Exact Static Solutions of a Generalized Discrete ϕ^4 Model Including Short-Periodic Solutions

Avinash Khare[1], Sergey V. Dmitriev[2], and Avadh Saxena[3]
[1]*Institute of Physics, Bhubaneswar, Orissa 751005, India*
[2]*Altai State Technical University, General Physics Department, Barnaul 656038, Lenin St. 46, Russia*
[3]*Center for Nonlinear Studies and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

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For a five-parameter discrete ϕ^4 model, we derive various exact static solutions, including the staggered ones, in the form of the basic Jacobi elliptic functions sn, cn, and dn, and also in the form of their hyperbolic function limits such as kink (tanh) and single-humped pulse (sech) solutions. Such solutions are admitted by the considered model in seven cases, two of which have been discussed in the literature, and the remaining five cases are addressed here. We also obtain sine, staggered sine as well as a large number of short-periodic static solutions of the generalized 5-parameter model. All the Jacobi elliptic, hyperbolic and trigonometric function solutions (including the staggered ones) are translationally invariant (TI), i.e., they can be shifted along the lattice by an arbitrary x_0, but among the short-periodic solutions there are both TI and non-TI solutions. The stability of these solutions is also investigated. Finally, the constructed Jacobi elliptic function solutions reveal four new types of cubic nonlinearity with the TI property.

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I. INTRODUCTION AND SETUP

In recent years there has been a growing interest in the analysis of new discrete nonlinear models since they play a very important role in many physical applications. For example, the question of mobility of solitonic excitations in discrete media is a key issue in many physical contexts; for example the mobility of dislocations, a kind of topological solitons, is of importance in the physics of plastic deformation of metals and other crystalline bodies [1]. Similar questions arise in optics for light pulses moving in optical waveguides or in photorefractive crystal lattices (see e.g., [2] for a relevant recent discussion) and in atomic physics for Bose-Einstein condensates moving through optical lattice potentials (see e.g., [3] for a recent review). These issues may prove critical in aspects related to the guidance and manipulation of coherent, nonlinear wavepackets in solid-state, atomic and optical physics applications.

In particular, the translationally invariant (TI) discrete models [4] have received considerable attention since they admit static solutions that can be placed anywhere with respect to the lattice. Such discretizations have been constructed and investigated for the Klein-Gordon field [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and for the nonlinear Schrödinger equation [16, 17, 18, 19, 20, 21]. For the Hamiltonian TI lattices [5, 8], this can be interpreted as the absence of the Peierls-Nabarro (PN) potential [1]. For the non-Hamiltonian lattices, the height of the Peierls-Nabarro barrier is path-dependent but there exists a continuous path along which the work required for a quasi-static shift of the solution along the lattice is zero [12].

In general, one can state that coherent structures in the TI models are not trapped by the lattice and they can be accelerated by even a weak external field. This particular property makes the TI discrete models potentially interesting for physical applications and one such physically meaningful model has been recently reported [13].

For some of the TI models it has been demonstrated that they conserve momentum [4] or energy (Hamiltonian) [5, 8] (see also [18, 19]). However, we do not know a TI model conserving both momentum and Hamiltonian and, for the Klein-Gordon lattices with classically defined momentum, it was proved that these two conservation laws are mutually exclusive [10].

It may be noted that the TI discrete models can support even moving solutions, but only for selected propagation velocities [19]. In some cases, the exact static or even moving solutions to the TI models can be expressed explicitly in terms of the Jacobi elliptic functions (JEF). Even in the cases when JEF solutions are impossible, static solutions to a TI model can always be obtained iteratively from a nonlinear map (first integral), solving at each step an algebraic equation.

While there has been no universally acceptable definition of TI models, it is fairly easy to describe what a TI solution is. It is a static solution which can be placed anywhere with respect to the lattice. In particular, if there is an analytic TI solution with an arbitrary shift x_0 along the chain or if one can show that there is a corresponding Goldstone mode with zero frequency for any x_0. As far as TI models are concerned, it is believed that they should possess following properties (i) they admit static solutions which can be placed anywhere with respect to the lattice, which can be associated with the absence of PN barrier. Note that in case analytical static solutions can be constructed with an arbitrary shift along the chain, x_0, that would automatically imply the absence of the PN barrier (ii) static version of TI discrete models are integrable, i.e. the static problems are reducible to a first-order difference equation which can be viewed as a nonlinear map from which static solutions can be constructed iteratively (in this study we will show that non-integrable three-point static problems also can have particular TI solutions derivable from a set of two lower-order finite-difference equations, and one of those equations is a two-point one while another is a three-point one) (iii) static solutions in TI models possess the translational Goldstone mode with zero frequency for any x_0.

A prototype class of discrete models, relevant to a variety of applications are the so called discrete ϕ^4 models which feature a cubic nonlinearity. The purpose of this paper is to study in detail several issues related to TI models. In particular we consider a rather general discrete ϕ^4 model with cubic nonlinearity which is invariant under the interchange of ϕ_{n+1} and ϕ_{n-1}

$$\ddot{\phi}_n = \frac{1}{h^2}(\phi_{n+1}+\phi_{n-1}-2\phi_n)+\lambda\phi_n - A_1\phi_n^3 - \frac{A_2}{2}\phi_n^2(\phi_{n+1}+\phi_{n-1})$$
$$-\frac{A_3}{2}\phi_n(\phi_{n+1}^2+\phi_{n-1}^2) - A_4\phi_n\phi_{n+1}\phi_{n-1} - \frac{A_5}{2}\phi_{n+1}\phi_{n-1}(\phi_{n+1}+\phi_{n-1}) - \frac{A_6}{2}(\phi_{n+1}^3+\phi_{n-1}^3), \tag{1}$$

with the model parameters satisfying the constraint

$$\sum_{k=1}^{6} A_k = \lambda\,. \tag{2}$$

In Eq. (1), $\phi_n(t)$ is the unknown function defined on the lattice $x_n = hn$ with the lattice spacing $h > 0$ and overdot means derivative with respect to time t. Without the loss of generality it is sufficient to consider $\lambda = 1$ or $\lambda = -1$.

If model parameters A_k are constant (i.e. independent of h), condition of Eq. (2) ensures that, in the continuum limit, Eq. (1) reduces to the ϕ^4 equation

$$\phi_{tt} = \phi_{xx} + \lambda\phi(1-\phi^2). \tag{3}$$

On the other hand, if model parameters A_k are functions of h, then the continuum limit can be different from Eq. (3) even when Eq. (2) holds.

Static form of Eq. (3) has the first integral

$$\phi_x^2 - \frac{\lambda}{2}\left(1-\phi^2\right)^2 + C = 0, \tag{4}$$

with the integration constant C.

So far as we are aware of, all the discrete ϕ^4 models discussed in the literature, under static consideration, are special cases of the general model, Eq. (1). Some of these models are

Model 1. Only A_1 nonzero with other A_k being equal to zero results in the classical discretization of Eq. (3) that has received a great deal of attention from the researchers in various fields. This model is not a TI one and it will not be further discussed here.

Model 2. $A_1 = A_3/2 = A_4 = \lambda\delta$, $A_5 = A_6 = 2\lambda\gamma$, $A_2 = \lambda(1-4\gamma-4\delta)$ with arbitrary γ and δ. This non-Hamiltonian TI model (for arbitrary γ and δ) conserves momentum [4]

$$P_1 = \sum_n \dot{\phi}_n(\phi_{n+1}-\phi_{n-1})\,. \tag{5}$$

Static version of this model (with the omitted inertia term $\ddot{\phi}_n$) has the first integral with the integration constant C [7],

$$U(\phi_{n-1},\phi_n) \equiv (\phi_n-\phi_{n-1})^2 + \Lambda\phi_n\phi_{n-1} - \Lambda\gamma\left(\phi_n^4+\phi_{n-1}^4\right)$$
$$-\Lambda\delta\phi_n\phi_{n-1}\left(\phi_n^2+\phi_{n-1}^2\right) - \Lambda\left(\frac{1}{2}-2\gamma-2\delta\right)\phi_n^2\phi_{n-1}^2 - \frac{C\Lambda}{2} = 0\,, \tag{6}$$

with Λ defined by

$$\Lambda = \lambda h^2\,, \tag{7}$$

from which any static solution to Eq. (1) can be constructed iteratively, starting from any admissible value of ϕ_0 and solving at each step the algebraic problem. This is so because Eq. (1) is nothing but

$$\ddot{\phi}_n = \frac{U(\phi_n,\phi_{n+1}) - U(\phi_{n-1},\phi_n)}{\phi_{n+1}-\phi_{n-1}}\,. \tag{8}$$

Equation (6) is the discretized first integral (DFI) [12], i.e., in the continuum limit ($h \to 0$) it reduces to Eq. (4). DFI of Eq. (6) is quartic in both ϕ_{n-1} and ϕ_n and thus, it cannot be reduced to the integrable nonlinear map reported in [22] where the corresponding first integral is quadratic in ϕ_{n-1} and ϕ_n. In this complete form the model was first constructed in [7] and almost concurrently in [9], although in the latter work any relation to the DFI was not observed. It is worth noting here that since γ and δ are arbitrary, one really has three distinct models each of which (as well as their sum) conserves momentum P_1 and there is a corresponding two-point map in each case.

Model 3. Model 2 with $\gamma = \delta = 0$ is the Bender-Tovbis model [6]. In the framework of the DFI approach [12], almost the entire space of static solutions supported by this model was described and many of those solutions were expressed in terms of the Jacobi elliptic functions (JEF) [12].

Model 4. Model 2 with $\delta = 0$ and $\gamma = 1/4$ is the Kevrekidis model [4].

Model 5. Discrete ϕ^4 model

$$\ddot{\phi}_n = \frac{1}{h^2}(\phi_{n-1} - 2\phi_n + \phi_{n+1}) + \frac{\lambda\left(\phi_n - \phi_n^3\right)}{1 - \Lambda\phi_n^2/2}, \tag{9}$$

discovered in [8] does not belong to Eq. (1) but its static problem coincides with that of the Bender-Tovbis Model 3 [8, 12]. Some very special features of this TI model are the conservation of energy and the on-site discretization of the nonlinear term. In all other TI ϕ^4 models derived so far the nonlinear term is discretized on the three neighboring nodes (i.e. lattice sites).

Model 6. With only A_4 nonzero and other A_k being equal to zero, one arrives at the non-Hamiltonian TI model derived by Barashenkov, Oxtoby, and Pelinovsky [9] and referred to as BOP. This model conserves the momentum defined as [14]

$$P_2 = \sum_n \dot{\phi}_n(\phi_{n+2} - \phi_{n-2}). \tag{10}$$

The first integral of the static version of this model has been found in [14], where an almost complete set of static solutions supported by this model were derived and many of those solutions were expressed in terms of JEF.

Model 7. Taking $A_1 = 2\lambda/9$, $A_2 = A_3 = \lambda/3$, $A_4 = A_5 = 0$, $A_6 = \lambda/9$ one gets the Hamiltonian of the Speight and Ward (SW) model [5]. For this model it is known that, it supports TI static kinks derivable from the two-point map

$$\phi_{n\pm1} = -\frac{\phi_n}{2} \mp \frac{3}{\sqrt{2}\,H} \pm \frac{\sqrt{3}}{2}\sqrt{-\phi_n^2 \pm \frac{6\sqrt{2}}{H}\phi_n + \frac{6}{H^2} + 4}, \quad H^2 = \frac{6\Lambda}{6-\Lambda}\,, \tag{11}$$

where one can take either the upper or the lower sign. Note that this map is defined in case $0 < \Lambda < 6$. To get this map, one has to set in Eq. (4) $C = 0$ and present it as

$$\phi_x \pm \sqrt{\frac{\lambda}{2}}(1 - \phi^2) = 0. \tag{12}$$

Discretizing Eq. (12) as

$$\frac{\phi_n - \phi_{n-1}}{H} - \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{2}}(\phi_{n-1}^2 + \phi_{n-1}\phi_n + \phi_n^2) = 0\,, \tag{13}$$

and solving the resulting quadratic equation we come to Eq. (11). It is not known if this model supports static TI solutions other than the kink. It is also not known if this model has the first integral of the static problem apart from the case of $C = 0$. In the present study we will give evidence that the answer to the second question is negative (see Sec. VII A) but we were able to find other TI solutions to this model, see Eq. (97) and Eq. (98).

Model 8. $A_1 = 4\lambda\alpha(\gamma+\beta)$, $A_2 = 4\lambda[2\alpha^2 + \gamma^2 + \beta(\gamma-\alpha)]$, $A_3 = 4\lambda\alpha(2\gamma+\beta)$, $A_4 = 4\lambda\gamma(\alpha-\beta)$, $A_5 = 4\lambda\alpha(\alpha-\beta)$, $A_6 = 4\lambda\alpha^2$, with two free parameters α and β with $\gamma = 1/2 - 2\alpha$. This model was also offered by Barashenkov, Oxtoby, and Pelinovsky [9]. This model includes as special cases the Model 3 (at $\alpha = \beta = 0$), the Model 6 (at $\alpha = 0$ and $\beta = -1/2$) and also the Model 7 (at $\alpha = \beta = 1/6$).

Model 9. $A_1 = 0$, $A_2 = 2\lambda(1/2 - \beta)$, $A_3 = \lambda\sigma(4 + h^2)$, $A_4 = 2\lambda\beta + \lambda\sigma(4 + h^2)$, $A_5 = -8\lambda\sigma$, $A_6 = 0$, with two free parameters β and σ. This model was also proposed by Barashenkov, Oxtoby, and Pelinovsky [9]. Note that the coefficients A_3 and A_4 in this model are h-dependent and that the constraint Eq. (2) for this model is satisfied only in the continuum limit (i.e. for $h = 0$).

In this paper we shall discuss four more TI models with cubic nonlinearities. The paper is organized as follows. In Sec. II we discuss the subclasses of model Eq. (1) that support different conservation laws. In Sec. III we report on a number of TI static solutions to Eq. (1) expressed in closed analytical form. All seven cases, when model Eq. (1) supports the exact static JEF solutions, are described. Two of these seven cases have been previously studied in the literature, and for the remaining five cases, basic JEF solutions are given here together with their hyperbolic function limits. We also obtain a periodic sine solution for the general 5-parameter model as given by Eq. (1). Section IV presents a number of short-periodic exact static solutions. In Sec. V we discuss the two-point maps for some of the TI models that have been reported in the literature and derive the map for the linear combination of Models 3 (Bender-Tovbis) and 6 (BOP). Goldstone translational modes of the TI static solutions are discussed in Sec. VI. Numerical results that illustrate some important properties of the TI static solutions are presented in Sec. VII. The discussion, conclusions, and future challenges are described in Sec. VIII. Finally in Appendix we spell out the short periodic solutions that are admitted by many of the models discussed in this paper.

II. MOMENTUM AND ENERGY CONSERVATION

As noted above, for the discrete model of Eq. (1), the momentum operator as given by Eq. (5) is conserved provided only A_2 is nonzero (Model 3), or if only A_5 and A_6 are nonzero with $A_5 = A_6$ (Model 4), or if only A_1, A_3, A_4 are nonzero with $A_1 = A_3/2 = A_4$ and for a linear combination of these models which is precisely the Model 2. It is also conserved in Model 8 (provided $\alpha = \beta = 0$) and in Model 9 (provided $\sigma = \beta = 0$). On the other hand, as was already mentioned, the momentum defined by Eq. (10) is conserved for Model 6 with only A_4 nonzero. It is also conserved in Model 8 (provided $\alpha = 0, \beta = -1/2$) and in Model 9 (provided $\sigma = 0, \beta = 1/2$).

Model 10. For

$$A_1 = 4\alpha_1\lambda, \quad A_2 = 6\alpha_2\lambda, \quad A_3 = 4\alpha_3\lambda, \quad A_6 = 2\alpha_2\lambda, \quad A_4 = A_5 = 0, \quad \text{with} \quad \alpha_1 + 2\alpha_2 + \alpha_3 = \frac{1}{4}\,, \tag{14}$$

the model Eq. (1) has the Hamiltonian

$$H = \sum_n \left[\frac{\dot{\phi}_n^2}{2} + \frac{(\phi_n - \phi_{n-1})^2}{2h^2} + \frac{\lambda}{4} - \frac{\lambda}{2}\phi_n^2 + \alpha_1\phi_n^4 + \alpha_2\phi_n\phi_{n-1}(\phi_n^2 + \phi_{n-1}^2) + \alpha_3\phi_n^2\phi_{n-1}^2 \right], \tag{15}$$

and hence energy is conserved in this model. As it can be seen, the Hamiltonian model has two free parameters. Note that the Speight-Ward model 7 is a special case of this model for $\alpha_1 = \alpha_2 = 1/18$ and $\alpha_2 = 1/12$. Similarly, energy is also conserved in Model 8 provided $\alpha = \beta = 1/6$.

III. TRANSLATIONALLY INVARIANT JEF, HYPERBOLIC, AND TRIGONOMETRIC STATIC SOLUTIONS

We shall first discuss the JEF solutions as well as the hyperbolic solutions which follow from the JEF solutions and later on we shall discuss the sine solutions which exist in almost all the models.

A. JEF and Hyperbolic Solutions

We shall now show that JEF solutions can be obtained for the discrete model of Eq. (1) in case $A_1 = A_6 = 0$ in the following seven cases: (i) only A_2 nonzero; (ii) only A_4 nonzero; (iii) only A_2 and A_4 nonzero; (iv) only A_3 and A_5 nonzero; (v) A_2, A_3, and A_5 nonzero; (vi) A_3, A_4, and A_5 nonzero; (vii) A_2, A_3, A_4, and A_5 nonzero.

The JEF solutions have already been reported in the case (i) in [12] and in the case (ii) in [14]. In this paper we report on the JEF solutions for the cases (iii) to (vii). In particular, in all these cases we obtain the sn solution

$$\phi_n = SA\mathrm{sn}[h\beta(n + x_0), m]\,, \tag{16}$$

the cn solution

$$\phi_n = SA\mathrm{cn}[h\beta(n + x_0), m]\,, \tag{17}$$

and the dn solution

$$\phi_n = SA\mathrm{dn}[h\beta(n + x_0), m]\,, \tag{18}$$

where x_0 is an arbitrary shift, $0 \leq m \leq 1$ is the JEF modulus, and $S = 1$ for the non-staggered and $S = (-1)^n$ for the staggered solutions.

We also derive the hyperbolic function limits ($m = 1$) of the above JEF solutions. In particular, while the sn solution reduces to the kink solution of the form

$$\phi_n = SA\tanh[h\beta(n + x_0)]\,, \tag{19}$$

both cn and dn solutions reduce to the single-humped pulse solution of the form

$$\phi_n = SA\mathrm{sech}[h\beta(n + x_0)]\,. \tag{20}$$

Expressions that relate the solution parameters A and β to the model parameters A_k and, where applicable, the relations between A_k, are given in what follows.

Note that if $\phi_n(t)$ is a solution to Eq. (1), then the staggered solution $(-1)^n\phi_n(t)$ is also a solution to the same equation, i.e. Eq. (1) with the coefficients A_1, A_3, and A_4 having the opposite signs and further $2-\Lambda$ is to be replaced by $\Lambda - 2$, where Λ is as given by Eq. (7). To make the presentation of the results as compact as possible, in most cases, we shall therefore not give the parameters for the staggered solutions. Only for the case (iii) with A_2 and A_4 nonzero and for hyperbolic solutions, parameters will be given for both the staggered and the nonstaggered solutions.

B. Case (iii): only A_2 and A_4 nonzero

Since only A_2, A_4 are nonzero, hence Eq. (2) reduces to

$$A_2 + A_4 = \lambda\,. \tag{21}$$

In this case, one has both nonstaggered sn and staggered sn solutions Eq. (16), with $S = 1$ and $S = (-1)^n$ respectively, provided the following relations are satisfied:

$$\frac{2m}{A^2h^2} = A_2\mathrm{ns}^2(h\beta, m) \pm 2A_4\mathrm{ns}(h\beta, m)\mathrm{ns}(2h\beta, m)\,, \tag{22}$$

$$\pm\frac{(2-\Lambda)m}{A^2h^2} = A_2\mathrm{cs}(h\beta)\mathrm{ds}(h\beta) \pm A_4\mathrm{ns}^2(h\beta, m)\,, \tag{23}$$

where $ns(x,m) = 1/sn(x,m)$, $cs(x,m) = cn(x,m)/sn(x,m)$ and $ds(x,m) = dn(x,m)/sn(x,m)$. In the above equations, the upper sign corresponds to the nonstaggered sn solution ($S = 1$), while the lower sign corresponds to the staggered case ($S = (-1)^n$).

In the limit of $m = 1$ the nonstaggered solution reduces to the kink solution, Eq. (19), with $S = 1$ and the relations (22) and (23) take the simpler form

$$A^2 = 1\,,\quad h^2A_4 = \frac{2\tanh^2(h\beta) - \Lambda}{\tanh^2(h\beta)}\,,\quad A_2 = \lambda - A_4\,. \tag{24}$$

Note that this solution is valid for any value of Λ including $\Lambda = 2$.

