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ON TWO-DIMENSIONAL DYNAMICAL DISLOCATIONS

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The problem of a two-dimensional dynamical dislocation in an isotropic, homogeneous, unbounded, elastic medium is discussed at length. Beginning with the known solution for a line force, the field due to a displacement dislocation is obtained through the Volterra relation. It is shown that the field due to a dip-slip source of arbitrary dip can be expressed in terms of the field due to a vertical dip-slip source and that due to a 45° dip-slip source. The representations of the two-dimensional dynamical sources are obtained in terms of the vector solutions of the Navier equation and in terms of the source potentials. The potential representation is used to get the field due to an arbitrary line source buried in a uniform half-space.

1. *Introduction*

Inspired by the pioneering work of STEKETEE (1958) on the geophysical applications of the elasticity theory of dislocations, MARUYAMA (1963) obtained the Green's functions for a three-dimensional dynamical dislocation in an infinite medium. BEN-MENAHEM and SINGH (1968) obtained the corresponding results for a layer over a half-space. Since then, several investigators have considered the problem of a three-dimensional dynamical dislocation in a layered half-space.

BOORE *et al.* (1971) discussed the antiplane strain problem of a two-dimensional moving dislocation representing a strike-slip fault. NIAZY (1973) obtained analytical expressions for the displacements in an infinite medium caused by a two-dimensional dislocation. GELLER (1974) integrated the three-dimensional solution to obtain the Green's function for a two-dimensional shear dislocation. The analytic solutions obtained have been evaluated numerically for a propagating ramp source function.

In the present paper, we discuss in detail the problem of a two-dimensional dynamical dislocation in infinite and semi-infinite media and indicate how the results for a multilayered half-space can be obtained. Both plane strain and antiplane strain problems are considered. The problem of a two-dimensional static dislocation has been discussed earlier by MARUYAMA (1966), RYBICKI (1971), SINGH and GARG (1985, 1986), and others.

2. *Basic Solution*

Let us consider a homogeneous, isotropic, unbounded, elastic medium of density ρ and Lamé's constants λ, μ. We are interested in a two-dimensional approximation in which the field equations, body force density and boundary conditions are all functions of only two Cartesian coordinates x_α $(\alpha=1, 2)$, so that $\partial/\partial_{x_3}\equiv 0$. Under this assumption, the plane strain problem $(u_3=0)$ and the antiplane strain problem $(u_\alpha=0)$ are decoupled and, therefore, can be tackled separately.

Assuming a harmonic time-dependence $\exp(-i\omega t)$, the equations of motion may be written in the form

$$(\lambda+\mu)\partial_\alpha\partial_\beta u_\beta+\mu\nabla^2 u_\alpha+f_\alpha+\rho\omega^2 u_\alpha=0\,, \tag{1}$$

$$\mu\nabla^2 u_3+f_3+\rho\omega^2 u_3=0\,, \tag{2}$$

where

$$\partial_i\equiv\partial/\partial_{x_i}\,, \qquad \nabla^2\equiv\partial_\alpha\partial_\alpha\,. \tag{3}$$

The summation over repeated suffixes is understood (the Greek subscripts can assume the values 1 and 2 only while the Latin subscripts can assume the values 1, 2, and 3), ω is the angular frequency and $f_i(x_1, x_2)$ is the body force per unit volume. For a line force of magnitude F_i per unit length acting at the point $P(y_1, y_2)$ in the x_i-direction, we have

$$f_i=F_i\delta(x_1-y_1)\delta(x_2-y_2)\,, \qquad (i=1, 2, 3) \tag{4}$$

where $\delta(x)$ is the Dirac delta function.

Let $G^\alpha_\beta(Q, P)$ denote the x_α-component of the displacement at the point $Q(x_1, x_2)$ caused by a line force of unit magnitude acting at the point $P(y_1, y_2)$ in the x_β-direction. $G^3_3(Q, P)$ is similarly defined. The solutions of Eqs. (1) and (2) with f_i of Eq. (4) are (ERINGEN and SUHUBI, 1975; p. 435)

$$G^\alpha_\beta(Q, P)=\frac{i}{4\mu}\left[K_2^{-2}\partial_\alpha\partial_\beta\{H_0(K_2R)-H_0(K_1R)\}+\delta_{\alpha\beta}H_0(K_2R)\right]\,, \tag{5}$$

$$G^3_3(Q, P)=\frac{i}{4\mu}H_0(K_2R)\,, \tag{6}$$

where $H_m(x)\equiv H^{(1)}_m(x)$ is the Hankel function of the mth order and first kind, $\delta_{\alpha\beta}$ is the Kronecker delta and

$$R^2=(x_1-y_1)^2+(x_2-y_2)^2\,, \qquad K_\alpha=\omega/c_\alpha\,,$$
$$c_1=[(\lambda+2\mu)/\rho]^{1/2}\,, \qquad c_2=(\mu/\rho)^{1/2}\,. \tag{7}$$

Equations (5) and (6) are the fundamental solutions of the equations of motion from which the field due to higher order sources, e.g., single couple, double couple, dipole, etc., can be obtained by simple differentiation and superposition.

