

# Exotic atoms in two dimensions\*

M Combescure<sup>1†</sup>, C Fayard<sup>1‡</sup>, A Khare<sup>2§</sup>, J-M Richard<sup>1¶</sup>

<sup>1</sup>Institut de Physique Nucléaire de Lyon, Université de Lyon, CNRS-IN2P3-UCBL,  
4, rue Enrico Fermi, Villeurbanne, France

<sup>2</sup>Raja Ramanna Fellow, IISER, Pune 411021, India

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## Abstract

We study the behavior of energy levels in two dimensions for exotic atoms, i.e., when a long-range attractive potential is supplemented by a short-range interaction, and compare the results with these of the one- and three-dimensional cases. The energy shifts are well reproduced by a scattering length formula  $\delta E = A_0^2 / \ln(a/R)$ , where  $a$  is the scattering length in the short-range potential,  $A_0^2/(2\pi)$  the square of the wave function at the origin in the external potential, and  $R$  is related to the derivative with respect to the energy of the solution that is regular at large distances.

## 1 Introduction

Hadronic atoms give valuable information about strong interactions at low energy. For a review, see, e.g., [1]. They have also motivated several studies on the behavior of the energy levels in a Schrödinger operator, with a potential  $V_1 + \lambda V_2$ , where  $V_1$  dominates at large distances, but is superseded by  $V_2$  at short distances. The case of exotic atoms corresponds to a world with three dimensions, where  $V_1 = -1/r$  (as a negatively-charged hadron orbits near the nucleus and is almost unscreened by the remaining electrons, if any), and  $V_2$  describes the short-range hadronic interaction. But the situation is far more general, and many features do not depend on the Coulomb character of  $V_1$ . Nevertheless, we shall use the word “exotic atom” for such a system, “atomic” for the energy domain of the eigenstates of  $V_1$  alone, and “nuclear” for any typical energy within  $V_2$  alone, for the sake of simplicity.

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†m.combescure@ipnl.in2p3.fr

‡c.fayard@ipnl.in2p3.fr

§khare@iiserpune.ac.in

¶j-m.richard@ipnl.in2p3.fr

The spectral problem of exotic atoms [1–3] differs significantly from the ordinary perturbation theory, for which an expansion of the eigenenergies in powers of  $\lambda$  is attempted. For exotic atoms, the energies for  $V_1 + \lambda V_2$  are often very close to the ones for  $V_1$  alone, but perturbation theory usually does not hold. For instance, if  $\lambda V_2$  is an infinite hard core of small radius, the energies are slightly shifted upwards, but the ordinary perturbative expansion diverges already at the first order. The proper concept here is “radius perturbation theory”, as described by Mandelszweig [4].

In this paper, we discuss how exotic atoms behave in  $d = 2$  dimensions. It may be noted that the study of exotic atoms in  $d = 1$  is more straightforward, and already discussed in the literature [3]. The  $d = 2$  case is more delicate. The leading order term for the energy shift is easily identified, and linked to  $\ln a$ , where  $a$  is the scattering length in the short-range potential. As in the  $d = 3$  case, the overall coefficient is the square of the wave-function at the origin in the external potential. However, the scale regularizing this leading term, i.e., the radius  $R$  leading to  $\ln a \rightarrow \ln(a/R)$  is not immediate, but it can be derived from a matching of the solution of  $V_1$  which is normalizable to the asymptotic solution emerging from the short-range term  $\lambda V_2$ .

The case of  $d = 2$  dimensions is rather special in spectral problems, as it corresponds to the largest value of  $d$  for which an attractive potential, however weak, always holds at least one bound state, see, e.g., [5, 6].<sup>1</sup> Hence, for  $d \leq 2$ , if  $V_2$  is attractive,  $\lambda V_2$  immediately develops its own bound state, which becomes the ground state of the Hamiltonian. However, this process is less effective for  $d = 2$  than for  $d = 1$ , and the spectrum, as a function of  $\lambda$  evolves more slowly.

This paper is organized as follows. In Sec. 2 and Sec. 3, we give a brief reminder about the cases of  $d = 3$  and  $d = 1$  space dimensions respectively, with particular emphasis on the phenomenon of level rearrangement and on the scattering length (hereafter referred to as SL) formula for the energy shifts. In Sec. 4, we present the results for the case of  $d = 2$  dimensions. The theoretical framework is presented in Sec. 5, before the final discussion in Sec. 6.