On the other hand, in the limit of $m = 1$, the staggered sn solution reduces to the staggered kink solution, Eq. (19), with $S = (-1)^n$ and the relations (22) and (23) reduce to

$$\mathrm{sech}^2(h\beta) = \frac{(8-3\Lambda)h^2A_4 + \Lambda(\Lambda-2)}{(2-h^2A_4)\Lambda}\,,\quad A^2 = \frac{(4-\Lambda)}{(2-h^2A_4)\Lambda}\,. \tag{25}$$

Another set of exact JEF solutions are the nonstaggered as well as the staggered cn solutions as given by Eq. (17) with $S = 1$ and $S = (-1)^n$, respectively, provided the following relations are satisfied:

$$\frac{2m}{A^2h^2} = -A_2\mathrm{ds}^2(h\beta, m) \mp 2A_4\mathrm{ds}(h\beta, m)\mathrm{ds}(2h\beta, m)\,, \tag{26}$$

$$\pm\frac{(2-\Lambda)m}{A^2h^2} = \mp A_4\mathrm{ds}^2(h\beta, m) - A_2\mathrm{cs}(h\beta, m)\mathrm{ns}(h\beta, m)\,. \tag{27}$$

In the above equations, the upper sign corresponds to the nonstaggered sn solution ($S = 1$), while the lower sign corresponds to the staggered case ($S = (-1)^n$).

Yet another exact JEF solutions are the nonstaggered as well as the staggered dn solutions as given by Eq. (18) with $S = 1$ and $S = (-1)^n$ respectively provided the following relations are satisfied:

$$\frac{2}{A^2h^2} = -A_2\mathrm{cs}^2(h\beta, m) \mp 2A_4\mathrm{cs}(h\beta, m)\mathrm{cs}(2h\beta, m)\,, \tag{28}$$

$$\pm\frac{(2-\Lambda)}{A^2h^2} = \mp A_4\mathrm{cs}^2(h\beta, m) - A_2\mathrm{ns}(h\beta, m)\mathrm{ds}(h\beta, m)\,. \tag{29}$$

In the above equations, the upper sign corresponds to the nonstaggered sn solution ($S = 1$), while the lower sign corresponds to the staggered case ($S = (-1)^n$).

In the limit of $m = 1$, the nonstaggered cn as well dn solutions reduce to the pulse solution Eq. (20) with $S = 1$ and the relations (26) and (27) as well as (28) and (29) take the simpler form

$$\Lambda = -2[\cosh(h\beta) - 1] < 0\,, \qquad A^2 = \frac{(|\Lambda| + 2)(|\Lambda| + 4)}{2[(|\Lambda| + 2) + h^2A_4]}\,. \tag{30}$$

Thus the pulse solution exists only if $\Lambda < 0$.

On the other hand, in the limit of $m = 1$ the staggered cn as well dn solutions reduce to the staggered pulse solution Eq. (20) with $S = (-1)^n$ and the relations (26) and (27) as well as (28) and (29) take the simpler form

$$\Lambda = 2[\cosh(h\beta) + 1] > 0\,, \qquad A^2 = \frac{(\Lambda - 2)(\Lambda - 4)}{2[2 + h^2A_4 - \Lambda]}\,. \tag{31}$$

Thus the staggered pulse solution (31) exists only if $\Lambda > 4$.

C. Case (iv): only A_3 and A_5 nonzero

Since only A_3, A_5 are nonzero, Eq. (2) reduces to

$$A_3 + A_5 = \lambda\,. \tag{32}$$

In this case, one has the sn solution Eq. (16) with $S = 1$ provided the following relations are satisfied:

$$A_5\mathrm{ns}(2h\beta, m) = -A_3\mathrm{ns}(h\beta, m)\,, \tag{33}$$

$$\frac{2m}{A^2h^2} = -A_3\mathrm{cs}(h\beta, m)\mathrm{ds}(h\beta, m) - A_5[\mathrm{cs}(2h\beta, m)\mathrm{ds}(2h\beta, m) - \mathrm{ns}^2(2h\beta, m)]\,, \tag{34}$$

$$\frac{(2-\Lambda)m}{A^2h^2} = -A_3\mathrm{ns}^2(h\beta, m)\,. \tag{35}$$

In the limit $m = 1$, this reduces to the kink solution Eq. (19) with $S = 1$ provided

$$A^2 = 1\,, \quad 2A_3 = -A_5[1 + \tanh^2(h\beta)]\,, \quad \Lambda = \frac{2\tanh^2(h\beta)\mathrm{sech}^2(h\beta)}{1 + 2\tanh^2(h\beta) - \tanh^4(h\beta)}\,. \tag{36}$$

From Eq. (36) it follows that $\Lambda < A_5h^2/2$. Further, using the fact that $\tanh^2(x) < 1$, it is easily checked that the kink solution exists in this model only if $0 < \Lambda < (2 - \sqrt{2})/2$.

In case only A_3 and A_5 are nonzero, the static Eq. (1) is equivalent to a highly nonlinear map which is even more complicated than the 18 parameter map of Quispel et al. [22]. In fact, this is also true for the remaining three cases discussed below.

The corresponding staggered solutions are easily found by changing the signs of terms with A_3 and $(2 - \Lambda)$. In particular, it is easily shown that in the limit $m = 1$, the staggered sn solution reduces to the staggered kink solution (19) with $S = (-1)^n$ provided

$$2A_3 = A_5[1 + \tanh^2(h\beta)]\,, \quad A_5h^2 = \frac{2\Lambda}{3 + \tanh^2(h\beta)}\,, \quad A^2 = \frac{[3 + \tanh^2(h\beta)]\tanh^2(h\beta)}{[2 + 3\tanh^2(h\beta) - \tanh^4(h\beta)]}\,. \tag{37}$$

One can show that this solution can only exist provided

$$\frac{6+\sqrt{2}}{2} < \Lambda < 4\,. \tag{38}$$

Another exact JEF solution is the nonstaggered dn solution as given by Eq. (18) with $S = 1$ provided the following relations are satisfied:

$$A_5 \mathrm{cs}(2h\beta, m) = -A_3 \mathrm{cs}(h\beta, m)\,, \quad \frac{2-\Lambda}{A^2h^2} = A_3 \mathrm{cs}^2(h\beta, m)\,, \tag{39}$$

$$\frac{2}{A^2h^2} = A_3 \mathrm{ds}(h\beta, m)\mathrm{ns}(h\beta, m) + A_5[\mathrm{ns}(2h\beta, m)\mathrm{ds}(2h\beta, m) - \mathrm{cs}^2(2h\beta, m)]\,. \tag{40}$$

Yet another exact JEF solution is the nonstaggered cn solution as given by Eq. (17) with $S = 1$ provided the following relations are satisfied:

$$A_5 \mathrm{ds}(2h\beta, m) = -A_3 \mathrm{ds}(h\beta, m)\,, \quad \frac{(2-\Lambda)m}{A^2h^2} = A_3 \mathrm{ds}^2(h\beta, m)\,, \tag{41}$$

$$\frac{2m}{A^2h^2} = A_3 \mathrm{cs}(h\beta, m)\mathrm{ns}(h\beta, m) + A_5[\mathrm{ns}(2h\beta, m)\mathrm{cs}(2h\beta, m) - \mathrm{ds}^2(2h\beta, m)]\,. \tag{42}$$

In the limit $m = 1$, both the cn and the dn solutions go over to the pulse solution Eq. (20) with $S = 1$, and the relations (39) and (40) as well as (41) and (42) take the simpler form

$$\begin{aligned} &\Lambda = -2[\cosh(h\beta) - 1] < 0\,, \quad A_5 < 0\,, \quad 2A_3 = -A_5 \mathrm{sech}(h\beta)\,, \quad |A_5|h^2 = |\Lambda| \frac{|\Lambda|+2}{|\Lambda|+1}\,, \\ &A^2 = \frac{(|\Lambda|+1)(|\Lambda|+2)(|\Lambda|+4)}{4}\,. \end{aligned} \tag{43}$$

One can also obtain the corresponding staggered cn and dn solutions by changing the signs of A_3 and $(2-\Lambda)$ in the relations (39) and (40) as well as (41) and (42). In the limit $m = 1$ one can show that *no* staggered pulse solution Eq. (20) with $S = (-1)^n$ exists in case only A_3 and A_5 are nonzero.

D. Case (v): only A_2, A_3, and A_5 are nonzero

Since only A_2, A_3, A_5 are nonzero, hence Eq. (2) reduces to

$$A_2 + A_3 + A_5 = \lambda\,. \tag{44}$$

In this case, one has the sn solution Eq. (16) with $S = 1$ provided the following relations are satisfied:

$$A_5 \mathrm{ns}(2h\beta, m) = -A_3 \mathrm{ns}(h\beta, m)\,, \quad \frac{(2-\Lambda)m}{A^2h^2} = A_2 \mathrm{cs}(h\beta)\mathrm{ds}(h\beta) - A_3 \mathrm{ns}^2(h\beta, m)\,, \tag{45}$$

$$\begin{aligned} \frac{2m}{A^2h^2} &= A_2 \mathrm{ns}^2(h\beta, m) - A_3 \mathrm{cs}(h\beta, m)\mathrm{ds}(h\beta, m) \\ &- A_5[\mathrm{cs}(2h\beta, m)\mathrm{ds}(2h\beta, m) - \mathrm{ns}^2(2h\beta, m)]\,. \end{aligned} \tag{46}$$

In the $m = 1$ case, for the kink solution Eq. (19) with $S = 1$, the relations (45) to (46) take the simpler form

$$\begin{aligned} &A^2 = 1\,, \quad h^2 A_3 = \frac{[\Lambda - 2\tanh^2(h\beta)][1+\tanh^2(h\beta)]}{\tanh^2(h\beta)[3-\tanh^2(h\beta)]}\,, \\ &A_2 h^2 = \frac{[\Lambda + 2(\Lambda-1)\tanh^2(h\beta) + (2-\Lambda)\tanh^4(h\beta)]}{\tanh^2(h\beta)[3-\tanh^2(h\beta)]}\,, \quad A_5 h^2 = -\frac{2[\Lambda - 2\tanh^2(h\beta)]}{\tanh^2(h\beta)[3-\tanh^2(h\beta)]}\,. \end{aligned} \tag{47}$$

Note that this solution is valid for any value of Λ including $\Lambda = 2$.

The corresponding staggered sn solution is easily obtained by changing the signs of terms with A_3 and $(2-\Lambda)$. In particular, it is easily shown that in the limit $m=1$, the staggered sn solution reduces to the staggered kink solution (19) with $S=(-1)^n$ provided

$$
\begin{aligned}
h^2A_3 &= \frac{\Lambda[1+\tanh^2(h\beta)][\Lambda-2(2-\tanh^2(h\beta)]}{[\Lambda\tanh^4(h\beta)+(8-\Lambda)\tanh^2(h\beta)+2(\Lambda-4)]}\,,\\
h^2A_5 &= \frac{2\Lambda[\Lambda-2(2-\tanh^2(h\beta)]}{[\Lambda\tanh^4(h\beta)+(8-\Lambda)\tanh^2(h\beta)+2(\Lambda-4)]}\,,\\
h^2A_2 &= \frac{\Lambda[(\Lambda-2)\tanh^4(h\beta)+2(3-\Lambda)\tanh^2(h\beta)+4-\Lambda]}{[\Lambda\tanh^4(h\beta)+(8-\Lambda)\tanh^2(h\beta)+2(\Lambda-4)]}\,,\\
A^2 &= \frac{(\Lambda-4)}{(2h^2A_3-\Lambda)}\,.
\end{aligned}
\tag{48}
$$

Note that this solution exists over a vast range of Λ values including $\Lambda=2$.

For the dn solution Eq. (18) with $S=1$ one has

$$
A_5\text{cs}(2h\beta,m)=-A_3\text{cs}(h\beta,m)\,,\quad \frac{2-\Lambda}{A^2h^2}=A_3\text{cs}^2(h\beta,m)-A_2\text{ds}(h\beta,m)\text{ns}(h\beta,m)\,,
\tag{49}
$$

$$
\frac{2}{A^2h^2}=-A_2\text{cs}^2(h\beta,m)+A_3\text{ds}(h\beta,m)\text{ns}(h\beta,m)+A_5[\text{ns}(2h\beta,m)\text{ds}(2h\beta,m)-\text{cs}^2(2h\beta,m)]\,.
\tag{50}
$$

Parameters of the cn solution Eq. (17) with $S=1$ satisfy

$$
A_5\text{ds}(2h\beta,m)=-A_3\text{ds}(h\beta,m)\,,\quad \frac{(2-\Lambda)m}{A^2h^2}=A_3\text{ds}^2(h\beta,m)-A_2\text{ns}(h\beta,m)\text{cs}(h\beta,m)\,,
\tag{51}
$$

$$
\frac{2m}{A^2h^2}=-A_2\text{ds}^2(h\beta,m)+A_3\text{cs}(h\beta,m)\text{ns}(h\beta,m)+A_5[\text{ns}(2h\beta,m)\text{cs}(2h\beta,m)-\text{ds}^2(2h\beta,m)]\,.
\tag{52}
$$

In the limiting case of $m=1$, from the relations (49) and (50) as well as (4.20) and (4.21), one finds the relations for the pulse solution Eq. (20) with $S=1$

$$
\begin{aligned}
&\Lambda=-2[\cosh(h\beta)-1]<0\,,\quad 2A_3=-A_5\text{sech}(h\beta)\,,\\
&A_2h^2=h^2A_3[2\cosh(h\beta)-1]-2[\cosh(h\beta)-1]\,, A^2=\frac{(|\Lambda|+2)(|\Lambda|+4)}{2[(|\Lambda|+2)-(|\Lambda|+3)h^2A_3]}\,.
\end{aligned}
\tag{53}
$$

Thus the pulse solution exists only if $\Lambda<0$.

One can also obtain the corresponding staggered dn and cn solutions. In the limit $m=1$, from these solutions we obtain the staggered pulse solution Eq. (20) with $S=(-1)^n$ satisfying

$$
\begin{aligned}
&\Lambda=2[1+\cosh(h\beta)]>4\,,\quad 2A_3=A_5\text{sech}(h\beta)]\,,\quad A_2h^2=-h^2A_3[2\cosh(h\beta)+1]+2[1+\cosh(h\beta)]\,,\\
&A^2=\frac{(\Lambda-2)(\Lambda-4)}{2[(\Lambda-3)h^2A_3-(\Lambda-2)]}\,.
\end{aligned}
\tag{54}
$$

Thus the staggered pulse solution exists only if $\Lambda>4$.

E. Case (vi): only A_3, A_4, and A_5 are nonzero

Since only A_3, A_4, A_5 are nonzero, hence Eq. (2) reduces to

$$
A_3+A_4+A_5=\lambda\,.
\tag{55}
$$

In this case, one has the sn solution Eq. (16) with $S=1$ satisfying

$$
A_5\text{ns}(2h\beta,m)=-A_3\text{ns}(h\beta,m)\,,\quad \frac{(2-\Lambda)m}{A^2h^2}=(A_4-A_3)\text{ns}^2(h\beta,m)\,,
\tag{56}
$$

$$\frac{2m}{A^2h^2} = 2A_4\text{ns}(h\beta,m)\text{ns}(2h\beta,m) - A_3\text{cs}(h\beta,m)\text{ds}(h\beta,m)$$
$$-A_5[\text{cs}(2h\beta,m)\text{ds}(2h\beta,m) - \text{ns}^2(2h\beta,m)]\,. \tag{57}$$

In the limit $m = 1$, we obtain from the sn solution the kink solution Eq. (19) with $S = 1$ with the parameters satisfying

$$A^2 = 1\,, \quad h^2A_3 = \frac{[1+\tanh^2(h\beta)][(\Lambda-2)\tanh^2(h\beta)+\Lambda]}{2\tanh^2(h\beta)}\,,$$
$$h^2A_4 = \frac{\Lambda + 2(\Lambda-1)\tanh^2(h\beta) - (\Lambda-2)\tanh^4(h\beta)}{2\tanh^2(h\beta)}\,, \quad h^2A_5 = -\frac{(\Lambda-2)\tanh^2(h\beta)+\Lambda}{\tanh^2(h\beta)}\,. \tag{58}$$

Note that this solution is valid for any Λ including $\Lambda = 2$ when conditions (58) take particularly simple form.

One can also obtain the corresponding staggered sn solution. In the limit $m = 1$, we obtain from the staggered sn solution the staggered kink solution Eq. (19) with $S = (-1)^n$ provided

$$A^2 = \frac{(3\Lambda-8)\tanh^2(h\beta)+2(\Lambda-4)}{\Lambda\tanh^2(h\beta)}\,, \tag{59}$$

$$A_3h^2 = \frac{\Lambda[1+\tanh^2(h\beta)][4-\Lambda+(2-\Lambda)\tanh^2(h\beta)]}{2[(8-3\Lambda)\tanh^2(h\beta)+2(4-\Lambda)]}\,. \tag{60}$$

From the above equations it follows that no solution exists in case $0 < \Lambda \le 3$. In particular, note that a solution does exist in case $\lambda < 0$ provided $A_3 < 0$.

Parameters of the dn solution Eq. (18) with $S = 1$ can be found from

$$A_5\text{cs}(2h\beta,m) = -A_3\text{cs}(h\beta,m)\,, \quad \frac{2-\Lambda}{A^2h^2} = (A_3 - A_4)\text{cs}^2(h\beta,m)\,, \tag{61}$$

$$\frac{2}{A^2h^2} = -2A_4\text{cs}(h\beta,m)\text{cs}(2h\beta,m) + A_3\text{ds}(h\beta,m)\text{ns}(h\beta,m) + A_5[\text{ns}(2h\beta,m)\text{ds}(2h\beta,m) - \text{cs}^2(2h\beta,m)]\,. \tag{62}$$

On the other hand, the parameters of the cn solution Eq. (17) with $S = 1$ satisfy

$$A_5\text{ds}(2h\beta,m) = -A_3\text{ds}(h\beta,m)\,, \quad \frac{(2-\Lambda)m}{A^2h^2} = (A_3 - A_4)\text{ds}^2(h\beta,m)\,, \tag{63}$$

$$\frac{2m}{A^2h^2} = -2A_4\text{ds}(h\beta,m)\text{ds}(2h\beta,m) + A_3\text{cs}(h\beta,m)\text{ns}(h\beta,m) + A_5[\text{ns}(2h\beta,m)\text{cs}(2h\beta,m) - \text{ds}^2(2h\beta,m)]\,. \tag{64}$$

In the limit $m = 1$, we obtain the parameters of the pulse solution Eq. (20) with $S = 1$

$$\Lambda = -2[\cosh(h\beta)-1] < 0\,, \quad 2A_3 = -A_5\text{sech}(h\beta)\,, \quad A_4h^2 = h^2A_3[2\cosh(h\beta)-1] - 2[\cosh(h\beta)-1]\,,$$
$$A^2 = \frac{(|\Lambda|+2)(|\Lambda|+4)}{4(1-h^2A_3)}\,. \tag{65}$$

Thus the pulse solution exists only if $\Lambda < 0$.

One can also obtain the corresponding staggered dn and cn solutions. In the limit $m = 1$, from these solutions we obtain the staggered pulse solution Eq. (20) with $S = (-1)^n$ satisfying

$$\Lambda = 2[1+\cosh(h\beta)] > 4\,, \quad 2A_3 = A_5\text{sech}(h\beta)\,, \quad A_4h^2 = -h^2A_3[2\cosh(h\beta)+1] + 2[1+\cosh(h\beta)]\,,$$
$$A^2 = \frac{(\Lambda-2)(\Lambda-4)}{4(1-h^2A_3)}\,. \tag{66}$$

Thus the staggered pulse solution exists only if $\Lambda > 4$.

F. Case (vii): nonzero A_2, A_3, A_4, and A_5 with $A_1 = A_6 = 0$

In this case, Eq. (2) reduces to

$$A_2 + A_3 + A_4 + A_5 = \lambda\,. \tag{67}$$

In this case, the sn solution Eq. (16) with $S = 1$ is characterized by

$$A_5 \text{ns}(2h\beta, m) = -A_3 \text{ns}(h\beta, m)\,, \tag{68}$$

$$\frac{2m}{A^2h^2} = 2A_4 \text{ns}(h\beta, m)\text{ns}(2h\beta, m) - A_3 \text{cs}(h\beta, m)\text{ds}(h\beta, m)$$
$$+A_2 \text{ns}^2(h\beta) - A_5[\text{cs}(2h\beta, m)\text{ds}(2h\beta, m) - \text{ns}^2(2h\beta, m)]\,, \tag{69}$$

$$\frac{(2-\Lambda)m}{A^2h^2} = (A_4 - A_3)\text{ns}^2(h\beta, m) + A_2 \text{cs}(h\beta)\text{ds}(h\beta)\,. \tag{70}$$

In the limit $m = 1$, this reduces to the kink solution satisfying

$$A^2 = 1, \quad \tanh^2(h\beta) = -\frac{2A_3 + A_5}{A_5}\,, \quad A_4 = \frac{2}{h^2} + \frac{\Lambda A_5}{h^2(2A_3 + A_5)} - A_3 - 2A_5\,. \tag{71}$$

Note that this solution is valid for any Λ including $\Lambda = 2$.

