $\chi_{10} = 1.0$, $G_{11} = -0.3$ (dashed line) BECs for the values of three body interaction potential coefficients (a). $\eta_1 = \eta_2 = -0.06$ (solid line) and $\eta_1 = -0.06$ (dashed line); (b). $\eta_1 = \eta_2 = 0$ (solid line) and $\eta_1 = 0$ (dashed line).
figure

VI. CONCLUSION

We have considered the quasi-one dimensional CGP equation for the dynamics of the coupled BECs in the presence of three body interaction term. We have investigated the MI of constant amplitude coupled BECs using the linear stability analysis. First, we have investigated the MI of uncoupled BECs when $G_{jl} = 0$ (coupling between the two BECs are zero). Therefore, the uncoupled BECs have been found to be stable for $G_{jj} > 0$ and unstable for $G_{jj} < 0$. Secondly, we investigated the MI of coupled BECs when $G_{jl} > 0$. According to this condition, the coupled BECs have been stable for $G_{jj} > 0$ and unstable for two different cases, one is $G_{11} < 0$ (or $G_{11} > 0$), $G_{22} > 0$ (or $G_{22} < 0$) (coupling between attractive(repulsive) and repulsive(attractive) BECs) and another $G_{jj} < 0$ (coupling between two attractive BECs). Modulationally unstable BECs may appear in the form of either periodic or localized solitary waves which can propagate through the atomic waveguide ('cigar shape' BEC). Finally we compared our results of both uncoupled and coupled BECs in the presence and absence of three body interaction potential.

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