## 2 Exotic atoms in three dimensions

There is an abundant literature about exotic atoms in three dimensions, motivated by experiments with pionic, kaonic and antiprotonic atoms [1, 2]. The simplest model consists of a two-component potential

$$V_1 + \lambda V_2, \quad (1)$$

where  $V_1$  is a long-range interaction with one or several bound states. Genuine exotic atoms correspond to  $V_1(r) \propto -1/r$ . The second term, with an explicit strength  $\lambda$  introduced for the ease of the discussion, accounts for the short-range interaction. The main results are:

- the shift is usually rather small, although  $\lambda V_2$  can be very large at short distances,
- the shift is usually well described by the approximate formula

$$\delta E = E(\lambda) - E(0) \simeq 4\pi |\phi(0)|^2 a, \quad (2)$$

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<sup>1</sup>More precisely, what is sufficient is that the integral of the potential over the whole space is positive.

where  $\phi$  is the normalized wave function for  $\lambda = 0$ , and  $a$  the scattering length in  $V_2$  alone. In case  $V_1$  is Coulombic, one recovers the well-known SL formula by Deser, Golberger, Bauman and Thirring, and Trueman [7, 8]

$$\frac{E_n - E_n^{(0)}}{E_n^{(0)}} \simeq -\frac{4a}{nB}, \quad (3)$$

where  $B$  is the Bohr radius and  $n$  the principal number for the energy  $E_n^{(0)}$  in  $V_1$  alone or  $E_n$  in the total potential. Many improvements and further corrections to this formula have been discussed in the literature [9, 10].

- When  $\lambda$  is varied, the shift usually varies very slowly, except near the specific values  $\lambda_1, \lambda_2, \dots$ , where the energy levels change very rapidly, and a *level rearrangement* occurs: near  $\lambda = \lambda_n$ , the  $n^{\text{th}}$  energy drops toward very large (negative) values in the nuclear domain, and is replaced in the upper part of the spectrum by the next level, which in turn is replaced by the next one, etc. An example is given in Fig. 1. Further examples are provided, e.g., in [3]. The critical values  $\lambda_n$  correspond to the coupling thresholds for which the short-range interaction  $\lambda V_2$  starts supporting a first or an additional bound state.

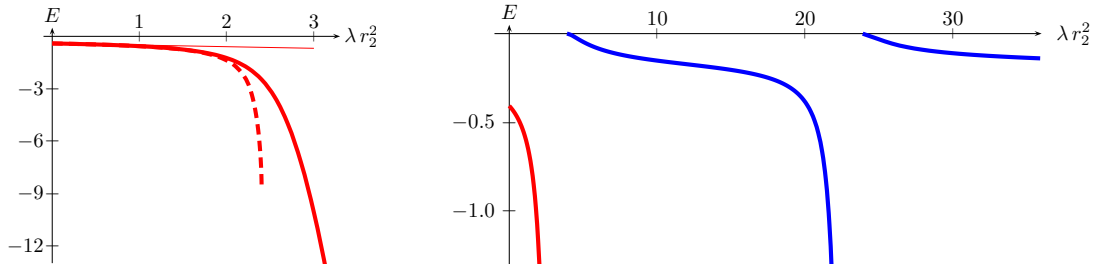


Figure 1: Level rearrangement in three dimensions. A wide and weak external square well  $V = -\lambda_1 \Theta(r_1 - r)$ , with  $\lambda_1 = 4$  and  $r_1 = 1$ , is supplemented by a short-range square well of increasing depth,  $\lambda V_2 = -\lambda \Theta(r - r_2)$ , with  $r_2 = 0.1$ , in units where  $\hbar^2/(2\mu) = 1$ ,  $\mu$  being the reduced mass. The first few energy levels are shown against  $\lambda r_2^2$ . Bottom: first three levels. Top: magnification for ground-state alone, with the exact value (thick line) compared to the first-order perturbation theory and SL formula.

### 3 Results in one dimension

An example of spectrum of exotic atom in  $d = 1$  is shown in Fig. 2. It consists again of a superposition of two square wells, the strength of the short-range one being varied. The main differences, as compared to the more familiar  $d = 3$  case are:

- As soon as  $\lambda$  slightly departs from zero, the atomic ground state energy immediately drops towards the range of the nuclear energies.