One can also work out the corresponding staggered sn solution Eq. (16) with $S = (-1)^n$. In the limit $m = 1$, we obtain the staggered kink solution Eq. (19) with $S = (-1)^n$ satisfying

$$A^2 = \frac{(4-\Lambda)[1 + \tanh^2(h\beta)]}{4A_3h^2 + (2A_2h^2 - \Lambda)[1 + \tanh^2(h\beta)]}\,, \tag{72}$$

$$A_2h^2[2(4-\Lambda) - \Lambda\tanh^2(h\beta)] + \frac{2A_3h^2[2(4-\Lambda) + (8-3\Lambda)\tanh^2(h\beta)]}{[1 + \tanh^2(h\beta)]} = \Lambda[4 - \Lambda + (2-\Lambda)\tanh^2(h\beta)]\,. \tag{73}$$

Another exact solution is the dn solution Eq. (18) with $S = 1$ provided the following relations are satisfied:

$$A_5 \text{cs}(2h\beta, m) = -A_3 \text{cs}(h\beta, m)\,, \tag{74}$$

$$\frac{2}{A^2h^2} = -2A_4 \text{cs}(h\beta, m)\text{cs}(2h\beta, m) + A_3 \text{ds}(h\beta, m)\text{ns}(h\beta, m)$$
$$-A_2 \text{cs}^2(h\beta) + A_5[\text{ns}(2h\beta, m)\text{ds}(2h\beta, m) - \text{cs}^2(2h\beta, m)]\,, \tag{75}$$

$$\frac{2-\Lambda}{A^2h^2} = (A_3 - A_4)\text{cs}^2(h\beta, m) - A_2 \text{ns}(h\beta)\text{ds}(h\beta)\,. \tag{76}$$

Yet another exact solution is the cn solution Eq. (18) with $S = 1$ satisfying

$$A_5 \text{ds}(2h\beta, m) = -A_3 \text{ds}(h\beta, m)\,, \tag{77}$$

$$\frac{2m}{A^2h^2} = -2A_4 \text{ds}(h\beta, m)\text{ds}(2h\beta, m) + A_3 \text{cs}(h\beta, m)\text{ns}(h\beta, m)$$
$$-A_2 \text{ds}^2(h\beta) + A_5[\text{ns}(2h\beta, m)\text{cs}(2h\beta, m) - \text{ds}^2(2h\beta, m)]\,, \tag{78}$$

$$\frac{(2-\Lambda)m}{A^2h^2} = (A_3 - A_4)\text{ds}^2(h\beta, m) - A_2 \text{ns}(h\beta)\text{cs}(h\beta)\,. \tag{79}$$

In the limit $m = 1$, we obtain the pulse solution Eq. (20) with $S = 1$

$$\Lambda = -2[\cosh(h\beta) - 1] < 0\,, \quad 2A_3 = -A_5 \mathrm{sech}(h\beta)\,, \quad (A_4 + A_2)h^2 = h^2 A_3[2\cosh(h\beta) - 1] - 2[\cosh(h\beta) - 1]\,,$$
$$A^2 = \frac{(|\Lambda| + 2)(|\Lambda| + 4)}{2(2 - 2h^2 A_3 - A_2 h^2)}\,. \tag{80}$$

Thus the pulse solution exists only if $\Lambda < 0$.

One can also work out the corresponding staggered dn and cn solutions. In the limit $m = 1$, we obtain the staggered pulse solution Eq. (20) with $S = (-1)^n$ with

$$\Lambda = 2[1 + \cosh(h\beta)] > 4\,, \quad 2A_3 = A_5 \mathrm{sech}(h\beta)\,, \quad (A_4 + A_2)h^2 = -h^2 A_3[2\cosh(h\beta) + 1] + 2[1 + \cosh(h\beta)]\,,$$
$$A^2 = \frac{(\Lambda - 2)(\Lambda - 4)}{2(2 - 2h^2 A_3 - A_2 h^2)}\,. \tag{81}$$

Thus the staggered pulse solution exists only if $\Lambda > 4$.

Summarizing, it is worth noting that in all the five cases discussed above (which is also true in the remaining two cases (i) and (ii)), for the kink solution, the amplitude A is always equal to 1. Further, in all the seven cases, while the pulse solution exists only if $\Lambda < 0$, the staggered pulse solution exists only if $\Lambda > 4$.

It is worth pointing out here that the trigonometric solutions for these models do not follow from above JEF solutions in the limit $m = 0$, since both left and right hand sides of the identities for the Jacobi elliptic functions sn and cn, which one has used in deriving these solutions, vanish identically in this limit [23]. However, trigonometric solutions can be derived for these models directly, as we show below.

G. Trigonometric solutions

We shall now show that unlike the JEF and the hyperbolic solutions, the static TI trigonometric solutions with an arbitrary shift along the chain x_0 exist even when all the six parameters A_i are nonzero. Ideally, the trigonometric solutions should be the $m \to 0$ limit of the JEF solutions. However, the relevant JEF identities we have used vanish in this limit and thus the trigonometric solutions must be derived separately.

We look for a solution of Eq. (1) of the form

$$\phi_n = SA \sin[h\beta(n + x_0)]\,, \tag{82}$$

where $S = 1$ or $(-1)^n$ depending on whether it is a nonstaggered or a staggered solution, and find that it exists under the following two conditions

$$\pm(\Lambda - 2) + 2\cos(h\beta) = h^2 A^2 \sin^2(h\beta)[\pm(A_3 - A_4) + (3A_6 - A_5)\cos(h\beta)]\,, \tag{83}$$

$$\pm[A_1 + A_4 + A_3\cos(2h\beta)] + (A_2 + A_5)\cos(h\beta) + A_6\cos(h\beta)[4\cos^2(h\beta) - 3] = 0\,, \tag{84}$$

where the upper (lower) sign corresponds to nonstaggered (staggered) sine solution. As mentioned before, if $\phi_n(t)$ is a solution to Eq. (1), then the staggered solution $(-1)^n\phi_n(t)$ is also a solution to Eq. (1) with the coefficients of A_1, A_3 and A_4 having opposite signs and with $2 - \Lambda$ replaced by $\Lambda - 2$, where $\Lambda = h^2\lambda$. To make the presentation compact, in most cases we shall therefore only give results for the nonstaggered solutions. We shall see that these solutions exist in all the models discussed in this paper except in the case of Model 1 (only A_1 nonzero) and Model 4 (only A_4 nonzero).

Using the well known addition theorem for sine, it is easily shown that the sine solution follows from the two-point quadratic map

$$\phi_{n+1}^2 + \phi_n^2 - 2\phi_{n+1}\phi_n\cos(h\beta) - A^2\sin^2(h\beta) = 0\,. \tag{85}$$

We now discuss these solutions in the several special cases.

Model 1: Only $A_1 \neq 0\,,\ (h^2 A_1 = \Lambda)$

From Eqs. (83) and (84) it follows that in this case there is neither a staggered nor a nonstaggered solution of the form of Eq. (82).

Model 6: Only $A_4 \neq 0\,,\ (h^2 A_4 = \Lambda)$

From Eqs. (83) and (84) it follows that in this case also there is neither a staggered nor a nonstaggered solution of the form of Eq. (82).

Model 3: Only $A_2 \neq 0\,,\ (h^2 A_2 = \Lambda)$

In this case, both staggered and nonstaggered solutions exist provided

$$\Lambda = 2\,, \quad h\beta = \pm\pi/2\,. \tag{86}$$

and in this case A is arbitrary.

Only $A_4, A_2 \neq 0\,,\ [h^2(A_4 + A_2) = \Lambda]$

In this case, the nonstaggered solution with $S = 1$ given by Eq. (82) exists provided

$$A_4 + A_2\cos(h\beta) = 0\,, \quad 2A^2\Lambda\cos(h\beta)\cos^2(h\beta/2) = \Lambda - 2[1 - \cos(h\beta)]\,. \tag{87}$$

On the other hand, the staggered solution with $S = (-1)^n$ exists in the same model (with only A_2 and A_4 nonzero) provided

$$A_4 + A_2\cos(h\beta) = 0\,, \quad 2A^2\Lambda\cos(h\beta)\sin^2(h\beta/2) = \Lambda - 2[1 + \cos(h\beta)]\,. \tag{88}$$

Only $A_4, A_3, A_5 \neq 0\,,\ [h^2(A_4 + A_3 + A_5) = \Lambda]$

In this case the nonstaggered solution exists provided

$$A_4 + A_5\cos(h\beta) + A_3\cos(2h\beta) = 0\,, \quad 2A^2h^2A_3\cos^2(h\beta)\sin^2(h\beta) = \Lambda - 2[1 - \cos(h\beta)]\,. \tag{89}$$

Only $A_2, A_3, A_5 \neq 0\,,\ [h^2(A_2 + A_3 + A_5) = \Lambda]$

In this case the nonstaggered solution exists provided

$$(A_2 + A_5)\cos(h\beta) + A_3\cos(2h\beta) = 0\,, \quad A^2h^2[A_3 - A_5\cos(h\beta)]\sin^2(h\beta) = \Lambda - 2[1 - \cos(h\beta)]\,. \tag{90}$$

Only $A_3, A_5 \neq 0\ [h^2(A_3 + A_5) = \Lambda]$

In this case the nonstaggered solution exists provided

$$A_5\cos(h\beta) + A_3\cos(2h\beta) = 0\,, \quad 4A^2\Lambda\cos^3(h\beta)\cos^2(\frac{h\beta}{2}) = (\Lambda - 2[1 - \cos(h\beta)])[1 + 2\cos(h\beta)]\,. \tag{91}$$

Case of $A_1 = A_6 = 0, A_2, A_3, A_4, A_5 \neq 0\,,\ [h^2(A_4 + A_2 + A_3 + A_5) = \Lambda]$

In this case the nonstaggered solution exists provided

$$A_4 + (A_2 + A_5)\cos(h\beta) + A_3\cos(2h\beta) = 0\,, \quad A^2h^2[A_3 - A_4 - A_5\cos(h\beta)]\sin^2(h\beta) = \Lambda - 2[1 - \cos(h\beta)]\,. \tag{92}$$

Model 2 with $\delta = 0, \gamma = 1/4$**: Only** $A_5 = A_6 \neq 0\,,\ 2h^2A_5 = \Lambda$

In this case the nonstaggered solution exists provided

$$\cos^2(h\beta) = \frac{1}{2}\,, \quad A^2\Lambda = 4 \pm 2\sqrt{2}(\Lambda - 2)\,. \tag{93}$$

Thus in this case one has a solution with period 8.

Model 2 with $\gamma = 0, \delta = 1/4$**: Only** $A_1 = A_3/2 = A_4 = \lambda/2$ **are nonzero**$,\ 2h^2A_3 = \Lambda$

In this case the nonstaggered solution exists provided

$$\sin^2(h\beta) = 1\,, \quad A^2\Lambda = 4(\Lambda - 2)\,. \tag{94}$$

Thus in this case one has a solution with period 4.

Model 2: Case of $A_3 = 2A_1 = 2A_4, A_6 = A_5, A_2$ **all nonzero ,** $[h^2(2A_3 + 2A_5 + A_2) = \Lambda]$

In this case the nonstaggered solution exists provided

$$\cos(h\beta)[A_2 + 2A_5\cos(2h\beta) + 2A_3\cos(h\beta)] = 0\,, \quad A^2h^2[A_3 + 4A_5\cos(h\beta)]\sin^2(h\beta) = 2[\Lambda - 2(1 - \cos(h\beta))]\,. \tag{95}$$

Thus in this case one has solution with period 4.

Model 10: Hamiltonian model given by Eq. (15) with $A_1 = 4\alpha_1\lambda\,, A_2 = 6\alpha_2\lambda\,, A_3 = 4\alpha_3\lambda\,, A_6 = 2\alpha_2\lambda\,, A_4 = A_5 = 0 \quad [4(\alpha_1 + 2\alpha_2 + \alpha_3) = 1]$

In this case the nonstaggered solution exists provided

$$2A^2\Lambda[2\alpha_2 + 3\alpha_2\cos(h\beta)]\sin^2(h\beta) = [\Lambda - 2(1 - \cos(h\beta))]\,,$$
$$\alpha_2\cos^3(h\beta) + \alpha_3\cos^2(h\beta) = \alpha_2 + \alpha_3 - \frac{1}{8}\,. \tag{96}$$

Speight and Ward Model 7: $\alpha_1 = \alpha_2 = \frac{2}{3}\alpha_3 = \frac{1}{18}$
Being a special case of the model 10, in this case Eq. (96) reduces to

$$4\cos^3(h\beta) + 6\cos^2(h\beta) - 1 = 0\,, \quad A^2\Lambda[1+\cos(h\beta)]\sin^2(h\beta) = 3[\Lambda - 2(1-\cos(h\beta))]\,. \tag{97}$$

The roots of the first equation in Eq. (97) are $h\beta = 2\pi/3$ and $h\beta \approx 1.196$. The first root corresponds to the three-periodic TI solution with an amplitude that can be found from the second equation in Eq. (97) as $A^2 = 8(\Lambda-3)/\Lambda$, so that the solution is defined either when $\Lambda > 3$ or when $\Lambda < 0$. The second root corresponds to the TI solution with the period approximately 5.253.

Instead, the staggered solution exists in this model provided

$$4\cos^3(h\beta) - 6\cos^2(h\beta) + 1 = 0\,, \quad A^2\Lambda[1-\cos(h\beta)]\sin^2(h\beta) = 3[\Lambda - 2(1+\cos(h\beta))]\,. \tag{98}$$

The roots of the first equation in Eq. (98) are $h\beta = \pi/3$ and $h\beta \approx 1.944$. The first root corresponds to the six-periodic TI solution with the amplitude which is same as for the three-periodic nonstaggered solution, i.e. $A^2 = 8(\Lambda-3)/\Lambda$, so that this solution is also defined either when $\Lambda > 3$ or when $\Lambda < 0$. The second root corresponds to the TI solution with the period approximately 3.023.

It is worth pointing out that in the case of Model 3 (only A_2 nonzero), both the staggered and the nonstaggered solutions are of period 4, i.e. $h\beta = \pi/4$. The same is also true of Model 2 in case $\delta = 1/4, \gamma = 0$, i.e. $A_1 = A_4 = A_3/2 = \lambda/4$. On the other hand, in the case of Model 2 with $\gamma = 1/4, \delta = 0$, i.e. $A_5 = A_6 = \lambda/2$, both the staggered and the nonstaggered solutions are of period 8. Finally, in the case of Model 7, i.e. Speight and Ward model with $A_1 = 2\lambda/9, A_2 = A_3 = \lambda/3, A_4 = A_5 = 0, A_6 = \lambda/9$, one of the nonstaggered solution is three-periodic while in the staggered case, one of the solution is of period 6.

IV. SHORT-PERIOD SOLUTIONS

We shall now show that apart from the JEF, hyperbolic and trigonometric, there are also several short period and even aperiodic solutions of Eq. (1). In order to obtain these solutions, it is useful to look at the symmetries of Eq. (1). In particular, notice that Eq. (1) is invariant under $\phi_{n-1} \to \phi_{n+1}$ and $\phi_{n+1} \to \phi_{n-1}$. Further, Eq. (1) is also invariant under $(\phi_{n-1}, \phi_n, \phi_{n+1}) \to (-\phi_{n-1}, -\phi_n, -\phi_{n+1})$. A consequence of these two symmetries is that if $(\phi_{n-1}, \phi_n, \phi_{n+1})$ is a solution to Eq. (1) under certain constraints, then $(-\phi_{n-1}, -\phi_n, -\phi_{n+1})$, $(\phi_{n+1}, \phi_n, \phi_{n-1})$ and $(-\phi_{n+1}, -\phi_n, -\phi_{n-1})$ are also solutions of Eq. (1) provided the same constraints are satisfied.

We list below several exact solutions to Eq. (1) satisfying the constraint (2). We shall only write down the exact solutions to the general model. IN Appendix we spell ot the short-period solutions that are admitted by many of the models discuused in this paper.

While obtaining the periodic solutions the following results have been used which have been derived by using Eqs. (1) and (2).

1. If $\phi_{n-1} = \phi_n = \phi_{n+1} = a$ then it follows that

$$a^2 = 1\,. \tag{99}$$

2. If $\phi_{n-1} = \phi_n = -\phi_{n+1} = a$ then it follows that

$$\Lambda - 2 = h^2 a^2 (A_1 + A_3 - A_4)\,. \tag{100}$$

3. If $\phi_{n-1} = -\phi_n = \phi_{n+1} = a$ then it follows that

$$\Lambda - 4 = h^2 a^2 (A_1 - A_2 + A_3 + A_4 - A_5 - A_6)\,. \tag{101}$$

4. If $\phi_{n-1} = \phi_n = a$, and $\phi_{n+1} = 0$ then it follows that

$$2(\Lambda - 1) = h^2 a^2 (2A_1 + A_2 + A_3 + A_6)\,. \tag{102}$$

5. If $\phi_{n-1} = \phi_{n+1} = a$, and $\phi_n = 0$ then it follows that

$$2 = h^2 a^2 (A_5 + A_6)\,. \tag{103}$$

6. If $\phi_{n-1} = -\phi_n = a$, and $\phi_{n+1} = 0$ then it follows that

$$2(\Lambda - 3) = h^2 a^2 (2A_1 - A_2 + A_3 - A_6) \,. \tag{104}$$

7. If $\phi_{n-1} = a$, and $\phi_n = \phi_{n+1} = 0$ then it follows that

$$2 = h^2 a^2 A_6 \,. \tag{105}$$

8. If $\phi_n = a$, and $\phi_{n-1} = \phi_{n+1} = 0$ then it follows that

$$\Lambda - 2 = h^2 a^2 A_1 \,. \tag{106}$$

We now discuss the various exact solutions.
(i) **Solution with period 2**: $\phi = (..., a, -a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 4 = h^2 a^2 [A_1 - A_2 + A_3 + A_4 - A_5 - A_6] \,. \tag{107}$$

In case $\Lambda = 4$, then one has a solution with a being an arbitrary real number provided $h^2(A_1 + A_3 + A_4) = h^2(A_2 + A_5 + A_6) = 2$.
(ii) **second solution with period 2**: $\phi = (..., a, 0, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 2 = h^2 a^2 A_1 \,, \quad 2 = (A_5 + A_6) h^2 a^2 \,. \tag{108}$$

One can in fact generalize this solution and show that even
$\phi = (..., a, 0, \ (a, 0 \ p \text{ times}), \ -a, 0...)$
is an exact solution with period $2p + 2$ $(p \geq 2)$ provided Eq. (108) is satisfied.
One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $2p+2$) and with "$-a, 0$" added randomly after any "0" is an exact solution to Eq. (1) provided Eq. (108) is satisfied.
(iii) **third solution with period 2**: $\phi = (..., a, \frac{1}{a}, a, \frac{1}{a}, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$A_1 = A_5 + A_6 = 0 \,, \quad h^2 A_2 = \Lambda - 2 \,, \quad h^2(A_4 + A_3) = 2 \,, \tag{109}$$

and in this case a is an arbitrary real number.
(iv) **Solution with Period 3**: $\phi = (..., a, -a, 0, ...)$
It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 h^2 [2A_1 + A_3 - A_2 - A_6] = 2(\Lambda - 3) \,. \tag{110}$$

In case $\Lambda = 3$, then one has a solution with a being arbitrary real number provided $2A_1 + A_3 = A_2 + A_6$.
(v) **second solution with Period 3**: $\phi = (..., a, a, -a, ...)$
It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 h^2 (A_1 + A_3 - A_4) = \Lambda - 2 \,, \quad a^2 h^2 (A_1 + A_3 + A_4 - A_2 - A_5 - A_6) = \Lambda - 4 \,. \tag{111}$$

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, \ (a, -a \ p \text{ times}), \ -a, ...)$
is an exact solution with period $2p + 1$ $(p \geq 1)$ provided Eq. (111) is satisfied.
One can also show that an *aperiodic solution* with any number of "a" and "$-a$" kept at random but with the constraint that at most two "a" or two "$-a$" are always together, is an exact solution to Eq. (1) provided Eq. (111) is satisfied.
(vi) **third solution with Period 3**: $\phi = (..., a, a, 0, ...)$
It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 h^2 (A_2 + 2A_1 + A_3 + A_6) = 2(\Lambda - 1) \,, \quad h^2 a^2 (A_5 + A_6) = 2 \,. \tag{112}$$

One can in fact generalize this solution and show that even
$\phi = (..., a, a, 0, -a, -a, 0, (a, a, 0, -a, -a, 0 \ p \text{ times}), \ a, a, 0, ...)$
is an exact solution with period $6p + 3$ $(p \geq 1)$ provided Eq. (112) is satisfied.

One can also show that an *aperiodic solution* can be constructed from the above periodic solution with period $6p+3$ with "$a,a,0$" or "$-a,-a,0$" added at random between "0" and "$-a$" or "0" and "a".

(vii) **Solution with Period 4**: $\phi = (...,a,a,a,0,...)$

It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 = 1\,, \quad h^2(A_2+2A_1+A_3+A_6) = 2(\Lambda-1)\,, \quad h^2(A_5+A_6) = 2\,. \tag{113}$$

From here it follows that such a solution is valid provided $A_2+A_3+2A_4+A_5 = 0$.

One can in fact generalize this solution and show that even

$\phi = (...,a,a,0,\ (a,a,0\ p\ \text{times}),\ a,...)$

is an exact solution with period $3p+1$ ($p \ge 1$) provided Eq. (113) is satisfied.

One can also show that an *aperiodic solution* with any number of "a" and "0" but with the constraint that at least two "a" are always together and no two "0" are either nearest or next-to-nearest neighbours is also an exact solution provided Eq. (113) is satisfied.

(viii) **second Solution with Period 4**: $\phi = (...,a,a,a,-a,...)$

It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 = 1\,, \quad h^2(A_1+A_3-A_4) = \Lambda-2\,, \quad h^2(A_1+A_3+A_4-A_2-A_5-A_6) = \Lambda-4\,. \tag{114}$$

In view of the constraint (2) this implies that such a solution is valid only if $A_4 = 0$.

One can in fact generalize this solution and show that even

$\phi = (...,a,a,a,\ (a\ p\ \text{times}),\ -a,...)$

is an exact solution with period $p+1$ ($p \ge 3$) provided Eq. (114) is satisfied.

One can also show that an *aperiodic solution* with any number of "a" and "$-a$" but with the constraint that at least once three or more "a" or "$-a$" are together.

(ix) **third solution with Period 4**: $\phi = (...,a,b,-a,-b,...)$, where $a^2 \ne b^2$.

