PRAMANA — journal of physics © Indian Academy of Sciences Vol. 73, No. 2 August 2009 pp. 375–385

Compactons in $\mathcal{PT}$-symmetric generalized Korteweg–de Vries equations

CARL M BENDER[1], FRED COOPER[2,5], AVINASH KHARE[3,*], BOGDAN MIHAILA[4] and AVADH SAXENA[5]
[1]Department of Physics, Washington University, St. Louis, MO 63130, USA
[2]National Science Foundation, Division of Physics, Arlington, VA 22230, USA and Santa Fe Institute, Santa Fe, NM 87501, USA
[3]Institute of Physics, Sachivalaya Marg, Bhubaneswar 751 005, India
[4]Material Science and Technology Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
[5]Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
[*]Corresponding author. E-mail: khare@iopb.res.in

Abstract. This paper considers the $\mathcal{PT}$-symmetric extensions of the equations examined by Cooper, Shepard and Sodano. From the scaling properties of the $\mathcal{PT}$-symmetric equations a general theorem relating the energy, momentum and velocity of any solitary-wave solution of the generalized KdV equation is derived. We also discuss the stability of the compacton solution as a function of the parameters affecting the nonlinearities.

Keywords. Compactons; $\mathcal{PT}$ symmetry; generalized KdV equations.

PACS Nos 03.65.Ge; 02.60.Lj; 11.30.Er; 52.35.Sb

1. Introduction

A few years ago, Cooper *et al* [1] introduced and Khare and Cooper [2] studied further the first-order Lagrangian

$$L(l,p)=\int \mathrm{d}x\left[\frac{\varphi_x\varphi_t}{2}+\frac{(\varphi_x)^l}{l(l-1)}-\alpha(\varphi_x)^p(\varphi_{xx})^2\right]. \qquad (1)$$

This Lagrangian is a modification of the original compacton equations of Rosenau and Hyman [3]. This Lagrangian gives rise to a general class of KdV equations

$$u_t+u^{l-2}u_x+\alpha[2u^pu_{xxx}+4pu^{p-1}u_xu_{xx}+p(p-1)u^{p-2}(u_x)^3]=0, \qquad (2)$$

where $u(x,t)=\varphi_x(x,t)$. For $0<p\le 2$ and $l=p+2$ these models admit compacton solutions whose width is independent of the amplitude. For $p>2$ the derivatives of the solution are not finite at the boundaries of the compacton where $u\to 0$.

Cooper *et al* [4] analysed the stability of the general compacton solutions of this equation and showed that solutions are stable provided $2 < l < p + 6$.

Recently, there has been some interest in complex $\mathcal{PT}$-symmetric extensions of the ordinary KdV equation. Such extensions exist in the complex plane but also lead to new PDEs that are entirely real. The first extension of the KdV equation by Bender *et al* [5] was

$$u_t - iu(iu_x)^\epsilon + u_{xxx} = 0, \tag{3}$$

which reduces to the usual KdV equation when $\epsilon = 1$. This extension of the KdV equation is not a Hamiltonian dynamical system at arbitrary ϵ. A more recent study by Fring [6] was based on a Hamiltonian formulation. The Hamiltonian studied by Fring is related to a special case of the system of generalized KdV equations examined here.

To find extensions of the generalized KdV equation that are invariant under the joint operation of space reflection (parity) P and time reversal T, we make the following definitions: spatial reflection P consists of making the replacement $x \to -x$. Also, because u is velocity, under P we replace u by $-u$. The effect of the time reversal operation T is to change the signs of i, T, and u: $i \to -i$, $t \to -t$, and $u \to -u$. Therefore, the combination iu_x is $\mathcal{PT}$ even, so a $\mathcal{PT}$-symmetric generalization of the Lagrangian (1) is

$$L_{\mathcal{PT}} = \int \mathrm{d}x \left[\frac{\varphi_x\varphi_t}{2} + \frac{(\varphi_x)^l}{l(l-1)} + \alpha(\varphi_x)^p(i\varphi_{xx})^m\right], \tag{4}$$