- As a coupling threshold in  $\lambda V_2$  is reached and  $\lambda$  further increases, a plateau is observed; the corresponding energy drops, and, by rearrangement, a upper level makes another plateau near the same value. This plateau in the sector of the *even* parity states, corresponds to an unperturbed energy level in the *odd* sector of  $V_1$ . Indeed, the orthogonality with the ground state forces a zero in the wave function near  $x = 0$ , and mimics an odd state.
- The Deser–Trueman formula, if translated for  $d = 1$ , reads

$$\delta E \simeq -2 \frac{|\phi(0)|^2}{a}. \quad (4)$$

The presence of the scattering length  $a$  in the denominator can be understood by dimensional analysis. Also, weaker the short-range interaction  $\lambda V_2$ , more flat the zero-energy wave function, and thus larger the scattering length  $a$ , defined (as for  $d = 3$ ) as the abscissa where the asymptotic zero-energy wave function vanishes.

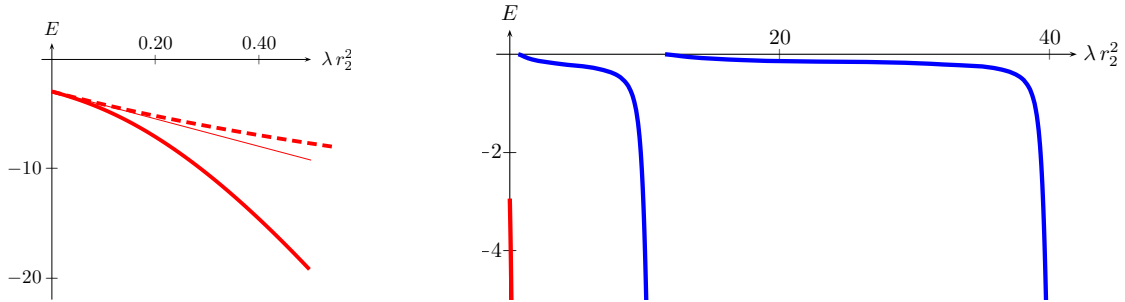


Figure 2: Level rearrangement in one dimension. Same interaction and same notation as Fig. 1.

## 4 Results for two dimensions

The calculation can be repeated for the isotropic (i.e., azimuthal quantum number  $m = 0$ ) states with  $d = 2$ . If the atomic spectrum is examined for increasing values of the strength of the short-range interaction, a pattern of level rearrangement is clearly identified, see Fig. 3.

The behavior of the ground state is displayed again in Fig. 4, where it is compared to the  $d = 1$  and  $d = 3$  cases. The trend is clearly intermediate between the plateau of  $d = 3$  and the immediate fall-off of  $d = 1$ .

For small values of  $\lambda$ , we can easily identify the following behavior for the  $d = 2$  energy shift  $\delta E$

$$\delta E = \frac{A_0^2}{\ln(a/R)}. \quad (5)$$

If one plots, as in the example shown in Fig. 5,  $-1/\delta$  as a function of  $\ln a$ , one hardly distinguishes the exact values from the results of a linear fit.

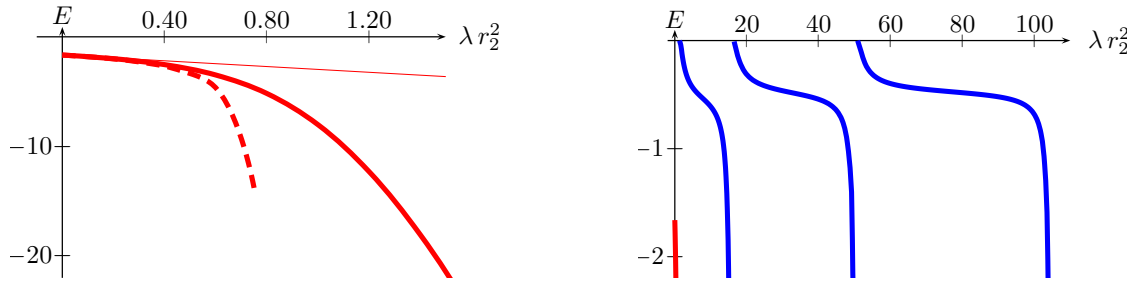


Figure 3: Level rearrangement in two dimensions, for the first four levels. Same interaction and same notation as Fig. 1.