It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$(a^2+b^2)h^2A_1 = \Lambda-2\,, \quad A_3 = A_1+A_4\,, \quad a^2 \ne b^2\,, \quad A_1 \ne 0\,. \tag{115}$$

Thus one has a one parameter family of solutions. It is easily seen that such a solution will exist in model 2 (in case δ is arbitrary but nonzero, while γ is arbitrary), model 8 (in case $\gamma = 0, \alpha = 1/4$) and Hamiltonian model 10 (in case $\alpha_1 = \alpha_3$).

In the special case of $A_3 = A_4$, $A_1 = 0$ and $\Lambda = 2$, one, in fact, has a two parameter family of solutions in the sense that now both a and b are arbitrary real numbers.

(x) **fourth solution with Period 4**: $\phi = (...,a,a,-a,-a,...)$

It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2h^2[A_1+A_3-A_4] = \Lambda-2\,. \tag{116}$$

In case $\Lambda = 2$, then one has a solution with a being an arbitrary real number provided $A_1+A_3 = A_4$.

(xi) **fifth solution with Period 4**: $\phi = (...,a,0,-a,0,...)$

It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2h^2A_1 = \Lambda-2\,. \tag{117}$$

In case $\Lambda = 2$, then one has a solution with a being an arbitrary real number provided $A_1 = 0$.

(xii) **sixth solution with Period 4**: $\phi = (...,a,0,-a,a,...)$

It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2h^2[A_1+A_3-A_4] = \Lambda-2\,, \quad a^2h^2[2A_1+A_3+A_2+A_6] = 2(\Lambda-1)\,, \quad a^2h^2[2A_1+A_3-A_2-A_6] = 2(\Lambda-3)\,. \tag{118}$$

From here it follows that such a solution is valid provided $A_3 = 2A_4$.

One can in fact generalize this solution and show that even

$\phi = (...,a,0,-a,\ (a,0,-a\ p\ \text{times}),\ a,...)$

is an exact solution with period $3p+1$ ($p \ge 1$) provided Eq. (118) is satisfied.

One can also show that an *aperiodic solution* with any number of "a" and "$-a$" but with the constraint that at most two "a" or two "$-a$" are always together and further "0" are added at random between "$-a$" and "a" or between "a" and "$-a$" but with the proviso that no two "0" are ever nearest or next-to-nearest neighbours, is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (118) is satisfied.

(xiii) **seventh solution with Period 4**: $\phi = (...,a,-a,0,a,...)$

It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2h^2[A_1 + A_3 + A_4 - A_2 - A_5 - A_6] = \Lambda - 4\,, \quad a^2h^2[2A_1 + A_3 - A_2 - A_6] = 2(\Lambda - 3)\,, \quad a^2h^2[A_6 + A_5] = 2\,. \tag{119}$$

From here it follows that such a solution is valid provided $A_2 + A_5 = A_3 + 2A_4$.

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, 0,\ (a, -a, 0,\ p \text{ times}),\ \ -a, ...)$
is an exact solution with period $3p + 1$ ($p \geq 1$) provided Eq. (119) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p + 1$) by adding at random "$a, -a$" between any 0 and "a" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (118) is satisfied.

(xiv) **Solution with Period 5**: $\phi = (..., a, a, a, -a, -a, ...)$

It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 = 1\,, \quad h^2[A_1 + A_3 - A_4] = \Lambda - 2\,. \tag{120}$$

One can in fact generalize this solution and show that even
$\phi = (..., a, a, a,\ (a\ p \text{ times}),\ \ -a, -a, (-a\ q \text{ times}), ...)$
is an exact solution with period $p + q$ ($p \geq 3, q \geq 2$ *or* $p \geq 2, q \geq 3$) provided Eq. (120) is satisfied.

One can also show that an *aperiodic solution* constructed from any numbers of "a" and "$-a$" with the constraint that two or more of "a" as well as two or more of "$-a$" are always together is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (120) is satisfied.

(xv) **second solution with period 5**: $\phi = (..., 0, a, 0, a, a, ...)$

It is easily shown that this is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$h^2a^2[2A_1 + A_3 + A_2 + A_6] = 2(\Lambda - 1)\,, \quad h^2a^2(A_6 + A_5) = 2\,, \quad h^2a^2A_1 = \Lambda - 2\,. \tag{121}$$

From here it follows that such a solution is valid provided $A_2 + A_3 = A_5$.

One can in fact generalize this solution and show that even
$\phi = (...0, a,\ (0, a\ p \text{ times}),\ a, ...)$
is an exact solution with period $2p + 1$ ($p \geq 2$) provided Eq. (121) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $2p + 1$) by adding at random "$-a, -a, 0$" after any "0" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (121) is satisfied.

(xvi) **third solution with period 5**: $\phi = (..., a, -a, a, -a, 0, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 4 = h^2a^2(A_4 + A_1 + A_3 - A_2 - A_5 - A_6)\,, \quad 2(\Lambda - 3) = h^2a^2(2A_1 + A_3 - A_2 - A_6)\,. \tag{122}$$

One can in fact generalize this solution and show that even
$\phi = (...a, -a,\ (a, -a\ p \text{ times}),\ 0, ...)$
is an exact solution with period $2p + 1$ ($p \geq 2$) provided Eq. (122) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $2p + 1$) by adding at random "0" between "$-a$" and "a" or "a" and "$-a$", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (122) is satisfied.

(xvii) **fourth solution with period 5**: $\phi = (..., a, 0, a, 0, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 2 = h^2a^2A_1\,, \quad 2 = h^2a^2(A_5 + A_6)\,, \quad 2(\Lambda - 3) = h^2a^2(2A_1 + A_3 - A_2 - A_6)\,. \tag{123}$$

From here it follows that such a solution is valid provided $A_2 = A_3 + A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, 0,\ (a, 0\ p \text{ times}),\ \ -a, ...)$
is an exact solution with period $2p + 1$ ($p \geq 2$) provided Eq. (123) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $2p + 1$) by adding randomly "$-a$" between "0" and "a" with a constraint that no two "$-a$" can be either nearest or next-to-nearest neighbors, is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (123) is satisfied.

(xviii) **fifth solution with period 5**: $\phi = (..., a, a, -a, -a, 0, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 2 = h^2a^2(A_1 + A_3 - A_4)\,, \quad 2(\Lambda - 1) = h^2a^2(2A_1 + A_3 + A_2 + A_6)\,. \tag{124}$$

One can in fact generalize this solution and show that even
$\phi = (..., a, a, -a, -a,\ (a, a, -a, -a\ p \text{ times}),\ 0, ...)$
is an exact solution with period $4p+1$ ($p \geq 1$) provided Eq. (124) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $4p+1$) by randomly adding "0" between "$-a$" and "a" or "a" and "$-a$" with a constraint that no two "0" can be either nearest or next-to-nearest neighbors, is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (124) is satisfied.

(xix) **sixth solution with period 5**: $\phi = (..., a, a, 0, -a, 0, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 2 = h^2 a^2 A_1\,, \quad 2(\Lambda - 1) = h^2 a^2 (2A_1 + A_3 + A_2 + A_6)\,. \tag{125}$$

One can in fact generalize this solution and show that even
$\phi = (..., a, a, -a, -a,\ (a, a, -a, -a\ p \text{ times}),\ 0, ...)$
is an exact solution with period $4p+1$ ($p \geq 1$) provided Eq. (125) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $4p+1$) by randomly adding "0" between "$-a$" and "a" or "a" and "$-a$" with a constraint that no two "0" can be either nearest or next-to-nearest neighbors, is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (125) is satisfied.

(xx) **seventh solution with period 5**: $\phi = (..., a, -a, 0, -a, a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 2 = h^2 a^2 (A_1 + A_3 - A_4)\,, \quad 2 = h^2 a^2 (A_5 + A_6)\,, \quad 2(\Lambda - 3) = h^2 a^2 (2A_1 + A_3 - A_2 - A_6)\,. \tag{126}$$

From here it follows that such a solution is valid provided $A_2 + A_3 = 2A_4 + A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, 0,\ (a, -a, 0\ p \text{ times}),\ -a, a, ...)$
is an exact solution with period $3p+2$ ($p \geq 1$) provided Eq. (126) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+2$) by randomly adding "$-a, a$" between "0" and "a" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (126) is satisfied.

(xxi) **eighth solution with period 5**: $\phi = (..., a, 0, -a, a, a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 = 1\,, \quad \Lambda - 2 = h^2 (A_1 + A_3 - A_4)\,, \quad 2(\Lambda - 3) = h^2 (2A_1 + A_3 - A_2 - A_6)\,, \quad 2(\Lambda - 1) = h^2 (2A_1 + A_3 + A_2 + A_6)\,. \tag{127}$$

In view of the constraint (2), such a solution is thus valid only if $A_3 = 2A_4 = -A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, 0, -a,\ (a, 0, -a\ p \text{ times}),\ a, a, ...)$
is an exact solution with period $3p+2$ ($p \geq 1$) provided Eq. (127) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+2$) by randomly adding "a, a" between "$-a$" and "a" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (127) is satisfied.

(xxii) **ninth solution with period 5**: $\phi = (..., a, a, 0, a, -a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\begin{aligned} &2 = h^2 a^2 (A_5 + A_6)\,, \quad \Lambda - 2 = h^2 a^2 (A_1 + A_3 - A_4)\,, \quad 2(\Lambda - 3) = h^2 a^2 (2A_1 + A_3 - A_2 - A_6)\,, \\ &2(\Lambda - 1) = h^2 a^2 (2A_1 + A_3 + A_2 + A_6)\,, \quad \Lambda - 4 = h^2 a^2 (A_1 + A_3 + A_4 - A_5 - A_6 - A_2)\,. \end{aligned} \tag{128}$$

From here it follows that such a solution is valid provided $A_2 = A_3 = 2A_4 = A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, a, 0,\ (a, a, 0\ p \text{ times}),\ a, -a, ...)$
is an exact solution with period $3p+2$ ($p \geq 1$) provided Eq. (128) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+2$) by randomly adding "$-a$" between two "a" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (128) is satisfied.

(xxiii) **Solution with period 6**: $\phi = (..., a, a, 0, -a, -a, 0, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$2(\Lambda - 1) = h^2 a^2 (2A_1 + A_3 + A_2 + A_6)\,. \tag{129}$$

In case $\Lambda = 1$, then one has a solution with a being an arbitrary real number provided $2A_1 + A_3 + A_2 + A_6 = 0$.

(xxiv) **second solution with period 6**: $\phi = (..., a, -a, 0, -a, a, 0, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$2(\Lambda - 3) = h^2 a^2 (2A_1 + A_3 - A_2 - A_6)\,, \quad 2 = h^2 a^2 (A_5 + A_6)\,. \tag{130}$$

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, 0,\ (a, -a, 0\ p \text{ times}),\ -a, a, 0, ...)$
is an exact solution with period $3p + 3$ ($p \geq 1$) provided Eq. (130) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+3$) by randomly adding "$-a, a, 0$" between "0" and "a" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (130) is satisfied.

(xxv) **third solution with period 6**: $\phi = (..., a, a, a, 0, -a, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 = 1\,, \quad 2(\Lambda - 1) = h^2 (2A_1 + A_3 + A_2 + A_6)\,, \quad \Lambda - 2 = h^2 (A_1 + A_3 - A_4)\,. \tag{131}$$

In view of the constraint (2) this implies that such a solution is valid only if $A_3 = -A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, a, a,\ (a\ p \text{ times}),\ 0, -a, -a, ...)$
is an exact solution with period $p + 3$ ($p \geq 3$) provided Eq. (131) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $p + 3$) by randomly adding "$0, -a, -a$" between any two "a" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (131) is satisfied.

(xxvi) **fourth solution with period 6**: $\phi = (..., a, a, 0, -a, 0, a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 = 1\,, \quad 2(\Lambda - 1) = h^2 (2A_1 + A_3 + A_2 + A_6)\,, \quad \Lambda - 2 = h^2 A_1\,. \tag{132}$$

In view of the constraint (2) this implies that such a solution is valid only if $A_4 = -A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, a, 0, -a, 0,\ (a, a, 0, -a, 0\ p \text{ times}),\ a, ...)$
is an exact solution with period $5p + 1$ ($p \geq 1$) provided Eq. (132) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $5p + 1$) by randomly adding as many "a" as one wants between any two "a" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (132) is satisfied.

(xxvii) **fifth solution with period 6**: $\phi = (..., a, -a, a, a, -a, 0, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$2(\Lambda - 3) = h^2 a^2 (2A_1 + A_3 - A_2 - A_6)\,, \quad \Lambda - 2 = h^2 a^2 (A_1 + A_3 - A_4)\,, \quad \Lambda - 4 = h^2 a^2 (A_1 + A_3 + A_4 - A_2 - A_5 - A_6)\,. \tag{133}$$

From here it follows that such a solution is valid provided $A_3 = A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, a, a, -a,\ (a, -a, a, a, -a\ p \text{ times}),\ 0, ...)$
is an exact solution with period $5p + 1$ ($p \geq 1$) provided Eq. (133) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $5p+1$) by randomly adding "$-a$" between any two "a" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (133) is satisfied.

(xxviii) **sixth solution with period 6**: $\phi = (..., a, a, 0, a, -a, 0, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$2(\Lambda - 1) = h^2 a^2 (2A_1 + A_3 + A_2 + A_6)\,, \quad 2(\Lambda - 3) = h^2 a^2 (2A_1 + A_3 - A_2 - A_6)\,, \quad 2 = h^2 a^2 (A_5 + A_6)\,. \tag{134}$$

From here it follows that such a solution is valid provided $A_2 = A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, a, 0,\ (a, a, 0\ p \text{ times}),\ a, -a, 0, ...)$
is an exact solution with period $3p + 3$ ($p \geq 1$) provided Eq. (134) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+3$) by randomly adding "$a, -a, 0$" between "0" and "a" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (134) is satisfied.

(xxix) **seventh solution with period 6**: $\phi = (..., a, a, -a, 0, -a, a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 = 1\,, \quad (\Lambda - 2) = h^2(A_1 + A_3 - A_4)\,, \quad 2(\Lambda - 3) = h^2(2A_1 + A_3 - A_2 - A_6)\,, \quad 2 = h^2(A_5 + A_6)\,. \tag{135}$$

In view of the constraint (2) this implies that such a solution is valid only if $A_2 = -2A_4, A_5 = 2A_2 + A_3$.
One can in fact generalize this solution and show that even
$\phi = (..., a, a, -a, 0, -a,\ (a, a, -a, 0, -a\ p \text{ times}),\ a, ...)$
is an exact solution with period $5p + 1$ ($p \geq 1$) provided Eq. (135) is satisfied.
One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $5p+1$) by randomly adding as many "a" as one wants between two "a", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (135) is satisfied.
(xxx) **eighth solution with period 6**: $\phi = (..., a, 0, a, 0, a, -a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\begin{aligned}&\Lambda - 2 = h^2a^2A_1\,, \quad (\Lambda - 4) = h^2a^2(A_1 + A_3 + A_4 - A_5 - A_6 - A_2)\,, \\ &2(\Lambda - 3) = h^2a^2(2A_1 + A_3 - A_2 - A_6)\,, \quad 2 = h^2a^2(A_5 + A_6)\,.\end{aligned} \tag{136}$$

From here it follows that such a solution is valid provided $A_4 = A_5\,, A_2 = A_3 + A_4$.
One can in fact generalize this solution and show that even
$\phi = (..., a, 0, a, 0,\ (a, 0,\ p \text{ times}),\ a, -a, ...)$
is an exact solution with period $2p + 2$ ($p \geq 2$) provided Eq. (136) is satisfied.
One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $2p+2$) by randomly adding "$-a$" between "0" and "a", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (136) is satisfied.
(xxxi) **ninth solution with period 6**: $\phi = (..., a, 0, a, 0, a, a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 2 = h^2A_1\,, \quad (\Lambda - 2) = h^2(A_1 + A_3 - A_4)\,, \ a^2 = 1\,, \quad 2 = h^2(A_5 + A_6)\,. \tag{137}$$

In view of the constraint (2) this implies that such a solution is valid only if $A_2 = -2A_3 = -2A_4$.
One can in fact generalize this solution and show that even
$\phi = (..., a, 0, a, 0,\ (a, 0,\ p \text{ times}),\ a, a, ...)$
is an exact solution with period $2p + 2$ ($p \geq 2$) provided Eq. (137) is satisfied. From here it follows that such a solution is valid provided $A_2 = -2A_3 = -2A_4$.
One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $2p+2$) by randomly adding "a, a" between "0" and "a" or "a" and "0", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (137) is satisfied.
(xxxii) **tenth solution with period 6**: $\phi = (..., a, -a, a, a, 0, a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\begin{aligned}&\Lambda - 4 = h^2a^2(A_3 + A_4 + A_1 - A_2 - A_5 - A_6)\,, \quad (\Lambda - 2) = h^2a^2(A_1 + A_3 - A_4)\,, \\ &2(\Lambda - 1) = h^2a^2(2A_1 + A_3 + A_2 + A_6)\,, \quad 2 = h^2a^2(A_5 + A_6)\,.\end{aligned} \tag{138}$$

From here it follows that such a solution is valid provided $A_3 + A_5 = 2A_2 = 4A_4$.
One can in fact generalize this solution and show that even
$\phi = (..., a, -a, a,\ (a, -a, a,\ p \text{ times}),\ a, 0, a, ...)$
is an exact solution with period $3p + 3$ ($p \geq 1$) provided Eq. (138) is satisfied.
One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+3$) by randomly adding "$-a$" between two "a", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (138) is satisfied.
(xxxiii) **eleventh solution with period 6**: $\phi = (..., a, 0, a, 0, -a, -a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\begin{aligned}&2(\Lambda - 3) = h^2a^2(2A_1 + A_3 + A_2 + A_6)\,, \quad (\Lambda - 2) = h^2a^2(A_1 + A_3 - A_4)\,, \\ &2(\Lambda - 1) = h^2a^2(2A_1 + A_3 + A_2 + A_6)\,, \quad 2 = h^2a^2(A_5 + A_6)\,, \quad \Lambda - 2 = h^2a^2A_1\,.\end{aligned} \tag{139}$$

From here it follows that such a solution is valid provided $A_2 = A_5\,, = A_3 = A_4 = 0$.
One can in fact generalize this solution and show that even

$\phi = (..., a, 0,\ (a, 0,\ p \text{ times}),\ -a, -a, ...)$
is an exact solution with period $2p+2$ ($p \geq 2$) provided Eq. (139) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $2p+2$) by randomly adding "$-a$" between "0" and "a" or "a" and "0", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (139) is satisfied.

(xxxiv) **Solution with period 7**: $\phi = (..., a, a, 0, -a, -a, 0, a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$2(\Lambda - 1) = h^2(2A_1 + A_3 + A_2 + A_6)\,, \quad a^2 = 1\,. \tag{140}$$

One can in fact generalize this solution and show that even
$\phi = (..., a, a, 0, -a, -a, 0,\ (a, a, 0, -a, -a, 0,\ p \text{ times}),\ a, ...)$
is an exact solution with period $6p+1$ ($p \geq 1$) provided Eq. (140) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $6p+1$) by randomly adding as many number of "a" ("$-a$") as one wants between two "a" ("$-a$"), is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (140) is satisfied.

(xxxv) **second solution with period 7**: $\phi = (..., a, 0, -a, a, 0, -a, 0, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$2(\Lambda - 3) = h^2 a^2(2A_1 + A_3 - A_2 - A_6)\,, \quad \Lambda - 2 = h^2 a^2 A_1\,. \tag{141}$$

One can in fact generalize this solution and show that even
$\phi = (..., a, 0, -a, a, o, -a,\ (a, 0, -a,\ p \text{ times}),\ 0, ...)$
is an exact solution with period $3p+1$ ($p \geq 2$) provided Eq. (141) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+1$) by randomly adding "0" between "$-a$" and "a" or "a" and "$-a$", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (141) is satisfied.

(xxxvi) **third solution with period 7**: $\phi = (..., a, a, 0, -a, -a, a, -a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$2(\Lambda - 1) = h^2 a^2(2A_1 + A_3 + A_2 + A_6)\,, \quad \Lambda - 2 = h^2 a^2 A_1\,, \quad \Lambda - 4 = h^2 a^2(A_3 + A_4 + A_1 - A_2 - A_5 - A_6)\,. \tag{142}$$

From here it follows that such a solution is valid provided $2A_3 + A_4 = A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, a, 0, -a, -a\ (a, a, 0, -a, -a,\ p \text{ times}),\ a, -a, ...)$
is an exact solution with period $5p+2$ ($p \geq 1$) provided Eq. (142) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $5p+2$) by randomly adding "$-a, a$" between "0" and "$-a$" or "a" and "0", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (142) is satisfied.

(xxxvii) **fourth solution with period 7**: $\phi = (..., a, -a, 0, a, a, 0, -a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$2(\Lambda-1) = h^2 a^2(2A_1+A_3+A_2+A_6)\,, \ 2(\Lambda-3) = h^2 a^2(2A_1+A_3-A_2-A_6)\,, \ \Lambda-4 = h^2 a^2(A_3+A_4+A_1-A_2-A_5-A_6)\,. \tag{143}$$

From here it follows that such a solution is valid provided $A_3 + 2A_4 = 2A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, 0,\ (a, -a, 0,\ p \text{ times}),\ a, a, 0, -a, ...)$
is an exact solution with period $3p+4$ ($p \geq 1$) provided Eq. (143) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+4$) by randomly adding "$a, -a$" between "0" and "a" or "$-a$" and "0", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (143) is satisfied.