3. *Displacement Dislocations*

Taking the x_3-axis along the strike of the fault and the x_2-axis vertically downwards, it has been shown by MARUYAMA (1966) that the displacement field due to a long displacement dislocation can be expressed as line integrals:

$$u_\gamma(Q)=\int_L \Delta u_\alpha G^\gamma_{\alpha\beta}(Q, P)n_\beta\,\mathrm{d}s\,, \tag{8}$$

$$u_3(Q)=\int_L \Delta u_3 G^3_{3\beta}(Q, P)n_\beta\,\mathrm{d}s\,, \tag{9}$$

where Δu_i is the relative slip vector and n_β is the unit normal to the fault section L. The Green's functions $G^\gamma_{\alpha\beta}$ and $G^3_{3\beta}$ are given by

$$G^\gamma_{\alpha\beta}(Q, P)=-\lambda\delta_{\alpha\beta}\partial_\varepsilon G^\gamma_\varepsilon-\mu(\partial_\alpha G^\gamma_\beta+\partial_\beta G^\gamma_\alpha)\,, \tag{10}$$

$$G^3_{3\beta}(Q, P)=-\mu\partial_\beta G^3_3\,. \tag{11}$$

From Eqs. (5), (6), (10), and (11), we find

$$G^\gamma_{\alpha\beta}=-\frac{i}{4}[2K_2^{-2}\partial_\alpha\partial_\beta\partial_\gamma\{H_0(K_2R)-H_0(K_1R)\}$$
$$+(\delta_{\beta\gamma}\partial_\alpha+\delta_{\gamma\alpha}\partial_\beta)H_0(K_2R)+(2A-1)\delta_{\alpha\beta}\partial_\gamma H_0(K_1R)]\,, \tag{12}$$

$$G^3_{3\beta}=-\frac{i}{4}\partial_\beta H_0(K_2R)\,, \tag{13}$$

where $A=(\lambda+\mu)/(\lambda+2\mu)$.

If, in Eq. (8), the slip vector Δu_α is along the normal n_β, we have a tensile dislocation while if Δu_α is perpendicular to n_β, we have a shear dislocation. In the latter case, the shear dislocation is of the dip-slip type because the slip is perpendicular to the strike direction. Denoting the dip angle by δ and the magnitude of the slip vector by b, we have, for a dip-slip dislocation

$$\Delta u_1=b\cos\delta\,,\qquad \Delta u_2=b\sin\delta\,,$$
$$n_1=-\sin\delta\,,\qquad n_2=\cos\delta\,. \tag{14}$$

Therefore, for a dip-slip dislocation, Eq. (8) takes the form

$$u_\gamma=\int_L b\left[G^\gamma_{12}\cos 2\delta+\frac{1}{2}(G^\gamma_{22}-G^\gamma_{11})\sin 2\delta\right]\mathrm{d}s\,. \tag{15}$$

It is now apparent that G_{12} represents a vertical dip-slip dislocation and $(1/2)(G_{22}-G_{11})$ represents a dip-slip dislocation on a 45° dipping fault.

Because the slip Δu_3 is perpendicular to the normal n_β, Eq. (9) corresponds to a strike-slip dislocation. Putting

$$\Delta u_3=b\,,\qquad n_1=-\sin\delta\,,\qquad n_2=\cos\delta\,, \tag{16}$$

Eq. (9) becomes

$$u_3=\int_L b(G_{32}^3 \cos\delta - G_{31}^3 \sin\delta)\, ds\,. \tag{17}$$

Therefore, the Green's function G_{31}^3 represents a vertical strike-slip dislocation and G_{32}^3 represents a horizontal strike-slip dislocation.

The stresses corresponding to the displacements of Eqs. (8) and (9) are given by

$$p_{\gamma\varepsilon}(Q)=\int_L \Delta u_\alpha G_{\alpha\beta}^{\gamma\varepsilon}(Q, P) n_\beta\, ds\,, \tag{18}$$

$$p_{3\alpha}(Q)=\int_L \Delta u_3 G_{3\beta}^{3\alpha}(Q, P) n_\beta\, ds\,, \tag{19}$$

where

$$G_{\alpha\beta}^{\gamma\varepsilon}=\lambda\delta_{\gamma\varepsilon}\partial_\theta G_{\alpha\beta}^{\theta}+\mu(\partial_\gamma G_{\alpha\beta}^{\varepsilon}+\partial_\varepsilon G_{\alpha\beta}^{\gamma})\,, \tag{20}$$

$$G_{3\beta}^{3\alpha}=\mu\partial_\alpha G_{3\beta}^{3}\,. \tag{21}$$

From Eqs. (12), (13), (20), and (21), we obtain

$$\begin{aligned} G_{\alpha\beta}^{\gamma\varepsilon}= -\frac{i}{4}\mu[&4K_2^{-2}\partial_\alpha\partial_\beta\partial_\gamma\partial_\varepsilon\{H_0(K_2R)-H_0(K_1R)\} \\ &+(\delta_{\alpha\gamma}\partial_\beta\partial_\varepsilon+\delta_{\alpha\varepsilon}\partial_\beta\partial_\gamma+\delta_{\beta\gamma}\partial_\alpha\partial_\varepsilon+\delta_{\beta\varepsilon}\partial_\alpha\partial_\gamma)H_0(K_2R) \\ &+2(2A-1)(\delta_{\alpha\beta}\partial_\gamma\partial_\varepsilon+\delta_{\gamma\varepsilon}\partial_\alpha\partial_\beta)H_0(K_1R)-(2A-1)^2\delta_{\alpha\beta}\delta_{\gamma\varepsilon}K_2^2H_0(K_1R)]\,, \end{aligned} \tag{22}$$

$$G_{3\beta}^{3\alpha}=-\frac{i}{4}\mu\partial_\alpha\partial_\beta H_0(K_2R)\,. \tag{23}$$