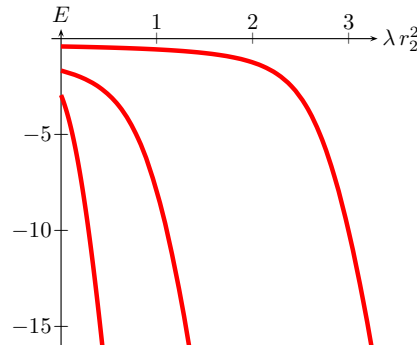


Figure 4: Ground-state energy for  $d = 1$  (left),  $d = 2$  (middle) and  $3$  (right) dimensions, with the same interaction as in the previous figures.

As discussed below,  $A_0^2 \simeq 2\pi |\phi(0)|^2$  and  $a$  is the  $d = 2$  scattering length, as recently revisited [11, 12]. The value of  $R$  is found of the order of magnitude of the “Bohr radius” of the wave function in the external potential, that is to say, the average radius. Its expression is derived in the next section.

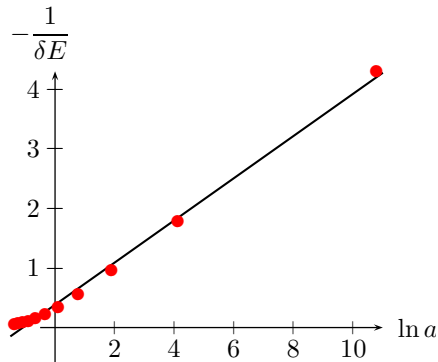


Figure 5:  $1/\delta E$  against  $\ln a$  for the ground-state energy in the double square-well of Fig. 4. The linear fit cannot be distinguished from the exact results.

## 5 Derivation of the energy shift

### 5.1 General formula

There are many approaches to the SL formula for  $d = 3$ , and various corrections and generalizations, see, e.g., [3,4,7–10] and references there. For the  $d = 2$  case, the following simple minded derivation is just based on the matching condition between solutions of the Schrödinger equation that are regular at short and at large distances.

For the sake of clarity, one can identify several approximations that are made when solving the bound-state problem in the potential  $V_1 + \lambda V_2$ :

1.  $V_1$  dominates at large distances
2. the energy  $E$  in  $V_1$  alone is a smooth function of the boundary condition enforced at  $r = 0$ ,
3.  $V_1$  and the energy term  $E u$  can be neglected at very small distances, where  $\lambda V_2$  dominates.

Let us start with Schrödinger equation for the external potential alone, i.e.,

$$-u'' - \frac{u}{4r^2} + V_1 u - E u = 0, \quad (6)$$

where  $u$  is the reduced radial wave function and we are working in the units with  $\hbar = 2m = 1$ . We denote  $h(E, r)$  the solution that is regular at infinity, i.e.,  $h(E, r) \propto \sqrt{r} K_0(kr)$  at large distance, with  $E = -k^2$  and  $K_0$  the usual Bessel function. The case of a confining interaction  $V_1$  is treated later. At short distance, this solution behaves as

$$\begin{aligned} h(E, r) &= B(E) \sqrt{r} \ln r + A(E) \sqrt{r} + \dots, \\ h(E_0, r) &= A_0 \sqrt{r} + \dots \end{aligned} \quad (7)$$

for the modified energy  $E$  and the unperturbed one  $E_0$ . The unperturbed energy corresponds to  $B(E_0) = 0$ , i.e., a solution that is regular as  $r \rightarrow 0$ , and is normalized, leading to a real value  $A_0 = A(E_0)$  at energy  $E_0$ , that can be chosen to be positive. For  $E \neq E_0$  in the neighborhood of  $E_0$ , we impose that the solution remains normalized, i.e.,

$$\int_0^{+\infty} h(E, r)^2 r = 1. \quad (8)$$

By combining (6) for  $h(E, r)$  and  $h(E_0, r)$ , one obtains the exact relation

$$A_0 B(E) = (E - E_0) \int_0^{+\infty} h(E, r) h(E_0, r) r \quad (9)$$

which gives for the energy shift  $\delta E = E - E_0$  a first relation  $\delta E \simeq B A_0$ . It is rather precise. Indeed, if the solution is kept to be normalized as per (8), and if  $h(E, r) \rightarrow h(E_0, r)$  as  $E \rightarrow E_0$ , the integral of  $h(E, r) h(E_0, r)$  entering (9) is also equal to 1, up to second order in  $\delta E$ .