(xxxviii) **fifth solution with period 7**: $\phi = (..., a, a, 0, a, a, 0, -a, ...)$
This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\begin{aligned} &2(\Lambda - 1) = h^2 a^2(2A_1 + A_3 + A_2 + A_6)\,, \quad 2(\Lambda - 3) = h^2 a^2(2A_1 + A_3 - A_2 - A_6)\,, \\ &\Lambda - 2 = h^2 a^2(A_3 + A_1 - A_4)\,, \quad 2 = h^2 a^2(A_5 + A_6)\,. \end{aligned} \tag{144}$$

From here it follows that such a solution is valid provided $A_3 = 2A_4\,, A_2 = A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, a, 0,\ (a, a, 0,\ p \text{ times}),\ -a, ...)$

is an exact solution with period $3p+1$ ($p \geq 2$) provided Eq. (144) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+1$) by randomly adding "$-a$" between "0" and "a" or "a" and "0", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (144) is satisfied.

(xxxix) **sixth solution with period 7**: $\phi = (..., a, a, -a, a, a, -a, 0, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$2(\Lambda - 1) = h^2 a^2 (2A_1 + A_3 + A_2 + A_6)\,, \quad 2(\Lambda - 3) = h^2 a^2 (2A_1 + A_3 - A_2 - A_6)\,,$$
$$\Lambda - 2 = h^2 a^2 (A_3 + A_1 - A_4)\,, \quad \Lambda - 4 = h^2 a^2 (A_3 + A_4 + A_1 - A_2 - A_5 - A_6)\,. \tag{145}$$

From here it follows that such a solution is valid provided $A_3 = 2A_4 = A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, a, -a,\ (a, a, -a,\ p \text{ times}),\ 0, ...)$
is an exact solution with period $3p+1$ ($p \geq 2$) provided Eq. (145) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+1$) by randomly adding "0" between "$-a$" and "a" or "a" and "$-a$" such that no two "0" are ever nearest or next-to-nearest neighbors, is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (145) is satisfied.

(xxxx) **seventh solution with period 7**: $\phi = (..., a, 0, -a, a, a, a, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 = 1\,, \quad 2(\Lambda - 3) = h^2 (2A_1 + A_3 - A_2 - A_6)\,,$$
$$\Lambda - 2 = h^2 (A_3 + A_1 - A_4)\,, \quad \Lambda - 4 = h^2 (A_3 + A_4 + A_1 - A_2 - A_5 - A_6)\,. \tag{146}$$

In view of the constraint (2), such a solution is valid only if $A_4 = 0, A_3 = A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, 0, -a,\ (a, 0, -a,\ p \text{ times}),\ a, a, a, -a, ...)$
is an exact solution with period $3p+4$ ($p \geq 1$) provided Eq. (146) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p+4$) by randomly adding "$-a, a$" between "a" and "0" or "0" and "$-a$", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (146) is satisfied.

(xxxxi) **eighth solution with period 7**: $\phi = (..., a, a, 0, a, 0, -a, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 2 = h^2 a^2 A_1\,, \quad 2(\Lambda - 1) = h^2 a^2 (2A_1 + A_3 + A_2 + A_6)\,,$$
$$\Lambda - 2 = h^2 a^2 (A_3 + A_1 - A_4)\,, \quad 2 = h^2 a^2 (A_5 + A_6)\,. \tag{147}$$

From here it follows that such a solution is valid provided $A_3 = A_4\,, A_5 = A_2 + A_3$.

One can in fact generalize this solution and show that even
$\phi = (..., a, a, 0, a, 0,\ (a, a, 0, a, 0,\ p \text{ times}),\ \ -a, -a, ...)$
is an exact solution with period $5p+2$ ($p \geq 1$) provided Eq. (147) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $5p+2$) by randomly adding "0" between any two "a", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (147) is satisfied.

(xxxxii) **ninth solution with period 7**: $\phi = (..., a, a, -a, 0, -a, 0, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 2 = h^2 A_1\,, \quad 2(\Lambda - 3) = h^2 (2A_1 + A_3 - A_2 - A_6)\,,$$
$$\Lambda - 2 = h^2 (A_3 + A_1 - A_4)\,, \quad 2 = h^2 (A_5 + A_6)\,. \tag{148}$$

In view of the constraint (2), such a solution is valid only if $A_2 = -2A_3 = -2A_4\,, A_5 = -3A_3$.

One can in fact generalize this solution and show that even
$\phi = (..., a, a, -a, 0, -a,\ (a, a, -a, 0, -a,\ p \text{ times}),\ 0, -a, ...)$
is an exact solution with period $5p+2$ ($p \geq 1$) provided Eq. (148) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $5p+2$) by randomly adding "0" between any two "a", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (148) is satisfied.

(xxxxiii) **tenth solution with period 7**: $\phi = (..., a, -a, a, a, a, 0, a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 = 1\,, \quad \Lambda - 4 = h^2(A_1 + A_3 + A_4 - A_2 - A_5 - A_6)\,, \quad 2(\Lambda - 1) = h^2(2A_1 + A_3 + A_2 + A_6)\,,$$
$$\Lambda - 2 = h^2(A_3 + A_1 - A_4)\,, \quad 2 = h^2(A_5 + A_6)\,. \tag{149}$$

In view of the constraint (2), such a solution is valid only if $A_4 = A_2 = 0, A_3 = -A_5$.

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, a, a, a, 0,\ (a, -a, a, a, a, 0,\ p \text{ times}),\ a, ...)$
is an exact solution with period $6p + 1$ ($p \geq 1$) provided Eq. (149) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $5p + 2$) by randomly adding as many "a" as one wants between two "a", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (149) is satisfied.

(xxxxiv) **eleventh solution with period 7**: $\phi = (..., a, 0, a, 0, a, a, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 2 = h^2a^2A_1\,, \quad \Lambda - 4 = h^2a^2(A_1 + A_3 + A_4 - A_2 - A_5 - A_6)\,, \quad 2(\Lambda - 1) = h^2a^2(2A_1 + A_3 + A_2 + A_6)\,,$$
$$\Lambda - 2 = h^2a^2(A_3 + A_1 - A_4)\,, \quad 2(\Lambda - 3) = h^2a^2(2A_1 + A_3 - A_2 - A_6)\,, \quad 2 = h^2a^2(A_5 + A_6)\,. \tag{150}$$

From here it follows that such a solution is valid provided $A_2 = A_3 = A_4 = A_5 = 0$.

One can in fact generalize this solution and show that even
$\phi = (..., a, 0,\ (a, 0,\ p \text{ times}),\ a, a, -a, ...)$
is an exact solution with period $2p + 3$ ($p \geq 2$) provided Eq. (150) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $2p + 3$) by randomly adding "$-a$" between "a" and "0" or "0" and "a", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (150) is satisfied.

(xxxxv) **Solution with period 8**: $\phi = (..., a, -a, 0, a, a, a, 0, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 4 = h^2(A_1 + A_3 + A_4 - A_2 - A_5 - A_6)\,, \quad 2(\Lambda - 1) = h^2(2A_1 + A_3 + A_2 + A_6)\,,$$
$$a^2 = 1\,, \quad 2(\Lambda - 3) = h^2(2A_1 + A_3 - A_2 - A_6)\,. \tag{151}$$

In view of the constraint (2), such a solution is valid only if $A_5 = 0, A_3 = -2A_4$.

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, 0,\ (a, -a, 0,\ p \text{ times}),\ a, a, a, 0, -a, ...)$
is an exact solution with period $3p + 5$ ($p \geq 1$) provided Eq. (151) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $3p + 5$) by randomly adding "$a, -a$" between "$-a$" and "0" or "0" and "a", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (151) is satisfied.

(xxxxvi) **second solution with period 8**: $\phi = (..., a, 0, a, 0, a, a, a, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 4 = h^2(A_1 + A_3 + A_4 - A_2 - A_5 - A_6)\,, \quad \Lambda - 2 = h^2A_1\,, \quad 2(\Lambda - 1) = h^2(2A_1 + A_3 + A_2 + A_6)\,,$$
$$a^2 = 1\,, \quad 2 = h^2(A_5 + A_6)\,, \quad \Lambda - 2 = h^2(A_1 + A_3 - A_4)\,, \quad 2(\Lambda - 3) = h^2(2A_1 + A_3 - A_2 - A_6)\,. \tag{152}$$

In view of the constraint (2), such a solution is valid only if $A_2 = A_3 = A_4 = A_5 = 0$.

One can in fact generalize this solution and show that even
$\phi = (..., a, 0, a, 0, a, a, a,\ (a, 0, a, 0, a, a, a,\ p \text{ times}),\ \ -a, ...)$
is an exact solution with period $7p + 1$ ($p \geq 1$) provided Eq. (152) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $7p + 1$) by randomly adding any combination of "$-a$", "0" and "a" at any place, is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (152) is satisfied.

(xxxxvii) **Solution with period 9**: $\phi = (..., a, a, 0, -a, -a, 0, a, -a, 0, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$2(\Lambda - 1) = h^2a^2(2A_1 + A_3 + A_2 + A_6)\,, \quad 2(\Lambda - 3) = h^2a^2(2A_1 + A_3 - A_2 - A_6)\,. \tag{153}$$

One can in fact generalize this solution and show that even
$\phi = (..., a, a, 0, -a, -a, 0,\ (a, a, 0, -a, -a, 0,\ p \text{ times}),\ a, -a, 0, ...)$
is an exact solution with period $6p + 3$ ($p \geq 1$) provided Eq. (153) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $6p+3$) by randomly adding "$a,-a,0$" between "0" and "$-a$", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (153) is satisfied.

(xxxxviii) **second solution with period 9**: $\phi = (..., a, -a, 0, a, 0, a, a, 0, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 4 = h^2a^2(A_1 + A_3 + A_4 - A_2 - A_5 - A_6)\,, \quad \Lambda - 2 = a^2h^2A_1\,, \quad 2(\Lambda - 1) = h^2a^2(2A_1 + A_3 + A_2 + A_6)\,,$$
$$2 = h^2a^2(A_5 + A_6)\,, \quad 2(\Lambda - 3) = h^2a^2(2A_1 + A_3 - A_2 - A_6)\,. \tag{154}$$

From here it follows that such a solution is valid provided $A_2 = A_4 = A_5\,, A_3 = 0$.

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, 0, a, 0, a, a, 0,\ (a, -a, 0, a, 0, a, a, 0,\ p \text{ times}),\ -a, ...)$
is an exact solution with period $8p+1$ ($p \geq 1$) provided Eq. (154) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $8p+1$) by randomly adding "0" between "a" and "$-a$", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (154) is satisfied.

(xxxxix) **Solution with period 10**: $\phi = (..., a, a, 0, -a, -a, 0, a, -a, 0, a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$a^2 = 1\,, \quad 2(\Lambda - 1) = h^2(2A_1 + A_3 + A_2 + A_6)\,, \quad 2(\Lambda - 3) = h^2(2A_1 + A_3 - A_2 - A_6)\,. \tag{155}$$

In view of the constraint (2), such a solution is valid only if $A_3 + 2A_4 + 2A_5 = 0$.

One can in fact generalize this solution and show that even
$\phi = (..., a, a, 0, -a, -a, 0, a, -a, 0,\ (a, a, 0, -a, -a, 0, a, -a, 0,\ p \text{ times}),\ a, ...)$
is an exact solution with period $9p+1$ ($p \geq 1$) provided Eq. (155) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $9p+1$) by randomly adding as many "a" as one wants between two "a" and by randomly adding as many "$-a$" between two "$-a$" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (155) is satisfied.

(xxxxx) **second solution with period 10**: $\phi = (..., a, -a, 0, a, 0, a, a, a, 0, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 4 = h^2(A_1 + A_3 + A_4 - A_2 - A_5 - A_6)\,, \quad \Lambda - 2 = h^2A_1\,, \quad 2(\Lambda - 1) = h^2(2A_1 + A_3 + A_2 + A_6)\,,$$
$$2 = h^2(A_5 + A_6)\,, \quad a^2 = 1\,, \quad 2(\Lambda - 3) = h^2(2A_1 + A_3 - A_2 - A_6)\,. \tag{156}$$

In view of the constraint (2), such a solution is valid only if $A_2 = A_3 = A_4 = A_5 = 0$.

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, 0, a, 0, a, a, a, 0,\ (a, -a, 0, a, 0, a, a, a, 0,\ p \text{ times}),\ -a, ...)$
is an exact solution with period $9p+1$ ($p \geq 1$) provided Eq. (156) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $9p+1$) by randomly adding as many "a" as one wants between two "a" and by randomly adding "0" between "$-a$" and "a" or "a" and "$-a$" is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (156) is satisfied.

(xxxxxi) **Solution with period 11**: $\phi = (..., a, -a, 0, a, a, a, 0, -a, a, a, -a, ...)$

This is an exact solution to Eq. (1) satisfying the constraint (2) provided

$$\Lambda - 4 = h^2(A_1 + A_3 + A_4 - A_2 - A_5 - A_6)\,, \quad 2(\Lambda - 1) = h^2(2A_1 + A_3 + A_2 + A_6)\,,$$
$$a^2 = 1\,, \quad 2(\Lambda - 3) = h^2(2A_1 + A_3 - A_2 - A_6)\,, \quad \Lambda - 2 = h^2(A_1 + A_3 - A_4)\,. \tag{157}$$

In view of the constraint (2), such a solution is valid only if $A_3 = A_4 = A_5 = 0$.

One can in fact generalize this solution and show that even
$\phi = (..., a, -a, 0, a, a, a, 0, -a, a, a,\ (a, -a, 0, a, a, a, 0, -a, a, a,\ p \text{ times}),\ -a, ...)$
is an exact solution with period $10p+1$ ($p \geq 1$) provided Eq. (157) is satisfied.

One can also show that an *aperiodic solution* constructed from the above periodic solution (with period $10p+1$) by randomly adding as many "a" as one wants between two "a", is an exact solution to Eq. (1) satisfying the constraint (2) provided Eq. (157) is satisfied.

V. TWO-POINT MAPS

A. General maps that include integration constant

As pointed out in Sec. I, while Model 2, for any value of δ and γ, has the first integral with the integration constant C expressed by Eq. (6), for the Speight and Ward Model 7, the known two point map Eq. (11) does not contain the integration constant C.

Remarkably, in case only A_2 and A_4 are nonzero (combination of Models 3 and 6, i.e. an admixture of Bender-Tovbis and BOP nonlinearities), the discrete model Eq. (1) has the following first integral with the integration constant C

$$\phi_{n+1} = (2-\Lambda)\frac{Z\phi_n \pm \sqrt{f(\phi_n)}}{2-\Lambda-Y\phi_n^2}, \qquad f(\phi_n) = \frac{Y}{2-\Lambda}(C - X\phi_n^2 + \phi_n^4), \tag{158}$$

where

$$Z = \frac{(2-\Lambda)^2 - Ch^4A_4^2}{2\,(2-\Lambda) + Ch^4A_2A_4}, \quad X = \frac{CY^2 + (2-\Lambda)^2\left(1-Z^2\right)}{(2-\Lambda)\,Y}, \quad Y = h^2(A_4 + A_2 Z). \tag{159}$$

As expected, in the limit of either $A_2 = 0$ or $A_4 = 0$, the nonlinear map reduces to the maps derived earlier in [12, 14]. Thus the nonlinear map defined by Eqs. (158) and (159) generalizes the maps considered in [12, 14] and it allows one to construct static solutions to Eq. (1) iteratively in a way similar to those studies.

Remarkably, in case $\Lambda \neq 2$, this first integral can be further factorized as

$$W(\phi_n, \phi_{n+1})(2-\Lambda)(2-\Lambda-Y\phi_n^2) = 0\,, \tag{160}$$

where

$$W(\phi_n, \phi_{n+1}) = \phi_n^2 + \phi_{n+1}^2 - \frac{Y}{2-\Lambda}\phi_n^2\phi_{n+1}^2 - 2Z\phi_n\phi_{n+1} - \frac{CY}{2-\Lambda}\,. \tag{161}$$

Since the vanishing of the third bracket in Eq. (160) is a trivial possibility, for $\Lambda \neq 2$, effectively the first integral in this case acquires rather simple form, $W(\phi_n, \phi_{n+1}) = 0$, which is precisely of the Quispel form as given in [22]. As expected, in the special case when $A_4 = 0$ and hence $h^2 A_2 = \Lambda$, the first integral $W(\phi_n, \phi_{n+1}) = 0$ agrees with that obtained from the map Eq. (6) in case $\gamma = \delta = 0$. We also note that Eq. (1) with only A_2 and A_4 nonzero can be expressed in terms of Eq. (161) as follows

$$\ddot{\phi}_n = \frac{2-\Lambda}{2Z(\phi_{n+1} - \phi_{n-1})}\left\{W(\phi_n, \phi_{n+1}) - W(\phi_{n-1}, \phi_n) + \frac{h^2A_4}{2-\Lambda}\left[\phi_{n+1}^2 W(\phi_{n-1}, \phi_n) - \phi_{n-1}^2 W(\phi_n, \phi_{n+1})\right]\right\}. \tag{162}$$

It is now clear that the static solutions to Eq. (1) can be found from the two-point map $W(\phi_n, \phi_{n+1}) = 0$. Note that for small h one has $Z \approx (2-\Lambda)/2$ and the last term in the curly bracket of Eq. (162) can be neglected and one obtains the equation similar to Eq. (8) of Model 2. In other words, the model with only A_2 and A_4 nonzero can be regarded as the Model 2 modified by the O-term (the last term in the curly bracket), i.e. the term which disappears in the continuum limit and vanishes upon substituting $W(\phi_n, \phi_{n+1}) = 0$ [12].

We have checked that from the two-point map $W(\phi_n, \phi_{n+1}) = 0$ one can obtain the staggered as well as the nonstaggered JEF solutions dn, cn and sn derived in Sec. III from the three-point static Eq. (1) in case only A_2 and A_4 are nonzero. The following local identities are helpful in these derivations [23]

$$\mathrm{dn}^2(x,m)\mathrm{dn}^2(x+a,m) + \mathrm{cs}^2(a)[\mathrm{dn}^2(x,m) + \mathrm{dn}^2(x+a,m)] - 2\mathrm{ds}(a,m)\mathrm{ns}(a,m)\mathrm{dn}(x,m)\mathrm{dn}(x+a,m) + 1 - m = 0\,, \tag{163}$$

$$m^2\mathrm{cn}^2(x,m)\mathrm{cn}^2(x+a,m) + m\mathrm{ds}^2(a)[\mathrm{cn}^2(x,m) + \mathrm{cn}^2(x+a,m)] - 2m\mathrm{cs}(a,m)\mathrm{ns}(a,m)\mathrm{cn}(x,m)\mathrm{cn}(x+a,m) - m(1-m) = 0\,, \tag{164}$$

$$m^2\mathrm{sn}^2(x,m)\mathrm{sn}^2(x+a,m) - m\mathrm{ns}^2(a)[\mathrm{sn}^2(x,m) + \mathrm{sn}^2(x+a,m)] + 2m\mathrm{cs}(a,m)\mathrm{ds}(a,m)\mathrm{sn}(x,m)\mathrm{sn}(x+a,m) + m = 0\,. \tag{165}$$

It is worth emphasizing here that as demonstrated in [14], first integral can always be constructed from the known JEF solutions. In the continuum limit, the DFI Eq. (158) and the DFI $W(\phi_n, \phi_{n+1}) = 0$ as given by Eq. (161), reduce to Eq. (4), which is the first integral of the static continuum ϕ^4 equation. Relation between the solutions obtained from the map $W(\phi_n, \phi_{n+1}) = 0$ iteratively and JEF solutions reported in Sec. III B can be established as follows. JEF solutions include two integration constants, the arbitrary shift x_0 and the modulus m. In the map $W(\phi_n, \phi_{n+1}) = 0$, the role of m is played by C and the role of x_0 is played by the initial value of the map ϕ_0. Relation between m and C can be found if one obtains the JEF solutions not only from the three-point static problem of Eq. (1) but also from the two-point problem $W(\phi_n, \phi_{n+1}) = 0$ [12, 14].

B. Two-point maps for the cases (iv) to (vii)

What about a universal map for the cases (iv) to (vii)? Unfortunately, so far we have not been able to find one for any of these cases. However, as we show now, corresponding to any static JEF solution, we can always generate a map. In particular, we now obtain maps corresponding to the three JEF solutions sn, cn and dn and show that they can always be factorized and reduced to Quispel [22] form.

1. sn *solution*

As shown in the last section, one of the exact JEF solution in cases (iv) to (vii) is

$$\phi_n = A\mathrm{sn}[h\beta(n+x_0), m]\,, \tag{166}$$

with arbitrary x_0 and $0 \le m \le 1$. On using the well known identity

$$\mathrm{sn}(u+v,m) = \frac{\mathrm{sn}(u,m)\,\mathrm{cn}(v,m)\,\mathrm{dn}(v,m) + \mathrm{sn}(v,m)\,\mathrm{cn}(u,m)\,\mathrm{dn}(u,m)}{1 - m\,\mathrm{sn}^2(u,m)\ \mathrm{sn}^2(v,m)}\,, \tag{167}$$

it immediately follows that ϕ_{n+1} and ϕ_n are related by the map

$$\phi_{n+1} = \frac{\phi_n \mathrm{cn}(h\beta,m)\mathrm{dn}(h\beta,m) + \mathrm{sn}(h\beta,m)\sqrt{A^2 - (1+m)\phi_n^2 + m(\phi_n^4/A^2)}}{1 - m\,\mathrm{sn}^2(h\beta,m)\phi_n^2/A^2}\,. \tag{168}$$

This can be simplified and put in the following factorized form

$$[A^2 - m\mathrm{sn}^2(h\beta,m)\phi_n^2]W_{sn}(\phi_n,\phi_{n+1}) = 0\,, \tag{169}$$

where

$$W_{sn}(\phi_n,\phi_{n+1}) = A^2(\phi_n^2 + \phi_{n+1}^2) - m\mathrm{sn}^2(h\beta,m)\phi_n^2\phi_{n+1}^2 - 2A^2\mathrm{cn}(h\beta,m)\mathrm{dn}(h\beta,m)\phi_n\phi_{n+1} - A^4\mathrm{sn}^2(h\beta,m)\,. \tag{170}$$

Since the vanishing of the first bracket in Eq. (169) is a trivial possibility, effectively the map in this case is given by $W_{sn}(\phi_n,\phi_{n+1}) = 0$ which is precisely of the Quispel form as given in [22].