Using the recurrence and differential relations for the Hankel functions, Eqs. (5), (6), (12), (13), (22), and (23) yield

$$\begin{aligned} G_\alpha^\beta=\frac{i}{8\mu}\Bigg[&\left(\frac{c_2}{c_1}\right)^2\{\delta_{\alpha\beta}H_0(K_1R)+(\delta_{\alpha\beta}-2X_\alpha X_\beta)H_2(K_1R)\} \\ &+\delta_{\alpha\beta}H_0(K_2R)-(\delta_{\alpha\beta}-2X_\alpha X_\beta)H_2(K_2R)\Bigg]\,, \end{aligned} \tag{24}$$

$$G_3^3=\frac{i}{4\mu}H_0(K_2R)\,, \tag{25}$$

$$\begin{aligned} G_{\alpha\beta}^{\gamma}=\frac{i}{8}K_2\Bigg[&\left(\frac{c_2}{c_1}\right)^3\Bigg\{\left(2\frac{2A-1}{1-A}\delta_{\alpha\beta}X_\gamma+\sum\delta_{\beta\gamma}X_\alpha\right)H_1(K_1R) \\ &+(\sum\delta_{\beta\gamma}X_\alpha-4X_\alpha X_\beta X_\gamma)H_3(K_1R)\Bigg\} \\ &+(\sum\delta_{\beta\gamma}X_\alpha-2\delta_{\alpha\beta}X_\gamma)H_1(K_2R)-(\sum\delta_{\beta\gamma}X_\alpha-4X_\alpha X_\beta X_\gamma)H_3(K_2R)\Bigg]\,, \end{aligned} \tag{26}$$

$$G^{3}_{3\beta}=\frac{i}{4}K_2X_\beta H_1(K_2R)\,, \tag{27}$$

$$\begin{aligned}G^{\gamma\varepsilon}_{\alpha\beta}=\frac{i}{24}\mu K_2^2\Bigg[&3\left(\frac{c_2}{c_1}\right)^4\cdot\left\{2\frac{2A-1}{1-A}\left(\frac{c_1}{c_2}\right)^2\delta_{\alpha\beta}\delta_{\gamma\varepsilon}+\sum\delta_{\alpha\beta}\delta_{\gamma\varepsilon}\right\}H_0(K_1R)\\&+4\left(\frac{c_2}{c_1}\right)^4\left\{3\frac{2A-1}{1-A}(\delta_{\alpha\beta}\delta_{\gamma\varepsilon}-\delta_{\alpha\beta}X_\gamma X_\varepsilon-\delta_{\gamma\varepsilon}X_\alpha X_\beta)\right.\\&\left.+\sum\delta_{\alpha\beta}\delta_{\gamma\varepsilon}-\sum\delta_{\alpha\beta}X_\gamma X_\varepsilon\right\}H_2(K_1R)\\&+\left(\frac{c_2}{c_1}\right)^4(\sum\delta_{\alpha\beta}\delta_{\gamma\varepsilon}-4\sum\delta_{\alpha\beta}X_\gamma X_\varepsilon+24X_\alpha X_\beta X_\gamma X_\varepsilon)H_4(K_1R)\\&+3(\sum\delta_{\alpha\beta}\delta_{\gamma\varepsilon}-2\delta_{\alpha\beta}\delta_{\gamma\varepsilon})H_0(K_2R)\\&+2(\sum\delta_{\alpha\beta}\delta_{\gamma\varepsilon}-3\delta_{\alpha\beta}\delta_{\gamma\varepsilon}+3\delta_{\alpha\beta}X_\gamma X_\varepsilon+3\delta_{\gamma\varepsilon}X_\alpha X_\beta-\sum\delta_{\alpha\beta}X_\gamma X_\varepsilon)H_2(K_2R)\\&-(\sum\delta_{\alpha\beta}\delta_{\gamma\varepsilon}-4\sum\delta_{\alpha\beta}X_\gamma X_\varepsilon+24X_\alpha X_\beta X_\gamma X_\varepsilon)H_4(K_2R)\Bigg]\,,\end{aligned} \tag{28}$$

$$G^{3\alpha}_{3\beta}=\frac{i}{8}\mu K_2^2[\delta_{\alpha\beta}H_0(K_2R)+(\delta_{\alpha\beta}-2X_\alpha X_\beta)H_2(K_2R)]\,, \tag{29}$$

where

$$X_\alpha=(x_\alpha-y_\alpha)/R\,,\qquad X_\alpha X_\alpha=1\,, \tag{30}$$

$$\sum\delta_{\beta\gamma}X_\alpha=\delta_{\beta\gamma}X_\alpha+\delta_{\gamma\alpha}X_\beta+\delta_{\alpha\beta}X_\gamma\,, \tag{31}$$

$$\sum\delta_{\alpha\beta}\delta_{\gamma\varepsilon}=\delta_{\alpha\beta}\delta_{\gamma\varepsilon}+\delta_{\alpha\gamma}\delta_{\beta\varepsilon}+\delta_{\alpha\varepsilon}\delta_{\beta\gamma}\,, \tag{32}$$

$$\sum\delta_{\alpha\beta}X_\gamma X_\varepsilon=\delta_{\alpha\beta}X_\gamma X_\varepsilon+\delta_{\alpha\gamma}X_\beta X_\varepsilon+\delta_{\alpha\varepsilon}X_\beta X_\gamma+\delta_{\beta\gamma}X_\alpha X_\varepsilon+\delta_{\beta\varepsilon}X_\alpha X_\gamma+\delta_{\gamma\varepsilon}X_\alpha X_\beta\,. \tag{33}$$

4. *Potential Representation*

Let

$$X_1=(x_1-y_1)/R=\cos\theta\,,\qquad X_2=(x_2-y_2)/R=\sin\theta\,, \tag{34}$$

and

$$\Phi^{c,s}_m(R,\theta)=H_m(K_1R)(\cos m\theta,\sin m\theta)\,, \tag{35}$$

$$\Psi^{c,s}_m(R,\theta)=H_m(K_2R)(\cos m\theta,\sin m\theta)\,. \tag{36}$$

The potentials Φ^σ_m and Ψ^σ_m ($\sigma=c$ or s) satisfy the two-dimensional Helmholtz equation:

$$(\nabla^2+K_1^2)\Phi^\sigma_m=0\,,\qquad(\nabla^2+K_2^2)\Psi^\sigma_m=0\,. \tag{37}$$