(with α real). For this Lagrangian we must find the correct $\mathcal{PT}$-symmetric contour that lies on the real axis when $m = 2$. For $\mathcal{PT}$ to be a good symmetry, branch cuts must be taken along the positive imaginary axis in the complex-x plane. The Hamiltonian resulting from the above Lagrangian is

$$H = \int \mathrm{d}x \left[-\frac{u^l}{l(l-1)} - \alpha u^p(iu_x)^m\right]. \tag{5}$$

In this paper we will restrict ourselves to m an even integer, and a convenient choice for α that allows for solitary-wave solutions and that gives a real equation for the generalized KdV system is

$$-\alpha(m-1)i^m = 1. \tag{6}$$

The $\mathcal{PT}$ generalization of (2) has the same canonical structure as the KdV equation. From Hamilton's equations we obtain the equations of motion for $u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\frac{\delta H}{\delta u} = \{u, H\}, \tag{7}$$

where the Poisson bracket structure is [7] $\{u(x), u(y)\} = \partial_x\delta(x-y)$. The resulting equation becomes

$$\begin{aligned} 0 = u_t + u_x u^{l-2} + u^{p-2}u_x^{m-3}\big[&(m-2)mu^2u_{xx}^2 \\ &+2mpuu_{xx}u_x^2 + mu^2u_{xxx}u_x + (p-1)pu_x^4\big]. \end{aligned} \tag{8}$$

This system of equations has three obvious conservation laws: conservation of mass M, momentum P and energy E, where the energy is the value of the Hamiltonian (5) and

$$M = \int \mathrm{d}x\, u(x,t), \quad P = \int \mathrm{d}x\, \frac{1}{2}u^2(x,t). \tag{9}$$

The case $m = 2$ leads to the well-known compacton solutions.

2. Scaling properties

If we require that solutions of (8) transform into solutions under the scaling

$$x \to \lambda x, \quad t \to \lambda^{\eta} t, \quad u \to \lambda^{\beta} u, \tag{10}$$

we find that

$$\beta = \frac{m}{p+m-l}; \quad 1 - \eta = \beta(l-2) \tag{11}$$

for a travelling solitary wave of the form $f(x - ct)$. Then c scales as x/t or as $\lambda^{1-\eta}$ or $c \propto \lambda^{\beta(l-2)}$. Since x scales as λ and $\lambda \propto c^{i_1}$, $i_1 = \frac{p+m-l}{m(l-2)}$ we find that the width of the solitary wave is independent of the velocity for $l = p + m$. We also have from the definitions:

$$P \propto c^{i_2}, \quad i_2 = \frac{3m - l + p}{m(l-2)}; \quad E \propto c^{(ml+p+m-l)/(p+m-l)} \tag{12}$$

so that

$$E \propto P^{-r}; \quad r = -\frac{lm + p + m - l}{p + 3m - l}. \tag{13}$$

In ref. [4] a general theorem was derived that relates the energy, momentum and velocity of solitary waves of the generic form

$$u(x,t) = AZ[\beta(x - q(t))] \tag{14}$$

for $m = 2$. Using the same ansatz, eq. (14), and following the approach of ref. [4], this is easily generalized to arbitrary m. For our ansatz, the value of the Hamiltonian (5) is

$$H = -C_1(l)\frac{A^l}{\beta l(l-1)} + A^{p+m}\beta^{m-1}C_2(p,m), \tag{15}$$

where

$$C_1(l) = \int \mathrm{d}z\, Z^l(z), \quad C_2(p,m) = \int \mathrm{d}z\, [Z'(z)]^m Z^p. \tag{16}$$

One can show that the exact solutions have the property that they are the functions of the parameter β that minimize the Hamiltonian with respect to β when the momentum P is fixed. Minimizing the Hamiltonian as a function of β keeping the conserved momentum fixed we find that $H = f(l,p,m)P^{-r}$, where r is given by (13), and $f(l,p,m)$ is a function of the quantities $C_2(p,m)$, and $C_1(l)$ which is given in [8]. One also finds

$$P \propto c^{\frac{p+3m-l}{m(l-2)}}, \quad A \propto c^{\frac{1}{l-2}}, \quad \beta \propto c^{\frac{l-p-m}{m(l-2)}}, \tag{17}$$

where the proportionality constants again depend on $C_2(p,m)$ and $C_1(l)$ as defined above. Once an exact solution is obtained, these constants can be calculated as will be shown below.