For any of the cases (iv) to (vii), the sn solution Eq. (166) can be derived from this map for any initial value taken from $-A \le \phi_0 \le A$. In this map, $0 \le m \le 1$ is the integration constant. For given h, parameters of the map β and A are related to the model parameters.

2. dn *solution*

Similarly, on using the identity

$$\mathrm{dn}(u+v,m) = \frac{\mathrm{dn}(u,m)\,\mathrm{dn}(v,m) - m\mathrm{sn}(v,m) + \mathrm{cn}(v,m)\,\mathrm{sn}(u,m)\,\mathrm{cn}(u,m)}{1 - m\,\mathrm{sn}^2(u,m)\ \mathrm{sn}^2(v,m)}\,, \tag{171}$$

we find the map for the dn solution

$$\phi_n = A\mathrm{dn}[h\beta(n+x_0), m]\,, \tag{172}$$

in the form

$$\phi_{n+1} = \frac{\phi_n \mathrm{dn}(h\beta,m) - \mathrm{sn}(h\beta,m)\mathrm{cn}(h\beta,m)\sqrt{A^2(m-1) + (2-m)\phi_n^2 - (\phi_n^4/A^2)}}{1 - \ \mathrm{sn}^2(h\beta,m)(1 - \phi_n^2/A^2)}\,. \tag{173}$$

This can be simplified and put in the following factorized form

$$[A^2\mathrm{dn}^2(h\beta,m) + m\mathrm{sn}^2(h\beta,m)\phi_n^2]W_{dn}(\phi_n,\phi_{n+1}) = 0\,, \tag{174}$$

where

$$W_{dn}(\phi_n,\phi_{n+1}) = A^2\mathrm{dn}^2(h\beta,m)(\phi_n^2 + \phi_{n+1}^2) + m\mathrm{sn}^2(h\beta,m)\phi_n^2\phi_{n+1}^2 - 2A^2\mathrm{cn}(h\beta,m)\phi_n\phi_{n+1} - A^4(1-m)\mathrm{sn}^2(h\beta,m)\,. \tag{175}$$

Since the vanishing of the first bracket in Eq. (174) is a trivial possibility, effectively the map in this case is given by $W_{dn}(\phi_n,\phi_{n+1}) = 0$ which is precisely of the Quispel form as given in [22].

3. cn *solution*

Similarly, on using the identity

$$\mathrm{cn}(u+v,m)=\frac{\mathrm{cn}(u,m)\,\mathrm{cn}(v,m)-\mathrm{sn}(v,m)+\mathrm{dn}(v,m)\,\mathrm{sn}(u,m)\,\mathrm{dn}(u,m)}{1-m\,\mathrm{sn}^2(u,m)\;\mathrm{sn}^2(v,m)}\,, \tag{176}$$

we find the map for the cn solution

$$\phi_n = A\mathrm{cn}[h\beta(n+x_0),m]\,, \tag{177}$$

in the form

$$\phi_{n+1}=\frac{\phi_n\mathrm{cn}(h\beta,m)-\mathrm{sn}(h\beta,m)\mathrm{dn}(h\beta,m)\sqrt{A^2(1-m)+(2m-1)\phi_n^2-m(\phi_n^4/A^2)}}{1-m\mathrm{sn}^2(h\beta,m)(1-\phi_n^2/A^2)}\,. \tag{178}$$

This can be simplified and put in the following factorized form

$$[A^2\mathrm{cn}^2(h\beta,m)+\mathrm{sn}^2(h\beta,m)\phi_n^2]W_{cn}(\phi_n,\phi_{n+1})=0\,, \tag{179}$$

where

$$W_{cn}(\phi_n,\phi_{n+1})=A^2\mathrm{cn}^2(h\beta,m)(\phi_n^2+\phi_{n+1}^2)+\mathrm{sn}^2(h\beta,m)\phi_n^2\phi_{n+1}^2-2A^2\mathrm{dn}(h\beta,m)\phi_n\phi_{n+1}+A^4(1-m)\mathrm{sn}^2(h\beta,m)\,. \tag{180}$$

Since the vanishing of the first bracket in Eq. (179) is a trivial possibility, effectively the map in this case is given by $W_{cn}(\phi_n,\phi_{n+1})=0$ which is precisely of the Quispel form as given in [22].

Summarizing, unlike the cases (i) to (iii), we do not have a universal map for the cases (iv) to (vii). However, in these four cases, one can always obtain a map by starting from sn, cn or dn solutions and remarkably, the maps in all three cases is effectively of the Quispel form. Obviously, one can also obtain the staggered as well as the nonstaggered JEF solution from the relevant two-point map.

It is clear from these arguments that for any discrete model that admits a JEF solution, one can easily construct the corresponding two-point map from which that JEF solution can be iteratively generated [14]. Further, such a map is effectively of the Quispel form. However, the problem of finding a universal map from which *any* static solution can be generated is not trivial even when the JEF solutions are known (note that not all TI models admit JEF solutions).

C. Particular factorized static problems

In some cases, the static problem can be factorized and one can obtain some of the exact solutions, such as those presented in Sec. IV, from this lower order algebraic equation. As an illustration, we now present a few examples of such factorized static problems.

As an example, we note that the two-point map for the Model 2 with $\gamma=0$ and $\delta=1/4$ is given by Eq. (6)

$$\frac{1}{h^2}\left(\phi_{n+1}-\phi_n\right)^2+\lambda\phi_n\phi_{n+1}-\frac{\lambda}{4}\phi_n\phi_{n+1}\left(\phi_n^2+\phi_{n+1}^2\right)-\frac{\lambda}{2}+C=0\,. \tag{181}$$

If the integration constant is chosen as $Ch^2=(\Lambda-4)^2/(2\Lambda)$, then Eq. (181) factorizes as

$$-\frac{4}{\Lambda}\left(1-\frac{\Lambda}{4}\phi_n\phi_{n+1}\right)\left[\Lambda-2-\frac{\Lambda}{4}\left(\phi_n^2+\phi_{n+1}^2\right)\right]=0\,. \tag{182}$$

The last multiplier of Eq. (182) generates the TI four-periodic solution $\phi_n\equiv(...,a,b,-a,-b,...)$ with $\Lambda(a^2+b^2)=4(\Lambda-2)$. This solution can also be written in the form of the four-periodic TI sine solution as given by Eq. (82) and Eq. (94). Further, the last multiplier also generates the staggered TI sine solution with $h\beta=\pi/2$, although it can be noted that for the four-periodic solution, non-staggered and staggered forms coincide after n is substituted with $-n$.

As a second example we take the Bender-Tovbis model, i.e. the Model 2 with $\gamma=\delta=0$. In this case, the two-point map Eq. (6) with $\Lambda=2$ and $C=1$ is factorized as

$$(1-\phi_{n-1}^2)(\phi_n^2-1)=0\,. \tag{183}$$

This example explains how one can obtain aperiodic solutions from factorized maps. Indeed, any sequence of ±1 satisfies Eq. (183) and hence the Bender-Tovbis model at $\Lambda=2$ and $C=1$. To obtain this aperiodic solution one can use either of the multipliers of Eq. (183). This is possible because the solutions are derived from the two-point rather than the three-point map.

In the above examples the short-periodic solutions were obtained from the factorized two-point map. More examples of this sort can be found in [14] where short-periodic solutions are derived from the two-point map Eq. (158) factorized for $\Lambda = 2$ for the cases $A_2 = \lambda$ and $A_4 = \lambda$.

As another example, the Speight and Ward Model 7 can be written in the form [12]

$$\ddot{\phi}_n = -v(\phi_{n-1},\phi_n)\frac{\partial}{\partial\phi_n}v(\phi_{n-1},\phi_n) - v(\phi_n,\phi_{n+1})\frac{\partial}{\partial\phi_n}v(\phi_n,\phi_{n+1})\,, \tag{184}$$

where

$$v(\phi_{n-1},\phi_n) = \frac{\phi_n-\phi_{n-1}}{H} - \frac{1}{\sqrt{2}} + \frac{(\phi_{n-1}^2+\phi_{n-1}\phi_n+\phi_n^2)}{3\sqrt{2}}\,, \quad H^2 = \frac{6\Lambda}{6-\Lambda}\,, \tag{185}$$

from which it is clear that the equation $v(\phi_{n-1},\phi_n) = 0$ generates static kink and inverted kink solutions of their model.

The following two examples are interesting because they give the short-periodic solutions, not from a two-point map, but, from a set of *two* finite-difference equations.

We note that the Speight and Ward Model 7 can be written in the form

$$\ddot{\phi}_n = \frac{1}{h^2}(\phi_{n+1}+\phi_{n-1}-2\phi_n) + \lambda\phi_n$$
$$-\frac{\lambda}{18}(\phi_{n-1}+2\phi_n)(\phi_{n-1}^2+\phi_{n-1}\phi_n+\phi_n^2) - \frac{\lambda}{18}(\phi_{n+1}+2\phi_n)(\phi_n^2+\phi_n\phi_{n+1}+\phi_{n+1}^2). \tag{186}$$

If the following two-point equation holds

$$\phi_{n-1}^2+\phi_{n-1}\phi_n+\phi_n^2 = 6\frac{\Lambda-3}{\Lambda}, \tag{187}$$

then the static version of Eq. (186) reduces to

$$(\Lambda-6)(\phi_{n-1}+\phi_n+\phi_{n+1}) = 0\,. \tag{188}$$

Two-point map Eq. (187) generates the exact three-periodic sin as well as the six-periodic staggered sin solutions to the Speight and Ward model [see Eqs. (82), (97) and (98)] and for these solutions Eq. (188) is also satisfied. Further, Eqs. (187) and (188) for the case of $\Lambda = 6$ also generate the short periodic solutions $\phi_n \equiv (..., a, 0, ..)$, $(..., a, -a, 0, ...)$, $(..., a, -a, a, -a, 0, ...)$, $(..., a, 0, a, 0, -a, ...)$, $(..., a, -a, 0, -a, a, 0, ...)$, $(a, 0, a, 0, a, -a)$, and $(a, 0, -a, a, 0, -a, 0)$.

Similarly, the equation of motion for the Hamiltonian Model 10 can be written in the form

$$\ddot{\phi}_n = \frac{1}{h^2}(\phi_{n+1}+\phi_{n-1}-2\phi_n) + \lambda\phi_n$$
$$-\lambda(\alpha_2\phi_{n-1}+2\alpha_1\phi_n)(\phi_{n-1}^2+\frac{\alpha_2}{\alpha_1}\phi_{n-1}\phi_n+\phi_n^2) - \lambda(\alpha_2\phi_{n+1}+2\alpha_1\phi_n)(\phi_{n+1}^2+\frac{\alpha_2}{\alpha_1}\phi_{n+1}\phi_n+\phi_n^2)\,. \tag{189}$$

In case the following two-point equation holds

$$\phi_{n-1}^2+\frac{\alpha_2}{\alpha_1}\phi_{n-1}\phi_n+\phi_n^2 = H\,, \tag{190}$$

then the static version of Eq. (189) reduces to

$$(\phi_{n-1}+\phi_{n+1})(1-\Lambda H\alpha_2) + (\Lambda-2-4\alpha_1\Lambda H)\phi_n = 0\,, \quad H = \frac{\Lambda-2-\alpha_2/\alpha_1}{\Lambda(4\alpha_1-\alpha_2^2/\alpha_1)}\,, \tag{191}$$

provided

$$\alpha_3 = \alpha_1 + \frac{\alpha_2^2}{2\alpha_1}\,. \tag{192}$$

In the special case of $\alpha_1 = \alpha_2 = 1/18$, this reduces to the SW model. The two-point map Eq. (190) generates the exact sine as well as the staggered sine solutions to the Hamiltonian Model 10 with $\cos(h\beta) = \mp\alpha_2/(2\alpha_1)$, and for these solutions Eq. (191) is also satisfied.

Once again, it is interesting that for the factorization Eq. (187) and Eq. (188), and also for the factorization Eq. (190) and Eq. (191), one has to satisfy two lower-order finite-difference equations *simultaneously*, and one of those equations is a two-point one while another is a three-point one.

All factorized problems discussed in this section do not contain the integration constant and thus they generate only particular solutions. Some of them are TI solutions, for example, the three-periodic solution derivable from Eq. (187) and Eq. (188), while others are not, for example, arbitrary sequence of ± 1, derivable from Eq. (183).

Solutions constructed in this section from factorized problems do not survive in the continuum limit because the factorized problems do not reduce to Eq. (1) or Eq. (4) in the continuum limit.

VI. GOLDSTONE MODES

A. Goldstone mode of a TI static solution

Let ϕ_n^0 be a static solution to Eq. (1). To study the dynamics in the vicinity of this solution we substitute the ansatz $\phi_n(t) = \phi_n^0 + \varepsilon_n(t)$ into Eq. (1), and obtain the following linearized equation

$$\ddot{\varepsilon}_n = K_{n,n-1}\varepsilon_{n-1} + K_{n,n}\varepsilon_n + K_{n,n+1}\varepsilon_{n+1}\,, \tag{193}$$

with

$$\begin{aligned} K_{n,n-1} &= \frac{1}{h^2} - \frac{A_2}{2}(\phi_n^0)^2 - A_3\phi_{n-1}^0\phi_n^0 - A_4\phi_n^0\phi_{n+1}^0 - \frac{A_5}{2}\phi_{n+1}^0(2\phi_{n-1}^0 + \phi_{n+1}^0) - \frac{3A_6}{2}(\phi_{n-1}^0)^2, \\ K_{n,n} &= \lambda - \frac{2}{h^2} - 3A_1(\phi_n^0)^2 - A_2\phi_n^0(\phi_{n-1}^0 + \phi_{n+1}^0) - \frac{A_3}{2}\left[(\phi_{n-1}^0)^2 + (\phi_{n+1}^0)^2\right] - A_4\phi_{n-1}^0\phi_{n+1}^0, \\ K_{n,n+1} &= \frac{1}{h^2} - \frac{A_2}{2}(\phi_n^0)^2 - A_3\phi_n^0\phi_{n+1}^0 - A_4\phi_{n-1}^0\phi_n^0 - \frac{A_5}{2}\phi_{n-1}^0(\phi_{n-1}^0 + 2\phi_{n+1}^0) - \frac{3A_6}{2}(\phi_{n+1}^0)^2\,. \end{aligned} \tag{194}$$

Looking for solutions of Eq. (193) of the form $\varepsilon_n(t) = U_n \exp(\pm i\omega t)$ we come to the eigen-value problem

$$[K]\,\mathbf{U} = -\omega^2\mathbf{U}\,, \tag{195}$$

where vector $\mathbf{U}$ contains U_n and the nonzero coefficients of matrix $[K]$ are given by Eq. (194).

If ϕ_n^0 is a TI static solution then it can be shifted along the chain by an arbitrary x_0, $\phi_n^0 = \phi(n + x_0)$. The eigenvector corresponding to the zero-frequency translational Goldstone mode, $\mathbf{U}_{\rm G}$, has components ϕ_n', where prime means derivative of ϕ with respect to its argument. To confirm that, we substitute $\mathbf{U}_{\rm G} = \{\phi_n'\}$ into Eq. (195) with $\omega^2 = 0$ and obtain

$$K_{n,n-1}\phi_{n-1}' + K_{n,n}\phi_n' + K_{n,n+1}\phi_{n+1}' = 0\,. \tag{196}$$

The last expression is an identity because it coincides with the derivative of static version of Eq. (1) with respect to x_0, and such a derivation is possible for the TI static solution, which is an equilibrium solution for any x_0. We thus have proved that any TI static solution has the zero-frequency translational mode $\mathbf{U}_{\rm G} = \{\phi_n'\}$. Particularly, for any static JEF, hyperbolic or trigonometric function solutions given in Sec. III, one can easily find the corresponding TI mode as it is proportional to the derivative of the solution with respect to its argument.

Looking for solutions of Eq. (193) and Eq. (194) with $\phi_n^0 = 1$ of the form of small-amplitude phonons, $\varepsilon_n(t) \sim \exp(ikn \pm i\omega t)$, where k denotes wavenumber and ω is frequency, one obtains the spectrum of the vacuum for the discrete model of Eq. (1),

$$\omega^2 = 2\lambda + 2\left[\frac{2}{h^2} - A_2 - 2A_3 - 2A_4 - 3A_5 - 3A_6\right]\sin^2\left(\frac{k}{2}\right)\,. \tag{197}$$

B. Goldstone modes of some short-period static solutions

If a static solution does not possess the zero-frequency translational Goldstone mode, then this solution is not a TI one. The opposite, in general is not true, i.e., a particular static solution may have the Goldstone mode only at certain positions with respect to the lattice x_0, but a TI solution must have such a mode at any x_0. It is interesting to check whether the short-periodic solutions derived in Sec. IV can have the Goldstone mode.

Four periodic solution of the form $..., a, b, -a, -b, ...$, as found in Sec. IV, exists in case $A_1 \neq 0$ under the constraint $A_1 - A_3 + A_4 = 0$ and has $a^2 = (\Lambda-2)/(A_1 h^2) - b^2$. This one-parameter solution is a TI solution because it can be expressed in the form $\phi_n = A\cos[\pi(n+x_0)/2]$ with arbitrary shift x_0 and $A^2 = (\Lambda-2)/(A_1h^2)$. Being a TI solution, it possesses the Goldstone mode at any x_0, as it was demonstrated in Sec. VI A. Note that Model 2, Model 8, Model 10 and Speight and Ward Model 7 have $A_1 \neq 0$ but the constraint $A_1 - A_3 + A_4 = 0$ is not satisfied for the SW model 7, while it is satisfied for Model 2 (at arbitrary γ and arbitrary nonzero δ), Model 8 (at $\gamma = 0, \alpha = 1/4$) and Model 10 (at $\alpha_1 = \alpha_3$). Thus, while Models 2,8 and 10 support this TI four-periodic solution, Model 7 does not.

It may be noted here that in case $\Lambda = 2$ and $A_1 = 0, A_3 = A_4$, then the TI solution is a two-parameter solution which can be expressed in the form $\phi_n = A\cos[\pi(n+x_0)/2]$ with an arbitrary shift x_0 and an arbitrary amplitude A. This is satisfied by Model 9 at $\beta = 0$ as well as by models where only A_3, A_4, A_5 or where only A_2, A_3, A_4, A_5 are nonzero.

Four periodic solution of the form $..., a, 0, -a, 0, ...$, as found in Sec. IV, exists for $A_1 \neq 0$ and has $a^2 = (\Lambda-2)/\left(A_1h^2\right)$. Inserting this solution into Eq. (195) one finds

$$\begin{bmatrix} \alpha & \beta & 0 & \beta \\ \delta & \gamma & \delta & 0 \\ 0 & \beta & \alpha & \beta \\ \delta & 0 & \delta & \gamma \end{bmatrix} \begin{Bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{Bmatrix}, \tag{198}$$

where

$$\alpha = -2\frac{\Lambda-2}{h^2}, \quad \beta = \frac{1}{h^2} - A_2\frac{\Lambda-2}{2A_1h^2},$$
$$\gamma = (A_1 - A_3 + A_4)\frac{\Lambda-2}{A_1h^2}, \quad \delta = \frac{1}{h^2} + (A_5 - 3A_6)\frac{\Lambda-2}{2A_1h^2}. \tag{199}$$

Characteristic equation of Eq. (198) is

$$\left(\alpha+\omega^2\right)\left(\gamma+\omega^2\right)\left[\left(\alpha+\omega^2\right)\left(\gamma+\omega^2\right) - 4\beta\delta\right] = 0, \tag{200}$$

and zero-frequency modes are possible when any of the following three conditions is satisfied

$$\alpha = 0, \quad \gamma = 0, \quad \alpha\gamma - 4\beta\delta = 0. \tag{201}$$

The considered four-periodic solution can also be expressed as $\phi_n = A\cos(\pi n/2)$ with $A^2 = (\Lambda-2)/(A_1h^2)$. The expected Goldstone mode is proportional to the derivative of this function with respect to its argument, and thus, should have the form $U_0 = U_2 = 0$, $U_1 = -U_3 = k$ with arbitrary $k \neq 0$. Substituting this into Eq. (198) with $\omega^2 = 0$ one finds that the Goldstone mode corresponds to $\gamma = 0$. Taking into account Eq. (199), we conclude that the considered four-periodic solution possesses the Goldstone mode under the condition

$$A_1 - A_3 + A_4 = 0\,. \tag{202}$$

Particularly, the Speight and Ward Model 7 supports the considered four-periodic solution but it does not satisfy the condition Eq. (202) so that the solution is *not* a TI one. On the other hand, Model 2 supports the considered four-periodic solution and it satisfies the condition of Eq. (202) so that the solution possesses the Goldstone mode.

It may be noted here that the solution (...,a,0,-a,0,...) also exists in case $\Lambda = 2$ and $A_1 = 0$, and in that case a is any real number. In this case the solution can be expressed in the form $\phi_n = a\cos[\pi n/2]$. Further, such a solution is *always* a TI one since according to Eq. (201), γ is proportional to $\Lambda - 2$ and hence is equal to 0. Note that such a solution exists in almost all the models at $\Lambda = 2$. For example it exists in Model 2 at $\delta = 0$, Model 3, Model 4, Model 6, Model 8 (at $\alpha = 0$ or/and $\gamma = -\beta = 1/2$), Model 9 and Model 10 (at $\alpha_1 = 0$) as well as the five models discussed in Sec. III.