From Eqs. (24), (25), (34), and (35), we find

$$G_1^1 = \frac{i}{8\mu}\left[\left(\frac{c_2}{c_1}\right)^2 (\Phi_0^c - \Phi_2^c) + \Psi_0^c + \Psi_2^c\right], \tag{38}$$

$$G_1^2 = G_2^1 = \frac{i}{8\mu}\left[-\left(\frac{c_2}{c_1}\right)^2 \Phi_2^s + \Psi_2^s\right], \tag{39}$$

$$G_2^2 = \frac{i}{8\mu}\left[\left(\frac{c_2}{c_1}\right)^2 (\Phi_0^c + \Phi_2^c) + \Psi_0^c - \Psi_2^c\right], \tag{40}$$

$$G_3^3 = \frac{i}{4\mu}\Psi_0^c. \tag{41}$$

Similarly, Eqs. (26), (27), and (34)–(36) yield

$$G_{11}^1 = \frac{i}{8}K_2\left[\left(\frac{c_2}{c_1}\right)^3\left(\frac{1+A}{1-A}\Phi_1^c - \Phi_3^c\right) + \Psi_1^c + \Psi_3^c\right], \tag{42}$$

$$G_{11}^2 = \frac{i}{8}K_2\left[\left(\frac{c_2}{c_1}\right)^3\left(\frac{3A-1}{1-A}\Phi_1^s - \Phi_3^s\right) - \Psi_1^s + \Psi_3^s\right], \tag{43}$$

$$G_{22}^1 = \frac{i}{8}K_2\left[\left(\frac{c_2}{c_1}\right)^3\left(\frac{3A-1}{1-A}\Phi_1^c + \Phi_3^c\right) - \Psi_1^c - \Psi_3^c\right], \tag{44}$$

$$G_{22}^2 = \frac{i}{8}K_2\left[\left(\frac{c_2}{c_1}\right)^3\left(\frac{1+A}{1-A}\Phi_1^s + \Phi_3^s\right) + \Psi_1^s - \Psi_3^s\right], \tag{45}$$

$$G_{12}^1 = \frac{i}{8}K_2\left[\left(\frac{c_2}{c_1}\right)^3 (\Phi_1^s - \Phi_3^s) + \Psi_1^s + \Psi_3^s\right], \tag{46}$$

$$G_{12}^2 = \frac{i}{8}K_2\left[\left(\frac{c_2}{c_1}\right)^3 (\Phi_1^c + \Phi_3^c) + \Psi_1^c - \Psi_3^c\right], \tag{47}$$

$$G_{31}^3 = \frac{i}{4}K_2\Psi_1^c, \qquad G_{32}^3 = \frac{i}{4}K_2\Psi_1^s. \tag{48}$$

5. *Vector Representation*

For a plane strain problem ($u_\alpha = u_\alpha(x_1, x_2)$, $u_3 = 0$), two independent solutions of the homogeneous Navier equation

$$(\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \mu\nabla^2\boldsymbol{u} + \rho\omega^2\boldsymbol{u} = \boldsymbol{0} \tag{49}$$

may be taken in the form

$$\begin{aligned} \boldsymbol{L}_m^\sigma &= \frac{1}{K_1}\,\mathrm{grad}\,\Phi_m^\sigma \\ &= \left(\boldsymbol{e}_R\frac{\partial}{\partial K_1 R} + \boldsymbol{e}_\theta\frac{1}{K_1 R}\frac{\partial}{\partial\theta}\right)\Phi_m^\sigma, \end{aligned} \tag{50}$$

$$N_m^\sigma = \frac{1}{K_2}\,\mathrm{curl}(e_z \Psi_m^\sigma) = \left(e_R \frac{1}{K_2 R}\frac{\partial}{\partial\theta} - e_\theta \frac{\partial}{\partial K_2 R}\right)\Psi_m^\sigma\,, \tag{51}$$

where

$$\begin{aligned} e_R &= e_1 \cos\theta + e_2 \sin\theta\,, \\ e_\theta &= e_2 \cos\theta - e_1 \sin\theta\,, \\ e_z &= e_3\,, \end{aligned} \tag{52}$$

and e_i is the unit vector in the x_i-direction.

We define

$$G_\alpha = e_1 G_\alpha^1 + e_2 G_\alpha^2 = e_R G_\alpha^R + e_\theta G_\alpha^\theta\,, \tag{53}$$

with

$$G_\alpha^R = G_\alpha^1 \cos\theta + G_\alpha^2 \sin\theta\,, \qquad G_\alpha^\theta = G_\alpha^2 \cos\theta - G_\alpha^1 \sin\theta\,. \tag{54}$$

The vector $G_{\alpha\beta}$ is defined similarly. Equations (38)–(41), (42)–(47), and (50)–(54) yield

$$G_1 = \frac{i}{4\mu}\left[\left(\frac{c_2}{c_1}\right)^2 L_1^c + N_1^s\right], \tag{55}$$

$$G_2 = \frac{i}{4\mu}\left[\left(\frac{c_2}{c_1}\right)^2 L_1^s - N_1^c\right], \tag{56}$$

$$G_{11} = \frac{i}{4} K_2\left[\left(\frac{c_2}{c_1}\right)^3 \left(-\frac{A}{1-A} L_0^c + L_2^c\right) + N_2^s\right], \tag{57}$$

$$G_{22} = -\frac{i}{4} K_2\left[\left(\frac{c_2}{c_1}\right)^3 \left(\frac{A}{1-A} L_0^c + L_2^c\right) + N_2^s\right], \tag{58}$$

$$G_{12} = \frac{i}{4} K_2\left[\left(\frac{c_2}{c_1}\right)^3 L_2^s - N_2^c\right], \tag{59}$$

$$\frac{1}{2}(G_{22} - G_{11}) = -\frac{i}{4} K_2\left[\left(\frac{c_2}{c_1}\right)^3 L_2^c + N_2^s\right]. \tag{60}$$