3. Travelling-wave solutions

We begin with the wave equation (8) which can be re-expressed as

$$u_t + u^{l-2}u_x - \frac{p}{m-1}\left[u^{p-1}(u_x)^m\right]_x + \frac{m}{m-1}\left(u^p u_x^{m-1}\right)_{xx} = 0, \tag{18}$$

and assume that $u(x,t) = f(x-ct) \equiv f(y)$. Then,

$$cf' = f^{l-2}f' + \frac{1}{m-1}\left(p\left[f^{p-1}(f')^m\right]' - m\left[f^p(f')^{m-1}\right]''\right), \tag{19}$$

and integrating once we obtain

$$cf = \frac{f^{l-1}}{l-1} + mf^p(f')^{m-2}f'' + p(f')^m f^{p-1} + K_1. \tag{20}$$

For compact solutions $K_1 = 0$. Setting the integration constant K_1 to zero, multiplying this equation by f, and integrating over y, we obtain

$$cI_2 = \frac{1}{l-1}I_l - \frac{p+m}{m-1}J_{m,p}, \tag{21}$$

where

$$I_n = \int_{-\infty}^{\infty} \mathrm{d}y\, f^n(y), \quad J_{m,p} = \int_{-\infty}^{\infty} \mathrm{d}y\, (f')^m f^p(y). \tag{22}$$

We multiply (20) by f' and integrate again with respect to y to get the following nonlinear differential equation for the travelling waves:

$$\frac{c}{2}f^2 - \frac{f^l}{l(l-1)} - (f')^m f^p = K_1 f + K_2. \tag{23}$$

We see that K_2 must also be zero for solutions f that are compact. In what follows we will take as the compacton equation:

$$\frac{c}{2}f^2 - \frac{f^l}{l(l-1)} - (f')^m f^p = 0. \tag{24}$$

If we now integrate eq. (24) with respect to y, we obtain

$$J_{m,p} = \frac{c}{2}I_2 - \frac{1}{l(l-1)}I_l. \tag{25}$$

From (21) and (25) we can solve for $J_{m,p}$ and I_l in terms of I_2. We obtain

$$J_{m,p} = \frac{(l-2)(m-1)}{2[p+m+(m-1)l]}cI_2; \quad I_l = \frac{l(l-1)(p+3m-2)c}{2[p+m(l+1)-l]}I_2. \tag{26}$$

The energy of the solitary wave is given by

$$H = \frac{1}{m-1}J_{m,p} - \frac{1}{l(l-1)}I_l \tag{27}$$

and the momentum $P = I_2/2$. From eqs (21), (25) and (27) we deduce that H, P and c are related by $H = Pc/r$, where r is given in (13).

3.1 *Weak solutions*

We are interested in compacton solutions that are a combination of a compact function $f(x)$ confined to a region (initially $-x_0 < x < x_0$ and zero elsewhere). At the boundaries $\pm x_0$ the function $f(x)$ is assumed to be continuous but higher derivatives most likely are not. For there to be a weak solution the jump in lhs of eq. (24), $\frac{c}{2}f^2 - \frac{f^l}{l(l-1)} - (f')^m f^p$ needs to be zero when we cross from $x_0 - \epsilon$ to $x_0 + \epsilon$. Since $f(x_0)$ is assumed to vanish, the requirement for a weak solution is

$$\mathrm{Disc}[(f')^m(x)f^p(x)]_{x_0} = 0, \tag{28}$$

where 'Disc' is the discontinuity across the boundary x_0. This is always satisfied if there is no infinite jump in the derivative of the function. The solitary wave for (19) for $m = 2n$, n integer is obtained by joining the positive and negative solutions of the mth root of (19) appropriately shifted so that the maximum is at the origin $y = 0$.