Four periodic solution of the form $..., a, a, -a, -a, ...$, as found in Sec. IV, exists for $A_1 + A_3 - A_4 \neq 0$ and has $h^2a^2 = (\Lambda-2)/(A_1 + A_3 - A_4)$. The solution can also be presented as $\phi_n = \sqrt{2}\,A\cos(\pi n/2 + \pi/4)$ where $A^2 = (\Lambda-2)/[h^2(A_1 + A_3 - A_4)]$, so that the Goldstone mode is $U_0 = -U_1 = -U_2 = U_3 = k$ with an arbitrary $k \neq 0$. Performing the calculations similar to the previous case one finds that the considered four-periodic solution possesses the Goldstone mode under the same condition of Eq. (202). Again, this solution has the Goldstone mode in Model 2 but not in the Speight and Ward Model 7, so that in the latter model the solution is *not* a TI one.

It may be noted here that the solution (...,a,a,-a,-a,...) also exists in case $\Lambda = 2$ and $A_1 + A_3 = A_4$, and in this case a is any real number. In this case the solution can be expressed in the form $\phi_n = \sqrt{2}a\cos[\pi n/2 + \pi/4]$. Further, such a solution is *always* a TI one since according to Eq. (202), γ is proportional to $\Lambda - 2$ and hence is equal to 0. It is

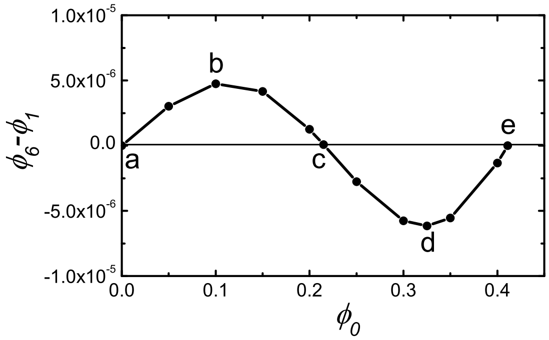

FIG. 1: Difference between ϕ_6 and ϕ_1 as function of ϕ_0 for the static solutions constructed for the Speight and Ward Model 7 from the three-point map $\phi_{n+1} = f(\phi_{n-1}, \phi_n)$ for chosen ϕ_0 and numerically found ϕ_1 such that $\phi_5 = \phi_0$. If the three-point problem is reducible to a two-point problem $\phi_{n+1} = g(\phi_n)$ then from having $\phi_5 = \phi_0$ one must also have $\phi_6 = \phi_1$, which is not the case and thus, for this solution the two-point reduction is impossible. Static solutions generated by the three-point map for the points marked with a to e are shown in the corresponding panels of Fig. 2.

easily checked that such a solution at $\Lambda = 2$ exists in Model 2 (at $\delta = 0$ and arbitrary γ), Model 8 (at $8\alpha^2 = 2\alpha + \beta$), Model 9 (at $\beta = 0$) and Model 10 (at $\alpha_1 = -\alpha_3$).

Three periodic solution of the form $..., a, 0, -a, ...$, as found in Sec. IV, exists for $2A_1 - A_2 + A_3 - A_6 \neq 0$ and has $h^2a^2 = 2(\Lambda - 3)/(2A_1 - A_2 + A_3 - A_6)$. Thus such a solution can be expressed by $\phi_n = (2/\sqrt{3})A\cos(2\pi n/3 - \pi/6)$ with $A^2 = [2(\Lambda - 2)]/[h^2(2A_1 - A_2 + A_3 - A_6)]$. The Goldstone mode is $U_0 = U_2 = k$, $U_1 = -p$, and the ratio k/p can be found but it is not important for our analysis. Calculations similar to that for the four-periodic solutions give that the considered three-periodic solution possesses the Goldstone mode or, in other words, is a TI solution under the condition

$$\Big(-4A_1^2 - A_3^2 + A_5^2 + 2A_6^2 - 2A_1A_6 + 2A_1A_2 + 4A_1A_3 - 4A_1A_4 - A_2A_6 + A_3A_6 + 2A_4A_6 - 3A_5A_6 - A_2A_3 + 2A_3A_4 + A_2A_5 - 2A_4A_5\Big)(\Lambda^2 - 3)^2 = 0\,. \quad (203)$$

It is easily checked that both Model 2 and Speight and Ward Model 7 support the three periodic solution and meet the condition Eq. (203). Thus the three-periodic solution possesses the Goldstone mode in both Model 2 and Model 7, but, as mentioned above, one cannot claim that this solution is a TI one in these models unless one demonstrates that the Goldstone mode exists for any shift of the three-periodic solution along the lattice.

It may be noted here that the solution (...,a,0,-a,...) also exists in case $\Lambda = 2$ and $2A_1 + A_3 = A_2 + A_6$, and in this case a is any real number. In this case the solution can be expressed in the form $\phi_n = (2/\sqrt{3})a\cos[2\pi n/3 - \pi/6]$. Further, such a solution is *always* a TI one since $\Lambda = 3$. Thus, such a solution exists at $\Lambda = 2$ in Model 2 (in case $2\gamma + 8\delta = 1$), Model 6, Model 8 (in case $4\alpha[\beta + \gamma] = 3\alpha^2 + \gamma^2 + \beta\gamma$), Model 10 (in case $2\alpha_1 + \alpha_3 = 2\alpha_2$), and in Models where only A_2, A_3, A_5 or only A_2, A_3, A_4, A_5 are nonzero.

VII. NUMERICAL RESULTS

A. Five-periodic static solution in Speight and Ward Model 7

Here we address the problem of integrability of the Speight and Ward (SW) Model 7. As it was mentioned, this model has the two-point map Eq. (11) to derive the kink solution but a general two-point map, that includes the integration constant as a free parameter, for this model is not known. We will give numerical evidence that the Model 7 is not integrable and the two-point map for obtaining a two-parameter set of static solutions cannot be constructed. For this purpose we will make an attempt to construct the static five-periodic solutions to this model and will observe that it can be constructed only for highly symmetric positions of the solution with respect to the lattice so that the solution is not a TI one and it possesses the Peierls-Nabarro potential. We have chosen the five-periodic solution for this study because it is relatively simple and it corresponds to a non-factorized static problem. Simple short-periodic

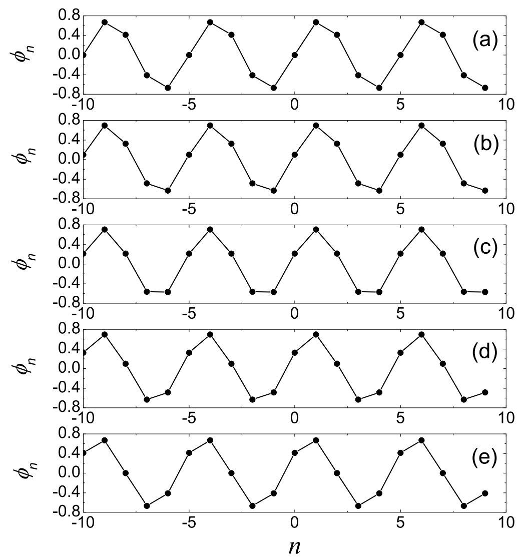

FIG. 2: Static solutions for the Speight and Ward Model 7 generated from the three-point map $\phi_{n+1} = f(\phi_{n-1}, \phi_n)$ for chosen ϕ_0 and numerically found ϕ_1 such that $\phi_5 = \phi_0$. Panels (a) to (e) show the results for the values of ϕ_0 marked with a to e in Fig. 1. Solutions in (b) and (d) are *not* five-periodic solutions because ϕ_6 differs from ϕ_1 by the amount shown in Fig. 1. Highly symmetric solutions in (a), (c), and (e) are the five-periodic ones and for them $\phi_6 = \phi_1$, as it can be seen from Fig. 1. Solution in (a) [and in (e)] corresponds to a minimum of the Peierls-Nabarro potential while the one in (c) to a maximum of this potential.

solutions described in Sec. IV are not suitable for this study because they are obtained from low-order algebraic equations, i.e., from factorized static problems that do not represent the model in its general formulation.

We set for the model parameters $\lambda = 1$, $h = 1.3$, and A_k corresponding to the Model 7 as given in the Introduction. The three-point static problem of Eq. (1) is written in the form of the map $\phi_{n+1} = f(\phi_{n-1}, \phi_n)$ which, for given ϕ_0 and ϕ_1, generates a static solution. For chosen ϕ_0 we numerically find ϕ_1 such that in the iteratively obtained solution $\phi_5 = \phi_0$ and check whether in this case we also have $\phi_6 = \phi_1$. If the three-point map $\phi_{n+1} = f(\phi_{n-1}, \phi_n)$ is reducible to a two-point map $\phi_{n+1} = g(\phi_n)$ then having $\phi_5 = \phi_0$ we must also have $\phi_6 = \phi_1$. However, as it can be seen from Fig. 1, $\phi_6 - \phi_1$ is equal to zero only for a discrete set of ϕ_0. In Fig. 2 (a) to (e) we plot the structures generated by the three-point map for different ϕ_0 indicated in Fig. 1 by, correspondingly, letters a to e. Recall that for all structures presented in Fig. 2 we have $\phi_5 = \phi_0$ but the condition $\phi_6 = \phi_1$ is fulfilled only for the highly symmetric structures shown in (a), (c), and (e) [structures in (a) and (e) are equivalent]. Solutions in (b) and (d) are *not* five-periodic solutions because ϕ_6 differs from ϕ_1 by the amount shown in Fig. 1. Solutions in (b) and (d) are modulated five-periodic structures but this cannot be seen in Fig. 2 because the period of the modulated structure is very large.

Static solutions shown in Fig. 2 (a) and (c) are the five-periodic equilibrium solutions for which the small-amplitude vibrational spectrum can be calculated as described in Sec. VI A. Doing so we find that these five-periodic structures do not possess the zero-frequency Goldstone mode, but they have a nearly translational mode with frequency $\omega = 0.0036$ for the structure in Fig. 2 (a) and purely imaginary frequency $\omega = 0.0036i$ for the structure in Fig. 2 (c). Thus, the five-periodic structure in the Speight and Ward model 7 is not a TI one and it experiences the Peierls-Nabarro potential with a minimum energy corresponding to the structure in (a) and a maximum energy corresponding to the structure in (c).

We have studied some other periodic solutions, for example, seven- and eight-periodic ones and have obtained the results qualitatively similar to that for the five-periodic structure. In all studied cases the structures had nonzero Peierls-Nabarro potential.

We conclude that the static solutions supported by the Speight and Ward Model 7, except for the kinks, anti-kinks, sine and staggered-sine solutions, usually have the Peierls-Nabarro potential. The corresponding three-point static problem is non-integrable and cannot be reduced to a two-point problem, again, except for the kink solutions.

This result is not surprising at all and it could be expected taking into account that the derivation of the two-point map Eq. (11), from which the kink solution can be derived, was done for the integration constant $C = 0$ in Eq. (4).

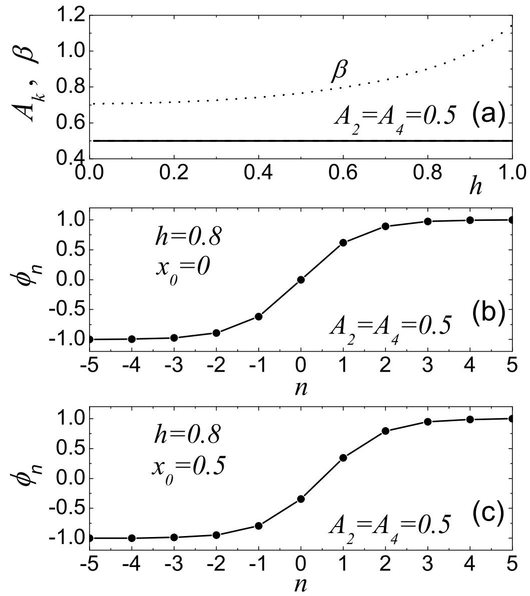

FIG. 3: Static kink in the case (iii) with only A_2 and A_4 nonzero: (a) model parameters A_k and kink parameter β as functions of h; (b) on-site kink at $h = 0.8$; (c) inter-site kink at $h = 0.8$. This model admits TI static solutions, including the kink solution, at constant (h-independent) model parameters A_k, as can be seen in (a).

The resulting discrete model supports the TI solutions only for this particular value of the integration constant, and those solutions are kinks.

Hamiltonian TI discretization of the Klein-Gordon field that generalizes the Speight and Ward model has been derived in Sec. II C of [12]. The model includes the integration constant and thus supports a two-parameter set of TI static solutions, although it is rather complex even for the cubic nonlinearity.

It is worth noting here that the Speight and Ward Model 7, apart from the five-periodic solutions shown in Fig. 2 (a), (c), and (e), supports the following non-TI five-periodic solutions:

$$\begin{aligned}
&\text{Solution (xiv)} \ : \quad \phi = (1,1,1,-1,-1) \text{ if } \Lambda = 9/2\,;\\
&\text{Solution (xix)} \ : \quad \phi = (1,1,0,-1,0) \text{ if } \Lambda = 18/7\,;\\
&\text{Solution (xvi)} \ : \quad \phi = (a,-a,a,-a,0) \text{ if } \Lambda = 6 \text{ and } a^2 = 3\,;\\
&\text{Solution (xvi)} \ : \quad \phi = (a,0,a,0-a) \text{ if } \Lambda = 6 \text{ and } a^2 = 3\,;\\
&\text{Solution (xvi)} \ : \quad \phi = (a,a,-a,-a,0) \text{ if } \Lambda = 12 \text{ and } a^2 = 3/2\,.
\end{aligned} \tag{204}$$

The solutions shown in Fig. 2 exist for a continuously varying Λ (at least within a range of Λ values) while the above five solutions exist for a fixed Λ.

B. Static kinks

Here, after a brief discussion on the JEF solutions, we focus on the analysis of the kink solutions because they are discussed in applications more often than the periodic solutions.

As it was mentioned, the JEF solutions and their hyperbolic function limit solutions like kink and pulse exist in the model Eq. (1) in the seven cases, of which the first three cases with (i) only A_2 nonzero, (ii) only A_4 nonzero, and (iii) only A_2 and A_4 nonzero are qualitatively different from the other four cases discussed in Sec. III. The difference is in that for the first three cases one has two conditions for finding the JEF solution parameters A and β while in the remaining four cases one has to satisfy one more condition. This additional constraint couples the model parameters A_k to the lattice spacing h. As a result, in the last four cases, for fixed A_k, one has TI solutions only at particular h, while in the first three cases, even for fixed A_k, one has TI solutions for any h.

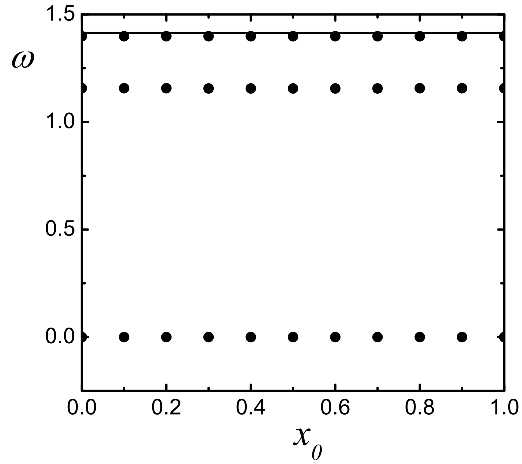

FIG. 4: Spectrum of the lattice with a kink in the case (iii) with only A_2 and A_4 nonzero. Straight horizontal line at $\omega = \sqrt{2}$ shows the lower bound of the phonon band while dots show the kink's internal modes calculated for the kink at various positions x_0 with respect to the lattice. At any position x_0 the kink possesses the zero-frequency Goldstone translational mode. The kink profiles at $x_0 = 0$ and $x_0 = 0.5$ are shown in Fig. 3 (b) and (c), respectively. Model parameters: $h = 0.8$, $A_2 = A_4 = 0.5$, $\lambda = 1$.

Let us demonstrate this qualitative difference between two groups of models by comparison of the properties of the static kinks. The first group of models will be represented by the case (iii) with only A_2 and A_4 nonzero while from the four models of the second group we will choose the case (iv) with only A_3 and A_5 nonzero, and the case (vi) with only A_3, A_4, and A_5 nonzero. For the kink solutions discussed below we will always set $\lambda = 1$.

Kink in the case (iii) *with only* A_2 *and* A_4 *nonzero.* Parameters of the kink solution Eq. (19) with $S = 1$ for this case are given by Eq. (24). For given model parameters h and A_4 one can find the inverse kink width β solving the second equation in Eq. (24). The model parameter A_2 must satisfy the continuity constraint given by the last expression in Eq. (24). Particular feature of this discrete ϕ_4 model is that it admits the TI solutions at constant A_k, see Fig. 3. Contrary to that, as it will be seen in the following examples, in the models (iv) to (vii), model parameters A_k are h-dependent.

Vibrational spectrum of the lattice containing a kink at different positions with respect to the lattice x_0 is shown in Fig. 4 for model parameters $h = 0.8$, $A_2 = A_4 = 0.5$, $\lambda = 1$. The corresponding kink profiles at $x_0 = 0$ and $x_0 = 0.5$ are shown in Fig. 3 (b) and (c), respectively. At any position x_0 the kink possesses the zero-frequency Goldstone translational mode. Straight horizontal line at $\omega = \sqrt{2}$ shows the lower bound of the phonon band, see Eq. (197).

Kink in the case (iv) *with only* A_3*, and* A_5 *nonzero.* Model parameters and kink parameters in this case are given by Eq. (36). For chosen A_3 (or A_5) one can find A_5 (or A_3) from the continuity constraint [Eq. (36)] and then find the inverse kink width β solving the second equation in Eq. (36). Finally, one of the Eq. (36) relates the model parameters A_k to the lattice spacing h. In Fig. 5 we show (a) the model parameters A_k and kink inverse width β as functions of h. In the region of lattice parameter around $h = 0.5$ it is possible to have two different static kink solutions at the same h which is illustrated in (b) and (c). In both cases $h = 0.5$, but model parameters A_k and the inverse kink width β are different (shown in each panel). Both kinks are stable and have a zero-frequency translational Goldstone mode at any position with respect to the lattice x_0.

Kink in the case (vii) *with only* A_3*,* A_4*, and* A_5 *nonzero.* Kink and model parameters are related by Eq. (58). In Fig. 6 we plot (a) model parameters A_k as the functions of h at *fixed* inverse kink width $\beta = 2$; (b) the on-site kink at $h = 1.3$; and (c) the inter-site kink at $h = 1.3$. Other model parameters for (b) and (c) are $A_3 = 0.8297$, $A_4 = 1.0092$, $A_5 = -0.8389$, and $\lambda = 1$.

Note that in Fig. 6 (a) all A_k vary with h but, interestingly, the inverse kink width β is constant ($= 2$) in a wide range of lattice spacing h. In the classical discrete ϕ^4 model and in the models with h-independent parameters A_k the kink width usually decreases with increase in h. On the other hand, it is possible to get from Eq. (58) one constant model parameter with two other model parameters A_k and kink parameter β being functions of h.

In Fig. 7 one can see the spectrum of the lattice with a kink. Straight horizontal line at $\omega = \sqrt{2}$ shows the lower bound of the phonon band [see Eq. (197)] while dots show the kink's internal modes calculated for the kink at various positions x_0 with respect to the lattice. At any position x_0 the kink possesses the zero-frequency Goldstone

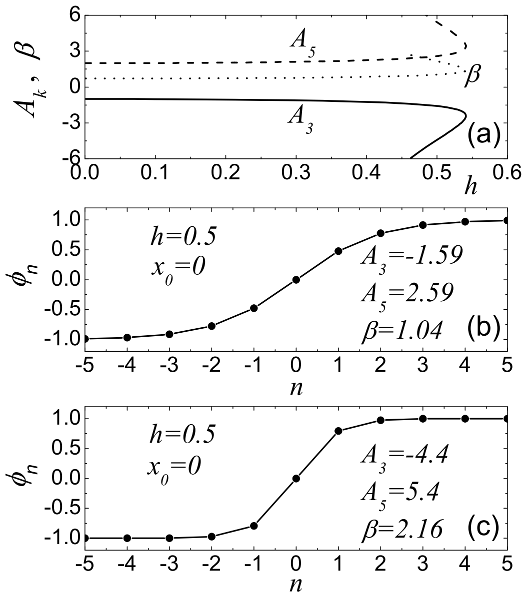

FIG. 5: Static kink in the case (iv) with only A_3 and A_5 nonzero. (a) Model parameters A_k and kink inverse width β as functions of h. In the region of lattice parameter around $h = 0.5$ it is possible to have two different static kink solutions at the same h which is illustrated in (b) and (c). In both cases $h = 0.5$, but model parameters A_k and the inverse kink width β are different (shown in each panel). Both kinks are stable and have zero-frequency translational Goldstone mode at any position with respect to the lattice x_0.

translational mode. The kink profiles at $x_0 = 0$ and $x_0 = 0.5$ are shown in Fig. 6 (b) and (c), respectively. Model parameters are: $h = 1.3$, $A_3 = 0.8297$, $A_4 = 1.0092$, $A_5 = -0.8389$, and $\lambda = 1$.

VIII. DISCUSSION AND CONCLUSIONS

In this paper we have introduced a rather general discrete ϕ^4 model of which all known models in the literature are special cases. We could find seven special cases when the model as given by Eq. (1) supports the exact static JEF and hence hyperbolic kink and pulse solutions. Two of those seven cases have been analyzed in [12] and [14], while for the remaining five cases, JEF solutions were given in Sec. III.