6. *Source Potentials*

For a plane strain problem, we define the potentials Φ_s and Ψ_s through the equations

$$G_s = \mathrm{grad}\,\Phi_s + \mathrm{curl}(e_z \Psi_s) \qquad (s = 1, 2, 11, 22 \text{ or } 12)\,. \tag{61}$$

Equations (55)–(61) then yield the following expressions for the potentials for various sources considered:

$$\Phi_1 = \frac{i}{4\mu K_1}\left(\frac{c_2}{c_1}\right)^2 \Phi_1^c, \tag{62}$$

$$\Psi_1 = \frac{i}{4\mu K_2} \Psi_1^s, \tag{63}$$

$$\Phi_2 = \frac{i}{4\mu K_1}\left(\frac{c_2}{c_1}\right)^2 \Phi_1^s, \tag{64}$$

$$\Psi_2 = -\frac{i}{4\mu K_2} \Psi_1^c, \tag{65}$$

$$\Phi_{11} = \frac{i}{4}\left(\frac{c_2}{c_1}\right)^2 \left(-\frac{A}{1-A}\Phi_0^c + \Phi_2^c\right), \tag{66}$$

$$\Psi_{11} = \frac{i}{4} \Psi_2^s, \tag{67}$$

$$\Phi_{22} = -\frac{i}{4}\left(\frac{c_2}{c_1}\right)^2 \left(\frac{A}{1-A}\Phi_0^c + \Phi_2^c\right), \tag{68}$$

$$\Psi_{22} = -\frac{i}{4} \Psi_2^s, \tag{69}$$

$$\Phi_{12} = \frac{i}{4}\left(\frac{c_2}{c_1}\right)^2 \Phi_2^s, \tag{70}$$

$$\Psi_{12} = -\frac{i}{4} \Psi_2^c, \tag{71}$$

$$\frac{1}{2}(\Phi_{22} - \Phi_{11}) = -\frac{i}{4}\left(\frac{c_2}{c_1}\right)^2 \Phi_2^c, \tag{72}$$

$$\frac{1}{2}(\Psi_{22} - \Psi_{11}) = -\frac{i}{4} \Psi_2^s, \tag{73}$$

where the functions Φ_m^σ and Ψ_m^σ are defined in Eqs. (35) and (36), respectively.

The potentials given in Eqs. (62)–(73) can be used for solving boundary-value problems involving cylindrical or plane-parallel boundaries in a plane strain problem. However, in an antiplane strain problem, Eqs. (41) and (48) can be used directly.

7. *Plane Waves*

The solutions obtained in Secs. 4–6 are in the form of cylindrical waves emanating from a line source. At large distances from the source, a cylindrical wave can be regarded as approximately a plane wave. In fact, it is possible to express the scalar and vector cylindrical wave functions as integrals over plane waves, the representation being valid for all distances.

A longitudinal plane wave whose propagation vector is

$$\boldsymbol{K}_1 = K_1 \boldsymbol{p} = K_1(\boldsymbol{e}_1 \cos\beta + \boldsymbol{e}_2 \sin\beta)$$

can be represented by

$$\phi(\beta) = e^{i\boldsymbol{K}_1 \cdot \boldsymbol{R}} = e^{iK_1(\boldsymbol{p}\cdot\boldsymbol{R})}, \tag{74}$$

where β is the angle which the direction of propagation makes with the x_1-axis and

$$\boldsymbol{R} = (x_1 - y_1)\boldsymbol{e}_1 + (x_2 - y_2)\boldsymbol{e}_2 = R(\boldsymbol{e}_1 \cos\theta + \boldsymbol{e}_2 \sin\theta).$$

Similarly, a transverse plane wave can be represented by

$$\psi(\beta) = e^{i\boldsymbol{K}_2 \cdot \boldsymbol{R}} = e^{iK_2(\boldsymbol{p}\cdot\boldsymbol{R})}. \tag{75}$$

The potentials ϕ and ψ satisfy the equations

$$(\nabla^2 + K_1^2)\phi = 0, \qquad (\nabla^2 + K_2^2)\psi = 0. \tag{76}$$

Two independent solutions of the vector Navier Eq. (49) representing plane longitudinal and transverse waves can be constructed from ϕ and ψ, respectively, as follows:

$$\boldsymbol{l}(\beta) = \frac{1}{K_1} \operatorname{grad} \phi = i\boldsymbol{p}\phi = i(\boldsymbol{e}_1 \cos\beta + \boldsymbol{e}_2 \sin\beta)\phi(\beta), \tag{77}$$

$$\boldsymbol{n}(\beta) = \frac{1}{K_2} \operatorname{curl}(\boldsymbol{e}_3\psi) = i(\boldsymbol{p}\boldsymbol{e}_3)\psi = i(\boldsymbol{e}_1 \sin\beta - \boldsymbol{e}_2 \cos\beta)\psi(\beta). \tag{78}$$