3.2 *Compacton solutions when m is an even integer*

Compacton solutions are constructed by patching a compact portion of a periodic solution that is zero at both ends to a solution that vanishes outside the compact region to give the weak solution described above. For $p \le 2, p \le l$, one has that the positive branch of the solution, satisfies the equation

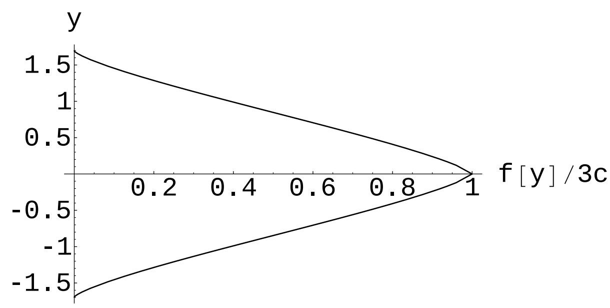

Figure 1. $f/(3c)$ vs. y for $m=4$, $l=3$ and $p=1$.

$$
\begin{aligned}
x-ct &= \int_0^f \mathrm{d}u \left(\frac{1}{2}cu^{2-p} - \frac{u^{l-p}}{(l-1)l}\right)^{-1/m} \\
&= 2^{-1/m}(-l+m+p)^{-1} f \left(f^{-l}\left(2f^l - cf^2(l-1)l\right)\right)^{1/m} \\
&\quad \times \left(f^{-p}\left(\frac{2f^l}{l-l^2} + cf^2\right)\right)^{-1/m} m \\
&\quad \times {}_2F_1\left(-\frac{-l+m+p}{(l-2)m}, \frac{1}{m}; \frac{ml+l-3m-p}{(l-2)m};\right. \\
&\qquad\qquad \left.\frac{1}{2}cf^{2-l}(l-1)l\right),
\end{aligned} \tag{29}
$$

where ${}_2F_1$ denotes the hypergeometric function.

For $p=1, l=3$, this simplifies to

$$
x-ct = 2^{\frac{1}{m}} 3^{\frac{m-1}{m}} c^{\frac{m-2}{m}} B_{\frac{f}{3c}}\left(\frac{m-1}{m}, \frac{m-1}{m}\right), \tag{30}
$$

where $B_n(x,y)$ is the incomplete beta function. For $m=2$ this further simplifies to the known result

$$
x-ct = 2\sqrt{6}\sin^{-1}\left[\sqrt{f/(3c)}\right] \to f = 3c\sin^2\left[\tfrac{1}{2\sqrt{6}}(x-ct)\right]. \tag{31}
$$

For $m=4$ we get for the positive real fourth root

$$
x-ct = 2^{1/4}\, 3^{3/4}\sqrt{c}\left[B_{\frac{f}{3c}}\left(\frac{3}{4}, \frac{3}{4}\right)\right]. \tag{32}
$$

In figure 1 we plot $B_{\frac{f}{3c}}\left(\frac{3}{4}, \frac{3}{4}\right) - B\left(\frac{3}{4}, \frac{3}{4}\right)$ and its mirror image as a function of $f/(3c)$. Here, $y=(x-ct)2^{-1/4}3^{-3/4}c^{-1/2}$.

We can simplify how we express the compacton solutions of eq. (29) by considering the generalization of the hyperelliptic compactons discussed in ref. [4]. Assuming we can parametrize the solutions to eq. (24) by $f = AZ^a[\beta(x-ct)]$, with the condition that Z obeys the hyperelliptic equation:

$$(Z')^m = 1 - Z^{2\tau}, \tag{33}$$

one immediately finds using eq. (24)

$$a = \frac{m}{m+p-2}, \quad \tau = \frac{m(l-2)}{2(m+p-2)}. \tag{34}$$

We also find that

$$A = [cl(l-1)/2]^{1/(l-2)},$$

$$\beta = \frac{1}{a[l(l-1)]^{1/m}}[cl(l-1)/2]^{(l-p-m)/(m(l-2))}. \tag{35}$$

Note that this ansatz gives the correct scaling behaviour of the amplitude parameter A and the width parameter β for the velocity c.