Now,  $A = A_0 + \tilde{A}_0 \delta E + \dots$ , one can identify the short-range behavior of  $h(E, r)$  with  $\sqrt{r} [\ln r - \ln a]$ , to obtain

$$\frac{B}{1} = \frac{A_0 + \tilde{A}_0 \delta E}{-\ln a}, \quad (10)$$

which when combined with  $B \simeq \delta E/A_0$  gives

$$\delta E \simeq -\frac{A_0^2}{\ln a + A_0 \tilde{A}_0}. \quad (11)$$

where the denominator can be cast as  $\ln a - \ln R$ . This relation gives explicitly the link between the energy and the boundary condition at  $r = 0$ , expressed by  $\ln a$ , where  $a$  is the scattering length in the short-range interaction alone, in terms of the quantities  $A_0$  and  $\tilde{A}_0$  linked to the value of the unperturbed solution at the origin.

The effective range correction to the scattering length approximation can be worked out explicitly, but turns out to be very small in most cases. For a positive energy  $E = k^2$ , the  $m = 0$  solution to the scattering problem in  $\lambda V_2$  alone is [11–13]

$$u_2(k, r) = \sqrt{r} [\cot \delta(k) J_0(kr) + Y_0(kr)], \quad (12)$$

where  $J_0$  and  $Y_0$  are Bessel functions, and

$$\cot \delta(k) = \frac{2}{\pi} [\ln(ak/2) + \gamma] + \frac{1}{2} r_0 k^2 + \dots \quad (13)$$

involving the scattering length  $a$  and effective range  $r_0$ . This expression is easily translated for negative energies  $E = -k^2$ , and, anyhow, the range of  $E$  which is explored is small as compared to the typical energy scale in the short-range potential, and thus the solution coming out is safely approximated by its small  $E$  limit

$$u_2 = \sqrt{r} (\ln r + \ln a \mp \pi r_0 E/4). \quad (14)$$

This means one can replace  $\ln a$  by  $\ln a + \pi r_0 E_0/4$  to probe the contribution of the effective range, which turns out negligible, provided the energy shift  $\delta E$  remains small as compared to  $E_0$ .

## 5.2 First example: double delta-shell

To illustrate (11), we consider as long range interaction an attractive delta-shell of strength  $g_1$  and radius that can be set to  $R_1 = 1$  to fix the length scale. The solution can be worked out analytically, in particular  $\sqrt{r} K_0(kr)$  is regular at large  $r$  and  $\sqrt{r} I_0(kr)$  at small  $r$ . The delta-shell interaction imposes the continuity of the radial solution  $u$  near  $r = R_1$  and the proper step in its derivatives, to fix the unperturbed energy  $E_0$ . A second delta-shell can be implemented at  $r = R_2 \ll R_1$ , leading to an explicit transcendental equation for the exact energy  $E$ , and shift  $\delta E = E - E_0$ , to be compared to the simple approximate value  $\delta E' = -A_0^2/\ln a$  and the improved  $\delta E''$  given by (11). For  $g_1 = 1/2$ ,  $R_2 = 0.04$ , and  $g_2 = 0.1$ , one gets

$$\begin{array}{ccc} \delta E & \delta E' & \delta E'' \\ -0.000833975 & -0.0008265 & -0.0008340 \end{array} \quad (15)$$

i.e., an almost perfect agreement, when the  $\ln R$  correction is taken into account.

### 5.3 Double exponential well

As an example involving smooth potentials, we consider the exponential potential  $V_1 = -g_1 \exp(-r/r_1)$  with  $g_1 = 1$  and take  $r_1 = 2$  for the long-range interaction, and study the changes due to another exponential interaction,  $-\lambda \exp(-r/r_2)$  with a much shorter range  $r_2 = 0.02$  and a variable strength. The results are displayed in Fig. 6. Again, there is a net gain as compared to the ordinary perturbation theory, and a good agreement with the exact calculation as long as the deviation from the unperturbed energy is not too large.