The cylindrical wave potentials Φ_m and Ψ_m defined in Eqs. (35) and (36) can be expressed in terms of the plane wave potentials ϕ and ψ. We have (STRATTON, 1941, p. 396; BOSTRÖM *et al.*, 1981)

$$H_m(K_\alpha R)(\cos m\theta, \sin m\theta) = \frac{i^{-m}}{\pi} \int_{C^\pm} e^{iK_\alpha(\boldsymbol{p}\cdot\boldsymbol{R})}(\cos m\beta, \sin m\beta)\,d\beta,$$

$$(\alpha = 1, 2;\ x_2 \gtrless y_2) \tag{79}$$

i.e.,

$$\Phi_m^{c,s} = \frac{i^{-m}}{\pi} \int_{C^\pm} \phi(\beta)(\cos m\beta, \sin m\beta)\,d\beta, \tag{80}$$

$$\Psi_m^{c,s} = \frac{i^{-m}}{\pi} \int_{C^\pm} \psi(\beta)(\cos m\beta, \sin m\beta)\,d\beta. \tag{81}$$

The contour C^+ (for $x_2 > y_2$) in the complex β-plane begins at $i\infty$, runs along the imaginary axis up to the origin, then along the real axis up to $\beta = \pi$ and finally parallel to the imaginary axis to $\pi - i\infty$. In contrast, the contour C^- (for $x_2 < y_2$) begins at $\pi + i\infty$, runs parallel to the imaginary axis up to $\beta = \pi$, then along the real axis up to $\beta=2\pi$ and finally parallel to the imaginary axis to $2\pi-i\infty$. Thus, if $\beta\in C^+$, then $\cos\beta$ is real and $Im(\sin\beta)\geq 0$ and, if $\beta\in C^-$, then $\cos\beta$ is real but $Im(\sin\beta)\leq 0$. Consequently, $\exp[iK_\alpha(x_2-y_2)\sin\beta]$ is bounded for $x_2\geq y_2$ $(x_2\leq y_2)$ when $\beta\in C^+$ $(\beta\in C^-)$.

Taking the gradient and the curl of the relevant functions, the vector cylindrical waves can be expressed in terms of vector plane waves. Equations (50), (51), and (77)–(81) yield

$$\boldsymbol{L}_m^{c,s}=\frac{i^{-m}}{\pi}\int_{C^\pm}\boldsymbol{l}(\beta)(\cos m\beta, \sin m\beta)\,\mathrm{d}\beta\,, \tag{82}$$

$$\boldsymbol{N}_m^{c,s}=\frac{i^{-m}}{\pi}\int_{C^\pm}\boldsymbol{n}(\beta)(\cos m\beta, \sin m\beta)\,\mathrm{d}\beta\,. \tag{83}$$

Let us introduce the change of variable $K=K_\alpha\cos\beta$. Then,

$$\begin{aligned}K_\alpha\sin\beta&=(K_\alpha^2-K^2)^{1/2} \qquad \text{for} \quad K<K_\alpha\\ &=\pm i(K^2-K_\alpha^2)^{1/2} \qquad \text{for} \quad K>K_\alpha\end{aligned} \tag{84}$$

on $C^\pm$. Thus, the exponential term in the plane wave solutions can be written as

$$\begin{aligned}\exp[iK_\alpha(\boldsymbol{p}\cdot\boldsymbol{R})]&=\exp[iK_\alpha\{(x_1-y_1)\cos\beta+(x_2-y_2)\sin\beta\}]\\ &=\exp[iK(x_1-y_1)-a_\alpha|x_2-y_2|]\,,\end{aligned} \tag{85}$$

where

$$\begin{aligned}a_\alpha&=(K^2-K_\alpha^2)^{1/2} \qquad \text{for} \quad K>K_\alpha\\ &=\mp i(K_\alpha^2-K^2)^{1/2} \qquad \text{for} \quad K<K_\alpha\,.\end{aligned} \tag{86}$$

Therefore, if we agree to write $\phi(K)$ for $\phi(\beta(K))$ and $\psi(K)$ for $\psi(\beta(K))$, we have

$$\phi(K)=\exp[iK(x_1-y_1)-a_1|x_2-y_2|]\,, \tag{87}$$

$$\psi(K)=\exp[iK(x_1-y_1)-a_2|x_2-y_2|]\,, \tag{88}$$

$$\boldsymbol{l}(K)=\frac{1}{K_1}(iK\boldsymbol{e}_1\mp a_1\boldsymbol{e}_2)\phi(K)\,, \tag{89}$$

$$\boldsymbol{n}(K)=\frac{1}{K_2}(\mp a_2\boldsymbol{e}_1-iK\boldsymbol{e}_2)\psi(K)\,, \tag{90}$$

valid for $x_2\gtrless y_2$.

With the change of variable $K=K_1\cos\beta$, Eq. (80) becomes

$$\Phi_m^{c,s}=\frac{i^{-m-1}}{\pi}\int_{-\infty}^{\infty}\phi(K)(\cos m\beta, \sin m\beta)\left(\frac{1}{a_1}\right)\mathrm{d}K\,, \tag{91}$$

and the path of integration in the complex K-plane is simply the real axis. Similarly, Eqs. (81)–(83) become

$$\Psi_m^{c,s}=\frac{i^{-m-1}}{\pi}\int_{-\infty}^{\infty}\psi(K)(\cos m\beta, \sin m\beta)\left(\frac{1}{a_2}\right)dK\,, \tag{92}$$

$$L_m^{c,s}=\frac{i^{-m-1}}{\pi}\int_{-\infty}^{\infty}l(K)(\cos m\beta, \sin m\beta)\left(\frac{1}{a_1}\right)dK\,, \tag{93}$$

$$N_m^{c,s}=\frac{i^{-m-1}}{\pi}\int_{-\infty}^{\infty}n(K)(\cos m\beta, \sin m\beta)\left(\frac{1}{a_2}\right)dK\,. \tag{94}$$

In the right-hand sides of Eqs. (91)–(94), $\cos m\beta$ and $\sin m\beta$ must be expressed in terms of K through the relations $K_\alpha\cos\beta=K$ and $K_\alpha\sin\beta=\pm ia_\alpha$ for $x_2\gtrless y_2$, where $\alpha=1$ for Eqs. (91) and (93) and $\alpha=2$ for Eqs. (92) and (94).