The solution to the differential equation (33) has m branches corresponding to the various values of $\mathrm{e}^{2i\pi n/m}$ when m is an integer and $n = 1, 2, \ldots, m$. For even integer m, the positive root can be integrated to give

$$y = \int_0^Z dx\, \left(1 - x^{2\tau}\right)^{-1/m} = Z\, {}_2F_1\left(\frac{1}{m}, \frac{1}{2\tau}; 1 + \frac{1}{2\tau}; Z^{2\tau}\right). \tag{36}$$

For even m we get the full solution for the compacton by adding the positive-real-root and the negative-real-root solutions to get the complete compacton profile. The compacton vanishes elsewhere.

For solutions satisfying (33) it is possible to determine explicitly the conserved quantities in terms of the velocity c of the wave and the parameters l, p and m of the differential equation. In terms of the parameters A and β given by (35) we find

$$M = \frac{A\Gamma\left(\frac{m-1}{m}\right)\Gamma\left(\frac{a+1}{2\tau}\right)}{\beta\tau\Gamma\left(1 - \frac{1}{m} + \frac{1+a}{2\tau}\right)}, \quad P = \frac{A^2\Gamma\left(\frac{m-1}{m}\right)\Gamma\left(\frac{2a+1}{2\tau}\right)}{2\beta\tau\Gamma\left(1 - \frac{1}{m} + \frac{1+2a}{2\tau}\right)}, \tag{37}$$

where $a = m/(m+p-2)$ and the energy $E = cP/r$.

4. Some particular solutions

There are two types of special cases. The first occurs when τ, l and p are integers. From eq. (34) it follows that for $m = 4$, $l = 2 + \tau(p+2)/2$, whereas for $m = 6$, $l = 2 + \tau(p+1)/3$. The other interesting case arises when the width of the solitary wave is independent of the velocity. This occurs when $l = p + m$.

Case $m = 4$

For this case

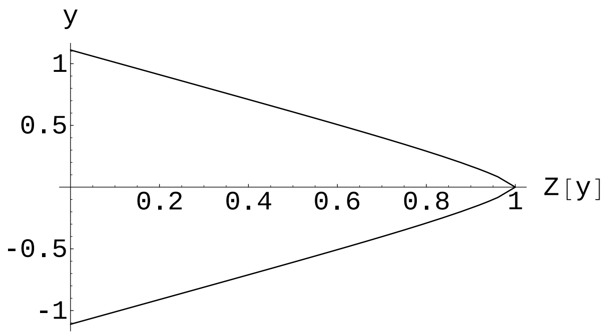

Figure 2. $Z[y]$ for $m = 4$, $l = p + 4$, $\tau = 2$.

$$\tau = \frac{2l - 4}{p + 2}, \quad a = \frac{4}{p + 2}. \tag{38}$$

For $\tau = 2$, $l = p + 4$ and the width is independent of velocity. The relevant function to invert is

$$y = \int_0^Z \frac{\mathrm{d}x}{(1 - x^4)^{1/4}} = Z\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1 + \tfrac{1}{2\tau}; Z^4\right). \tag{39}$$

For a compacton centred about the origin $y = 0$, the two halves of the compacton are given by

$$y_\pm = \pm f_1(Z) \mp f_1(Z = 1), \quad f_1(x) = x\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; \tfrac{5}{4}; x^4\right) \tag{40}$$

and

$$f_1(Z = 1) = \Gamma\left(\tfrac{3}{4}\right)\Gamma\left(\tfrac{5}{4}\right) = \tfrac{1}{4}\pi\sqrt{2} = 1.11072\ldots. \tag{41}$$

The result for $Z[y]$ is shown in figure 2.

For the case $p = 1$, $l = 5$, we get $A = (c/2)^{1/3}$ and the solution goes as $Z^{4/3}$. For the case $p = 2$, $l = 6$, we get $A = (c/2)^{1/4}$ and the solution is linear in Z.