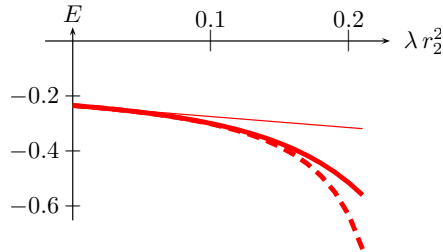


Figure 6: Exponential well supplemented by another exponential of shorter range. Thick line: exact, dashed line: SL formula, thin line: perturbation theory. We use here  $V = -g_1 \exp(-r/r_1) - \lambda \exp(-r/r_2)$  with  $r_1 = 2$ ,  $g_1 = 1$  and  $r_2 = 0.02$ .

### 5.4 Harmonic confinement

The problem is to study how the levels, in particular the ground state, are modified when a  $d = 2$  harmonic oscillator is supplemented by a short-range interaction. A recent contribution is by Farrell and van Zyl [14]. They first stressed the property of *universality*, namely that the energy shift does not depend on the details of the short-range potential, but instead is governed by the scattering length alone. This is, indeed, a very general property of the exotic atoms, in the general sense define in the introduction [3]. For  $V_1 = r^2$ , the general solution that is regular at large distances can be written as

$$h(k, r) = \exp(-r^2/2) \sqrt{r} U(1/2 - k^2/4, 1, r^2), \quad (16)$$

in terms of the confluent hypergeometric function  $U$ . This expression is simpler, but equivalent to the one given in [14]. From (16), one can calculate explicitly the normalization integral  $I(E)$  and its derivative. The short-range behavior of  $h(k, r)$  is known and if the ratio of the  $-\sqrt{r}$  to  $\sqrt{r}$   $\ln r$  coefficients is identified with  $\ln a$ , one recovers the formula given in [14]. Our prescription (11) corresponds to an approximate, but accurate, solution to the matching equation. For instance, using  $\lambda V_2 = -\lambda \exp(-r/R_2)$  as an additional potential, with  $R_2 = 0.02$  and  $\lambda = -80$ , one gets, using the same notation as above and  $\tilde{\delta}E$  for [14],

$$\begin{array}{cccc} \delta E & \tilde{\delta}E & \delta E' & \delta E'' \\ -0.06999 & -0.07017 & -0.07092 & -0.07020 \end{array} \quad (17)$$



Clearly, the main discrepancy comes from reducing this short-range interaction to a zero-range ansatz. Once this is accepted, our approximate treatment is nearly exact as compared to the precise matching of  $h(k, r)$  to the  $\sqrt{r} \ln(r/a)$  boundary condition.

## 6 Summary

In this note, we have studied how the energy levels in a wide potential are modified by a short-range attraction of increasing strength, focusing on the case of  $d = 2$  space dimensions, as compared to the  $d = 1$  and  $d = 3$  situations.

The energy shifts in a given external potential are well described by the following SL formulas,

$$\delta E = \begin{cases} A_0^2/a & (d = 1) \\ A_0^2/\ln(a/R) & (d = 2) \\ A_0^2 a & (d = 3) \end{cases} \quad (18)$$

i.e., a perfect fit is obtained if  $A_0^2$  (and  $R$  for  $d = 2$ ) are treated as free parameters. Moreover,  $A_0$  can be identified with the first non vanishing coefficient of the short-range expansion of the radial wave function and is thus proportional to  $\phi(0)$ , the wave function at the origin for the state in the external potential alone. The ratio is  $A_0^2/|\phi(0)|^2 = 2$  ( $d = 1$ ),  $2\pi$  ( $d = 2$ ),  $4\pi$  ( $d = 3$ ), the unit-sphere area in  $d$  dimensions.

In the  $d = 2$  case which is our main concern, a formula has been derived for  $R$ , namely  $\ln R = -A_0 A'(E_0)$ , where  $A(E)$  is the coefficient of  $\sqrt{r}$  in the normalized wave function, assumed to be real and positive and to match  $A_0$  at energy  $E_0$ .

This SL relation becomes more accurate when additional potential  $V_2$  becomes more short-ranged. In particular, it improves significantly the simple prediction from first order perturbation theory in  $d = 2$  and  $d = 3$ .

This study of exotic atoms is intimately linked to the statistical physics of bosons. The common tool is the *pseudo-potential*, which enables one to replace a finite (but short) range interaction by a contact interaction. Deriving the pseudo-potential as a function of the scattering length for different values of the space dimension  $d$  has been extensively discussed. The case of  $d = 2$  is notoriously delicate, see, e.g., [14–17] for recent contributions.

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