Equations (91) and (92) can be used to express the source potentials obtained in Sec. 6 in terms of the plane wave potentials $\phi(K)$ and $\psi(K)$. Similarly, Eqs. (93) and (94) can be used to express the displacement field due to various line sources obtained in Sec. 5 in terms of the plane wave solutions $\boldsymbol{l}(K)$ and $\boldsymbol{n}(K)$.

From Eqs. (62)–(73), (91), and (92), we find that the source potentials Φ_1, Ψ_1, etc., can be expressed as Fourier integrals in the form

$$\Phi^0=\frac{1}{\pi K_2^2}\int_0^{\infty}e^{-a_1|x_2-y_2|}[A^0\cos(Kx_1)+B^0\sin(Kx_1)]\,dK\,, \tag{95}$$

$$\Psi^0=\frac{1}{\pi K_2^2}\int_0^{\infty}e^{-a_2|x_2-y_2|}[C^0\cos(Kx_1)+D^0\sin(Kx_1)]\,dK\,, \tag{96}$$

where we have taken $y_1=0$. The source coefficients A^0, B^0, C^0, and D^0 obtained for various sources are given in Table 1. The results for the centre of dilatation are obtained by superposing the corresponding results for the sources (11) and (22). In fact, source (11) is equivalent to the centre of dilatation plus a dipole in the x_1-direction, of suitable strengths. A similar interpretation can be given for the source (22).

The displacement components u_1 and u_2 and the stress components p_{12} and p_{22} can be obtained from the potentials Φ^0 and Ψ^0 through the relations [cf., Eq. (61)]

$$u_1=\partial_1\Phi^0+\partial_2\Psi^0\,,\qquad u_2=\partial_2\Phi^0-\partial_1\Psi^0\,, \tag{97}$$

$$p_{12}=\mu(2\partial_1\partial_2\Phi^0-\partial_1\partial_1\Psi^0+\partial_2\partial_2\Psi^0)\,, \tag{98}$$

$$p_{22}=\mu[(c_1/c_2)^2\nabla^2\Phi^0-2(\partial_1\partial_1\Phi^0+\partial_1\partial_2\Psi^0)]\,. \tag{99}$$

Equations (95)–(99) yield

$$u_1=\frac{1}{\pi K_2^2}\int_0^{\infty}[K\,e^{-a_1|x_2-y_2|}(B^0\cos Kx_1-A^0\sin Kx_1)$$
$$\mp a_2\,e^{-a_2|x_2-y_2|}(C^0\cos Kx_1+D^0\sin Kx_1)]\,dK\,, \tag{100}$$

$$u_2=\frac{1}{\pi K_2^2}\int_0^\infty [\mp a_1\,\mathrm{e}^{-a_1|x_2-y_2|}(A^0\cos Kx_1+B^0\sin Kx_1)$$
$$-K\,\mathrm{e}^{-a_2|x_2-y_2|}(D^0\cos Kx_1-C^0\sin Kx_1)]\,\mathrm{d}K\,, \tag{101}$$
$$p_{12}=\frac{2\mu}{\pi K_2^2}\int_0^\infty [\mp Ka_1\,\mathrm{e}^{-a_1|x_2-y_2|}(B^0\cos Kx_1-A^0\sin Kx_1)$$
$$+b_2^2\,\mathrm{e}^{-a_2|x_2-y_2|}(C^0\cos Kx_1+D^0\sin Kx_1)]\,\mathrm{d}K\,, \tag{102}$$
$$p_{22}=\frac{2\mu}{\pi K_2^2}\int_0^\infty [b_2^2\,\mathrm{e}^{-a_1|x_2-y_2|}(A^0\cos Kx_1+B^0\sin Kx_1)$$
$$\pm Ka_2\,\mathrm{e}^{-a_2|x_2-y_2|}(D^0\cos Kx_1-C^0\sin Kx_1)]\,\mathrm{d}K\,, \tag{103}$$

where

$$2b_2^2=2K^2-K_2^2\,. \tag{104}$$

A close look at the coefficients given in Table 1 reveals that, in general, u_1, u_2, p_{12}, and p_{22} are discontinuous across the plane $x_2=y_2$. This discontinuity gives a representation of the source which can be conveniently used in the Thomson-Haskell matrix formulation of the problem of a line source in a multilayered half-space. Inserting the values of the coefficients A^0, B^0, C^0, and D^0 from Table 1 in Eqs. (100)–(103), we get the desired jumps in u_1, u_2, p_{12}, and p_{22} across the plane $x_2=y_2$.

The results for the antiplane strain case are much simpler. Equations (41), (48), (91), and (92) yield

$$G_3^3=\frac{1}{2\pi\mu}\int_0^\infty \mathrm{e}^{-a_2|x_2-y_2|}\cos Kx_1\left(\frac{1}{a_2}\right)\mathrm{d}K\,, \tag{105}$$
$$G_{31}^3=\frac{1}{2\pi}\int_0^\infty \mathrm{e}^{-a_2|x_2-y_2|}\sin Kx_1\left(\frac{K}{a_2}\right)\mathrm{d}K\,, \tag{106}$$

Table 1. Source coefficients.