5. Alternative generating function and stability

Solitary waves of the form $f(y) = f(x - ct)$ can be derived by considering the following function:

$$\Phi[f(y), f'(y)] = \int \mathrm{d}x\, (H[f, f'] + P[f]c) \equiv \int \mathrm{d}x\, \varphi[f, f']. \tag{42}$$

Notice that φ is the negative of the Lagrangian density. That is, the original equation for the solitary wave can be written as $\partial_x(\delta\Phi/\delta f) = 0$. The once-integrated equation (20) (with no integration constants) is obtained from

$$\frac{\delta\Phi}{\delta f} = 0, \quad \to \frac{\partial\varphi}{\partial f} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\varphi}{\partial f'}\right). \tag{43}$$

We have explicitly

$$\Phi = \int \mathrm{d}y \left[-\frac{f^l}{l(l-1)} + f^p(f')^m + \frac{1}{2}cf^2\right]. \tag{44}$$

The first variation after an integration by parts can be written as

$$\delta\Phi = \int \mathrm{d}y \left[-\frac{f^{l-1}}{l-1} + cf - pf^{p-1}(f')^m - mf^p(f')^{m-2}f''\right]\delta f. \tag{45}$$

The second variation which is important for the linear stability analysis can be written as

$$\delta^2\Phi = \int \mathrm{d}y\, \delta f\; L\; \delta f, \tag{46}$$

where L is the operator

$$\begin{aligned} L = {} & c - f^{l-2} - p(p-1)f^{p-2}(f')^m - mpf^{p-1}(f')^{m-2}f'' \\ & + \left(-mpf^{p-1}(f')^{m-1} - m(m-2)f^p(f')^{m-3}f''\right)\frac{\mathrm{d}}{\mathrm{d}y} \\ & - mf^p(f')^{m-2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}. \end{aligned} \tag{47}$$

When $m = 2$, this reduces to the result given in Dey and Khare [9].

One can write Φ in terms of I_l and $J_{m,p}$:

$$\Phi[f] = \frac{1}{m-1}J_{m,p} - \frac{1}{l(l-1)}I_l + \frac{c}{2}I_2. \tag{48}$$

Let us consider the scale transformation: $f(y) \to \lambda^\rho f(\lambda y)$. This leads to the equations of motion plus a boundary term because

$$\begin{aligned} \left.\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}\right|_{\lambda=1} = {} & \int \mathrm{d}y \left[\frac{\partial\varphi}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\varphi}{\partial f'}\right](\rho f + xf') \\ & + \left[\frac{\partial\varphi}{\partial f'}\right](\rho f + xf'')\Big|_{y_{\min}}^{y_{\max}} = 0. \end{aligned} \tag{49}$$

Assuming that the boundary term vanishes at the edges of the compacton, we recover the equation of motion

$$\Phi[\lambda^\rho f(\lambda y)] = \frac{1}{m-1}\lambda^{m-1+\rho(m+p)}J_{m,p} - \frac{I_l}{l(l-1)}\lambda^{l\rho-1} + \frac{c}{2}\lambda^{2\rho-1}I_2. \tag{50}$$

The condition for a minimum is

$$\left.\frac{\mathrm{d}\Phi(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=1} = 0 = \frac{(l\rho-1)I_l}{l(l-1)} - \frac{m-1+\rho(m+p)}{m-1}J_{m,p} - \frac{c}{2}(2\rho-1)I_2. \tag{51}$$

The particular case $\rho = 1/2$ is special in that P is invariant under this transformation; that is, $P[\lambda^{1/2}f(\lambda y)] = P[f(y)]$. If we choose $\rho = 1/2$, when we vary Φ, we are varying the Hamiltonian with the constraint that P is held fixed. (This particular variation was first considered by Kuznetsov [10] and was then elaborated by Karpman [11] and Dey and Khare [9].)

For $\rho = 1/2$ we obtain

$$J_{mp} = \frac{(l-2)(m-1)}{l(l-1)(3m+p-2)}I_l, \tag{52}$$

which is equivalent to (26). For arbitrary ρ the second derivative does not factor into a simple form that allows one to say when it changes sign. However, for $\rho = 1/2$ the answer does factor and the second derivative yields

$$\left.\Phi''(\lambda)\right|_{\lambda=1} = \frac{Pc(l-2)(3m+p-l)(3m+p-2)}{4l(-1+m)+m+p}. \tag{53}$$

Earlier we found that a weak solution that is compact can exist if $p \le 2,\ p \le l$. This leads to the statement that solitary waves will be unstable under this type of deformation when $l > p + 3m$.