Source	A^0	B^0	C^0	D^0
Force in the x_1-direction	0	$K/2\mu a_1$	$\pm 1/2\mu$	0
Force in the x_2-direction	$\pm 1/2\mu$	0	0	$-K/2\mu a_2$
Source (11)	$-(AK_2^2+b_2^2)/a_1$	0	0	$\pm K$
Source (22)	b_2^2/a_1	0	0	$\mp K$
Centre of dilatation	$-AK_2^2/a_1$	0	0	0
Dipole in the x_1-direction	$-K^2/2a_1$	0	0	$\pm K/2$
Dipole in the x_2-direction	$a_1/2$	0	0	$\mp K/2$
Vertical dip slip	0	$\pm K$	b_2^2/a_2	0
45^0 dip-slip	b_1^2/a_1	0	0	$\mp K$

[The upper sign is for $x_2>y_2$ and the lower sign is for $x_2<y_2$. Also, $2b_1^2=2K^2-K_1^2$, $2b_2^2=2K^2-K_2^2$, $A=(\lambda+\mu)/(\lambda+2\mu)$.]

$$G_{32}^{3} = \pm\frac{1}{2\pi}\int_{0}^{\infty} e^{-a_2|x_2-y_2|}\cos Kx_1\,dK\,. \tag{107}$$

Equations (105)–(107) represent displacements. The stresses p_{31} and p_{32} can be obtained by simple differentiation. The jumps in u_3 and p_{32} then follow immediately.

SINGH and GARG (1985, 1986) studied in detail the representation of two-dimensional static sources and obtained the jumps in the displacements and stresses across the horizontal plane passing through the line source. The jumps obtained from Eqs. (100)–(103) and (105)–(107) coincide with the corresponding results of SINGH and GARG (1985, 1986).

8. *A Line Source in a Uniform Half-Space*

Let a line source of unit strength be situated at the point (0, h) of a uniform half-space $x_2 \geq 0$. We assume plane strain and a stress-free boundary, resulting in the boundary conditions

$$p_{21} = p_{22} = 0 \qquad \text{at} \quad x_2 = 0\,. \tag{108}$$

The potentials Φ^0 and Ψ^0 for the line source in an unbounded medium are given by Eqs. (95) and (96) with $y_2 = h$. For the line source in the half-space, we assume

$$\Phi = \Phi^0 + \frac{1}{\pi K_2^2}\int_0^{\infty} e^{-a_1 x_2}(A\cos Kx_1 + B\sin Kx_1)\,dK\,, \tag{109}$$

$$\Psi = \Psi^0 + \frac{1}{\pi K_2^2}\int_0^{\infty} e^{-a_2 x_2}(C\cos Kx_1 + D\sin Kx_1)\,dK\,, \tag{110}$$

where $A(K)$, $B(K)$, $C(K)$, and $D(K)$ are unknown functions to be determined from the boundary conditions. We find

$$F(K)A = -E(K)A^{-}e^{-a_1 h} + 2Ka_2 b_2^2 D^{-} e^{-a_2 h}\,, \tag{111}$$

$$F(K)B = -E(K)B^{-}e^{-a_1 h} - 2Ka_2 b_2^2 C^{-} e^{-a_2 h}\,, \tag{112}$$

$$F(K)C = -2Ka_1 b_2^2 B^{-} e^{-a_1 h} - E(K)C^{-} e^{-a_2 h}\,, \tag{113}$$

$$F(K)D = 2Ka_1 b_2^2 A^{-} e^{-a_1 h} - E(K)D^{-} e^{-a_2 h}\,, \tag{114}$$

where

$$F(K) = b_2^4 - K^2 a_1 a_2\,, \qquad E(K) = b_2^4 + K^2 a_1 a_2\,. \tag{115}$$

It is noticed from Table 1 that the coefficients A^0, B^0, C^0, and D^0 might have different values for $x_2 \gtrless y_2$; A^-, B^-, C^-, and D^- are the values of A^0, B^0, C^0, and D^0, respectively, valid for $x_2 < y_2$.

Equations (109)–(114) give the formal solution of the problem. The surface displacements are found to be

$$u_1=\frac{1}{\pi}\int_0^\infty [Ka_1\,e^{-a_1h}(-B^-\cos Kx_1+A^-\sin Kx_1)$$

$$-b_2^2\,e^{-a_2h}(C^-\cos Kx_1+D^-\sin Kx_1)]\frac{a_2}{F(K)}\,dK\,, \tag{116}$$

$$u_2=\frac{1}{\pi}\int_0^\infty [-b_2^2\,e^{-a_1h}(A^-\cos Kx_1+B^-\sin Kx_1)$$

$$+Ka_2\,e^{-a_2h}(D^-\cos Kx_1-C^-\sin Kx_1)]\frac{a_1}{F(K)}\,dK\,. \tag{117}$$

Equations (116) and (117) are very general. On substituting the values of the source coefficients A^-, B^-, C^-, and D^- from Table 1, we get the surface displacements for different line sources. The integrals in Eqs. (116) and (117) can be evaluated by the Cagniard-de Hoop Technique. Alternatively, one can use Lapwood's method of integration (LAPWOOD, 1949) to get the various pulses generated by the line sources under consideration. These pulses are: the P-pulse, the S-pulse, the surface P-pulse, the surface S-pulse, and the Rayleigh-pulse.

9. *Conclusions*

Let us review the importance of the various results obtained. Equation (15) shows that the field due to an arbitrary dip-slip dislocation can be expressed in terms of the field due to a vertical dip-slip and that due to a 45° dip-slip. Equations (55)–(60) give an elegant and compact representation of various line sources in terms of the vector solutions of the Navier equation. Equations (62)–(73), (95), and (96) yield the source potentials for various sources which can be used in solving boundary-value problems involving cylindrical or plane-parallel boundaries. Finally, Eqs. (116) and (117) give the surface displacements due to an arbitrary line source buried in a uniform half-space.

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