6. Compacton stability

The stability of ordinary compactons has been discussed in detail by Dey and Khare [9]. By essentially extending their arguments we can discuss the stability of the $\mathcal{PT}$-invariant compacton solutions. In particular, to study linear stability one assumes that

$$u(x,t) = f(y) + v(x,t), \quad |v| \ll 1, \quad (u,v) = 0\,, \tag{54}$$

and then following the work of [9–11], one can show that $2 < l < p+3m$ (for details, see [8]).

On the other hand, Lyapunov stability uses sharp estimates and has been used by Weinstein [12] and Karpman [13]. By following their arguments and using the Holder inequality, one can again show that our $\mathcal{PT}$-invariant compacton solutions are stable so long as $2 < l < p + 3m$ [8].

6.1 *Approximate variational solutions*

To study stability it is useful to have approximate solutions that are close to the exact solutions to see if they relax to the exact solutions or become unstable. For this purpose it is useful to study the post-Gaussian trial functions

$$f_V(x-ct) \equiv g(x-ct) = A \exp\left[-|\beta(x-ct)|^{2n}\right], \tag{55}$$

where A, β and n are continuous variational parameters chosen to minimize the action. These trial wave functions have earlier been successfully used [14] to approximate various solitary waves in both KdV systems and NLSE applications.

The advantage of these trial functions is that the action as well as all of the conserved quantities can be explicitly evaluated. For solutions that are compact and cover half of the period of a positive periodic function, an alternative choice for a variational trial function is

$$u_2(x) = A[\cos(\beta x)]^{\gamma}, \tag{56}$$

where β and γ are the variational parameters. For integers p, m and l it is again possible to obtain an explicit expression for $H[\beta,\gamma]$. One can perform the minimization with respect to β explicitly. We have investigated both these types of variational solutions and they globally agree with the exact solutions quite well (except near the end points), see [8].

7. Open problems

One of the most challenging questions is if these generalized KdV equations holding compacton solutions are integrable or not. Recently, Assis and Fring [15] have analysed these generalized KdV equations and have shown that those cases in which the compacton width depends on the amplitude pass the Painlevé test while those where compacton width is independent of the amplitude do not pass the Painlevé test. It would be worthwhile to examine the cases which pass Painlevé test in great detail to see if these are truly integrable models.

References

[1] F Cooper, H Shepard and P Sodano, *Phys. Rev.* **E48**, 4027 (1993)
[2] A Khare and F Cooper, *Phys. Rev.* **E48**, 4843 (1993)
[3] P Rosenau and J M Hyman, *Phys. Rev. Lett.* **70**, 564 (1993)
[4] F Cooper, A Khare and A Saxena, *Complexity* **11**, 30 (2006)
[5] C M Bender, D C Brody, J-H Chen and E Furlan, *J. Phys. A: Math. Theor.* **40**, F153 (2007)
[6] A Fring, *J. Phys. A: Math. Theor.* **40**, 4215 (2007)
[7] A Das, *Integrable models* (World Scientific, Singapore, 1989)
[8] C M Bender, F Cooper, A Khare, B Mihaila and A Saxena, arXiv.org:0810.3460
[9] B Dey and A Khare, *Phys. Rev.* **E58**, R2741 (1998)
[10] E A Kuznetsov, *Phys. Lett.* **A101**, 314 (1984)
[11] V I Karpman, *Phys. Lett.* **A210**, 77 (1996)
[12] M I Weinstein, *Commun. Math. Phys.* **87**, 567 (1983)
[13] V I Karpman, *Phys. Lett.* **A215**, 254 (1996) and references therein.
[14] F Cooper, H Shepard, C Lucheroni and P Sodano, *Physica* **D68**, 344 (1993)
F Cooper, C Lucheroni, H Shepard and P Sodano, *Phys. Lett.* **A173**, 33 (1993)
[15] P E G Assis and A Fring, arXiv.org:0901.1267