The exact solutions constructed for the considered discrete ϕ^4 model are important for the theory of the TI lattices. Indeed, the JEF static solutions with an arbitrary shift along the lattice x_0 are the TI solutions with the zero-frequency Goldstone mode, i.e., solutions that are free of the Peierls-Nabarro potential. The discrete ϕ^4 model of Eq. (1) corresponding to the following four particular cases

1. case (iv) only A_3 and A_5 nonzero;

2. case (v) only A_2, A_3, and A_5 nonzero;

3. case (vi) only A_3, A_4, and A_5 nonzero;

4. case (vii) only A_2, A_3, A_4, and A_5 nonzero;

are the new TI models. Each of these models (like the other three) supports a two-dimensional set of TI static solutions that can be parameterized by the points of the plane (m, x_0). For *fixed* model parameters A_k these models support TI solutions only for a particular lattice spacing h. Note, however, that the other three models, i.e. cases (i) to (iii), for fixed model parameters A_2 and A_4, support the TI solutions for any arbitrary value of h. However, when A_k are considered to be functions of h, the TI solutions in (iv) to (vii) can also be constructed for continuously varying h (see numerical examples of Sec. VII B). In this context, it is worth noting that the TI Model 5 given by Eq. (9) and the TI Model 9 also have h-dependent nonlinearities.

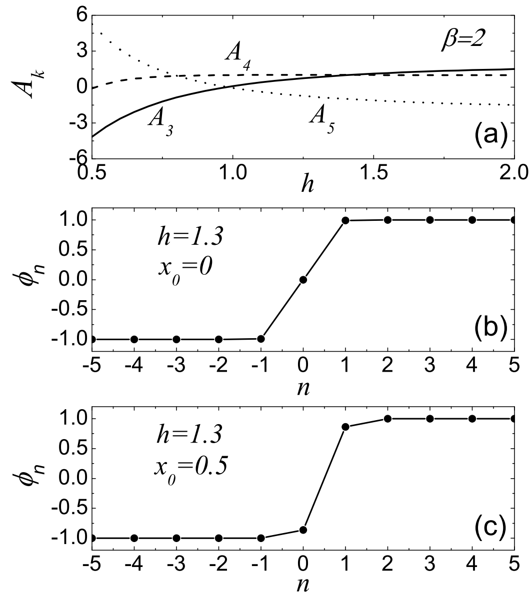

FIG. 6: Static kink in the case (vii) with only A_3, A_4, and A_5 nonzero. (a) Model parameters A_k as functions of h at *fixed* inverse kink width $\beta = 2$. (b) On-site kink at $h = 1.3$. (c) Inter-site kink at $h = 1.3$. Other model parameters for (b) and (c) are $A_3 = 0.8297$, $A_4 = 1.0092$, $A_5 = -0.8389$, and $\lambda = 1$.

In this paper, we also showed that the general model, Eq. (1) supports periodic sine and staggered sine solutions. Remarkably, almost all the known models (even those not supporting the JEF solutions), were found to support these solutions. Besides, a large number of exact, short-periodic and aperiodic static solutions admitted by Eq. (1) were obtained in Sec. IV. While we do not have a rigorous proof, but the few examples discussed in Sec. V C suggest that very likely, the short-periodic, aperiodic as well as trigonometric solutions, in fact follow from low order algebraic equations. In this context, it is worth pointing out that the sine solution does not follow from the map for Model 2 as well as the map for case (iii) when only A_2 and A_4 are nonzero. The factorization can also easily explain the appearance of the aperiodic solutions that can be regarded as the solutions obtained from different multipliers and linked together, as exemplified by the discussion below Eq. (183).

It is worth pointing out that none of the factorized problems discussed in Sec. V C contain an integration constant and thus they generate only particular solutions. Some of them are TI solutions, for example, the three-periodic solution to the Speight and Ward Model 7 derivable from Eq. (187) and Eq. (188), while others are not, for example, arbitrary sequence of ± 1, derivable from Eq. (183).

As it was shown in Sec. V C, in some cases the two-point map can be factorized and then the short-periodic solutions can also be derived from a two-point problem. We also discussed several examples in which the three-point problem can be reduced to a set of two lower-order finite-difference equations, and one of those equations is a two-point one while another is a three-point one. Based on these examples, we conjecture that all short-periodic, aperiodic as well as sine and staggered sine solutions should follow from lower order difference equations.

Note that the short-periodic solutions and, more generally, the solutions derived from factorized problems very often do not survive the continuum limit because factorized equations usually have a different continuum limit than the original, non-factorized one. In this context it is worth noting that sine is not a solution of the continuum ϕ^4 field equation. One exception to this rule is the kink solution to the Speight and Ward Model 7 for which the reduced two-point problem Eq. (11) in the continuum limit obtains a form which is equivalent to the first integral of the static ϕ^4 field.

Coming back to the exact JEF solutions, we emphasize that they are important because by using them one can construct the corresponding two-point maps from which the corresponding solutions can be obtained iteratively. Moreover, in some cases, the map obtained for a particular JEF solution can be transformed to the form of a general map from which majority of static solutions including other JEF solutions admitted by the model can be constructed. We conjecture that, except for those obtainable from special cases when the three-point problem can be substituted with a set of two equations, all other solutions can be obtained recursively from this general map. Following this way

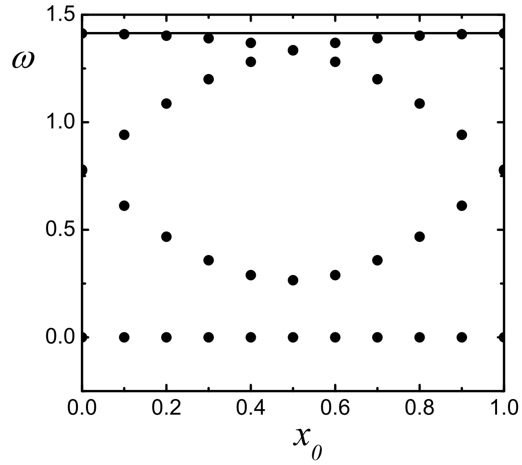

FIG. 7: Spectrum of the lattice with a kink in the case (vii) with only A_3, A_4, and A_5 nonzero. Straight horizontal line at $\omega = \sqrt{2}$ shows the lower bound of the phonon band while dots show the kink's internal modes calculated for the kink at various positions x_0 with respect to the lattice. At any position x_0 the kink possesses the zero-frequency Goldstone translational mode. The kink profiles at $x_0 = 0$ and $x_0 = 0.5$ are shown in Fig. 6 (b) and (c), respectively. Model parameters: $h = 1.3$, $A_3 = 0.8297$, $A_4 = 1.0092$, $A_5 = -0.8389$, and $\lambda = 1$.

we could construct the map Eq. (158) from which any static solution of Eq. (1) with only A_2 and A_4 nonzero can be constructed (again, except for the solutions that result from specially factorized three-point problems). On the other hand, for cases (iv) to (vii), while one can obtain a map from a JEF solution, so far we are unable to obtain a general map.

In Sec. VII A we provided numerical evidence that the Speight and Ward Model 7 does not support TI static solutions other than those derivable from reduced lower-order algebraic problems as discussed in Sec. V C. In addition to the well-known TI kink solution we have found the TI *sin* and staggered sin solutions to this model. We believe that in the general (non-factorized) formulation, the static Speight and Ward model is not integrable and a two-point map that includes the integration constant as a parameter cannot be constructed for this model.

Based on the results of the present study one can separate the TI models into two classes. In the first class belong the models that support a two-dimensional space of TI static solutions. These solutions, if they are derivable from a two-point nonlinear map, can be parameterized by the points of the plane (C, ϕ_0), where C is the integration constant that can vary continuously within certain range and ϕ_0 is the initial value of the map that can also vary continuously. Alternatively, if the JEF solutions are known, then the solutions can be parameterized by the points of the plane (m, x_0) so that m plays the role of the integration constant C while variation of x_0 plays the role of ϕ_0, and results in the shift of the solution along the lattice. The second class is formed by the models that admit TI static solutions with an arbitrary shift along the lattice (controlled by either x_0 or ϕ_0) but corresponding solutions do not include the integration constant as a parameter.

The TI models in the first class have been investigated in [12, 14, 19] and four more TI cubic nonlinearities belonging to this class are found in the present work. In particular, TI I models are cases (i) to (vii) and Model 2. In this context it is worth noting that while a universal two-point map is known for Model 2 (for arbitrary γ and δ), no JEF or any other analytical solutions are known so far which can be characterized by C and ϕ_0. On the other hand, no universal two-point map is known for cases (iv) to (vii).

It is likely that the Speight and Ward model 7 [5] belongs to the second class of TI models because it supports the well-known TI kink and the TI sine solutions derived in the present work [see Eq. (82) and Eq. (97)] but these solutions are derived from reduced equations, as shown in Sec. V C. The reduced equations do not contain the integration constant. On the other hand, e.g., the five-periodic solution derived from the non-factorized model possesses the Peierls-Nabarro potential, as shown in Sec. VII A.

At this stage, it may be worthwhile classifying the various known models. As noted above, cases (i) to (vii) and Models 2 and 9 (and hence Models 3 to 6) are TI I models. On the other hand, Models 7, 8 and 10 are in general TI II models even though in the special cases of $\alpha = \beta = 0$ or $\alpha = 0, \beta = -1/2$, Model 8 becomes a TI I model.

Note that while a TI I model can also be a TI II model, the converse is obviously not true. For example, except for case (ii) when only A_4 is nonzero, all other TI I models also admit sine and staggered sine solutions. Besides, Model 2, Model 8 (in case $\gamma = 0$) and Model 10 (in case $\alpha_1 = \alpha_3$) also satisfy the four-periodic TI solution $(..., a, b, -a, -b, ...)$. At the special value of $\Lambda = 2$, such a TI solution is admitted by Model 9 (at $\beta = 0$) as well as by Models where only A_3, A_4, A_5 or only A_2, A_3, A_4, A_5 are nonzero.

It is thus clear that the general Model as given by Eq. (1) is only a TI II model even though in few special cases, it could be elevated to a TI I model. There is one model, however, which is not a TI model with respect to any known solution. We have in mind the model where only A_1 is nonzero, i.e. the ϕ^4 model with standard discretization. For example, while it admits the short-period solutions $(..., a, 0, -a, ...)$, $(..., a, 0, -a, 0, ...)$, $(..., a, a, -a, -a, ...)$, neither of them possesses the Goldstone mode.

Before closing, we spell out some of the open problems.

1. For the cases (iv) to (vii), can one obtain a unified general two-point map from which all solutions, including the JEF solutions can be derived? Note that, at the moment by starting from sn, cn, dn solutions, one can obtain three different maps from which only the respective solution can be obtained.

2. For Model 2, while a general two-point map is known, to date no analytic solution is known which is characterized by the two parameters C and ϕ_0. Can one find few such analytic solutions?

3. Can one rigorously show that all short period, sine and staggered sine solutions for any TI I or TI II model, follow from the lower order equations?

4. Can one rigorously prove that Model 7 of Speight and Ward is only a TI II and not a TI I model?

5. There is a belief that all TI models (at least TI I models) must have some conserved quantity. Unfortunately, for cases (iii) to (vii) (which are all TI I models) no such conserved quantity is known at present. Can one find such a quantity or disprove the conjecture?

6. While it has been demonstrated that no discrete model can simultaneously have conservation of P_1 and energy E, it is not known whether one can have a model where both P_2 and E can be simultaneously conserved. The obvious guess would be *no*. It would be nice to prove or disprove this conjecture.

7. Can one find particular TI solutions for the discrete models that do not belong to the TI I class, i.e., finding the isolated TI solutions to the discrete models that are not considered by many researchers as the TI models. In the present study we have given several examples of such solutions, for instance, four-periodic TI solution $(..., a, b, -a, -b, ...)$ and the TI trigonometric solutions given in Sec. III G. It can be so that isolated TI solutions exist for many discrete models. Many of the isolated TI solutions result from factorized static problems and thus, finding various factorizations of the original static problem can be a method for their derivation.

Finally, the results obtained in this paper are easily extended to the case of the general nonlinear Schrödinger equation [21]. We hope to address these issues in a forthcoming publication.

Acknowledgements

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IX. APPENDIX

In this Appendix, using the various short-period solutions given in Sec. IV, we spell out the solutions which are admitted by some of the Models discussed in the paper. We shall only mention the allowed solutions with the lowest period. Needless to say that the corresponding solutions with arbitrarily large period or aperiodic solutions (if they exist) will also be valid in that case. For example, in case only A_2 is nonzero, as shown below, the solution $\phi_n = (..., a, a, -a, ...)$ is allowed provided $\Lambda = 2$ and $a^2 = 1$. It is then clear from the discussion in Sec. IV that in that case a solution of the form $\phi_n = (..., a, -a, (a, -a \; p \; times), -a, ...)$ with period $2p + 1$ $(p \geq 1)$ as well as an *aperiodic solution* with any number of "a" and "-a" kept at random but with the constraint that at most two "a" or two "-a" are always together, is also an exact solution.

A. Model 1: Only A_1 nonzero, $A_1 h^2 = \Lambda$

This is the case of the classical discretization of ϕ^4 term. In this case the admitted solutions are
(i) $\phi_n = (..., a, -a, ...)$ in case $a^2 = \frac{\Lambda-4}{\lambda}$. Thus such a solution is valid if $\Lambda > 4$ or if $\Lambda < 0$.
(ii) $\phi_n = (..., a, -a, 0, ...)$ in case $a^2 = \frac{\Lambda-3}{\lambda}$. Thus such a solution is valid if $\Lambda > 3$ or if $\Lambda < 0$.
(iii) $\phi_n = (..., a, a, -a, -a, ...)$ and $\phi_n = (..., a, 0, -a, 0, ...)$ in case $a^2 = \frac{\Lambda-2}{\lambda}$. Thus such a solution is valid if $\Lambda > 2$ or if $\Lambda < 0$.
(iv) $\phi_n = (..., a, a, 0, -a, -a, 0, ...)$ in case $a^2 = \frac{\Lambda-1}{\lambda}$. Thus such a solution is valid if $\Lambda > 1$ or if $\Lambda < 0$.

B. Model 3: Only A_2 nonzero, $A_2 h^2 = \Lambda$

In this case the admitted solutions are
(i) $\phi_n = (..., a, -a, ...)$ in case $a^2 = \frac{4-\Lambda}{\lambda}$. Thus such a solution is valid if $0 < \Lambda < 4$.
(ii) $\phi_n = (..., a, -a, 0, ...)$ in case $a^2 = \frac{3-\Lambda}{\lambda}$. Thus such a solution is valid if $0 < \Lambda < 3$.
(iii) $\phi_n = (..., a, a, -a, -a, ...)$ and $\phi_n = (..., a, 0, -a, 0, ...)$ in case $\Lambda = 2$ while a is *any* real number.
(iv) $\phi_n = (..., a, b, -a, -b, ...)$ in case $\Lambda = 2$ while a, b both are *any* arbitrary real numbers.
(v) $\phi_n = (..., a, a, 0, -a, -a, 0, ...)$ in case $a^2 = \frac{\Lambda-1}{\lambda}$. Thus such a solution is valid if $\Lambda > 1$ or if $\Lambda < 0$.
(vi) In addition, there are a large number of solutions which are admitted in case $\Lambda = 2$ and $a^2 = 1$. These allowed solutions are

$$\begin{aligned}
&\phi_n = (..., a, a, -a, ...)\,,\ (..., a, a, a, -a, ...)\,,\ (..., a, 0, -a, a, ...)\,,\ (..., a, a, a, -a, -a, ...)\,,\\
&(..., a, -a, a, -a, 0, ...)\,,\ (..., a, a, -a, -a, 0, ...)\,,\ (..., a, a, 0, -a, 0, ...)\,,\ (..., a, 0, -a, a, a, ...)\,,\\
&(..., a, a, a, 0, -a, -a, ...)\,,\ (..., a, a, 0, -a, 0, a, ...)\,,\ (..., a, -a, a, a, -a, 0, ...)\,,\ (..., a, a, 0, -a, -a, 0, a, ...)\,,\\
&(..., a, 0, -a, a, 0, -a, 0, ...)\,,\ (..., a, -a, 0, a, a, 0-, a, ...)\,,\ (..., a, a, -a, a, a, -a, 0, ...)\,,\ (..., a, 0, -a, a, a, a, -a, ...)\,,\\
&(..., a, -a, 0, a, a, a, 0, -a, ...)\,,\ (..., a, a, 0, -a, -a, 0, a, -a, 0, ...)\,,\\
&(..., a, a, 0, -a, -a, 0, a, -a, 0, a, ...)\,,\ (..., a, -a, 0, a, a, a, 0, -a, a, a, -a, ...)\,. \qquad (205)
\end{aligned}$$

C. Model 6: Only A_4 nonzero, $A_4 h^2 = \Lambda$

In this case the admitted solutions are
(i) $\phi_n = (..., a, -a, ...)$ in case $a^2 = \frac{\Lambda-4}{\lambda}$. Thus such a solution is valid if either $\Lambda > 4$ or $\Lambda < 0$.
(ii) $\phi_n = (..., a, a, -a, -a, ...)$ in case $a^2 = \frac{2-\Lambda}{\lambda}$. Thus such a solution is valid if $0 < \Lambda < 2$.
(iii) $\phi_n = (..., a, 1/a, a, 1/a, ...)$ and $\phi_n = (..., a, 0, -a, 0, ...)$ in case $\Lambda = 2$ while a is *any* real number.
(iv) $\phi_n = (..., a, -a, 0, ...)$ in case $\Lambda = 3$ while a is *any* real number.
In addition, there are a large number of solutions which are admitted in case $\Lambda = 1$ and $a^2 = 1$. These allowed solutions are

$$\begin{aligned}
&\phi_n = (..., a, a, a, -a, -a, ...)\,,\ (..., a, a, -a, -a, 0, ...)\,,\ (..., a, a, 0, -a, -a, 0, ...)\,,\ (..., a, a, a, 0, -a, -a, ...)\,,\\
&(..., a, a, 0, -a, -a, 0, a, ...)\,. \qquad (206)
\end{aligned}$$

D. Case (iii): Only A_2, A_4 nonzero, $(A_2 + A_4)h^2 = \Lambda$

In this case the admitted solutions are
(i) $\phi_n = (..., a, -a, ...)$ in case $h^2 a^2 (A_4 - A_2) = \Lambda - 4$.
(ii) $\phi_n = (..., a, 1/a, a, 1/a, ...)$ in case $h^2 A_4 = 2\,,\ h^2 A_2 = \Lambda - 2$ while a is any real number.
(iii) $\phi_n = (..., a, -a, 0, ...)$ in case $a^2 h^2 A_2 = 2(3 - \Lambda)$.
(iv) $\phi_n = (..., a, a, -a, -a, ...)$ in case $a^2 h^2 A_4 = 2 - \Lambda$.
(v) $\phi_n = (..., a, 0, -a, 0, ...)$ in case $\Lambda = 2$ while A_2, A_4 as well as a are arbitrary real numbers.
(vi) $\phi_n = (..., a, a, -a, -a, 0, ...)$ in case $a^2 h^2 A_2 = 2\Lambda - 2$.
(vii) $\phi_n = (..., a, a, 0, -a, 0, ...)\,,\ (..., a, 0, -a, a, 0, -a, 0, ...)\,,$ $(..., a, a, 0, -a, -a, 0, a, -a, 0, ...)$, in case $\Lambda = 2, h^2 a^2 A_2 = 2$.
(viii) $\phi_n = (..., a, a, a, -a, -a, ...)\,,\ (..., a, a, -a, -a, 0, ...)\,,\ (..., a, a, 0, -a, -a, 0, a, ...)$, in case $a^2 = 1, h^2 A_2 = 2\Lambda - 2, h^2 a^2 A_4 = 2 - \Lambda$.

(ix) $\phi_n = (..., a, a, -a, ...)\,,\ (..., a, -a, a, -a, 0, ...)\,,\ (..., a, -a, a, a, -a, 0, ...)\,,$ in case

$$a^2 = \frac{8-3\Lambda}{\Lambda}\,,\quad h^2 A_2 = \frac{2\Lambda(3-\Lambda)}{8-3\Lambda}\,,\quad h^2 A_4 = \frac{\Lambda(2-\Lambda)}{8-3\Lambda}\,. \tag{207}$$

Thus this solution is valid provided $0 < \Lambda < 8/3$.

E. Case (iv): Only A_3, A_5 nonzero, $(A_3 + A_5)h^2 = \Lambda$

In this case the admitted solutions are
(i) $\phi_n = (..., a, -a, ...)$ in case $h^2a^2(A_3 - A_5) = \Lambda - 4$.
(ii) $\phi_n = (..., a, 0, ...)$ in case $a^2h^2A_5 = 2\,, \Lambda = 2$.
(iii) $\phi_n = (..., a, -a, 0, ...)$ in case $a^2h^2A_3 = 2(\Lambda - 3)$.
(iv) $\phi_n = (..., a, a, 0, ...)$ in case $a^2 = 1, h^2A_3 = \Lambda - 1, h^2A_5 = 1$. (v) $\phi_n = (..., a, 0, -a, 0, ...)$ in case $\Lambda = 2$ and a is any real number.
(vi) $\phi_n = (..., a, a, -a, -a, ...)$ in case $a^2h^2A_3 = \Lambda - 2$.
(vii) $\phi_n = (..., a, a, 0, -a, 0, ...)$ in case $\Lambda = 2, a^2h^2A_3 = 2$.
(viii) $\phi_n = (..., a, a, 0, -a, -a, 0, ...)$ in case $a^2h^2A_3 = 2\Lambda - 2$.
(ix) $\phi_n = (..., a, a, a, 0, -a, -a, 0, ...)$ in case $a^2 = 1, h^2A_3 = 2(\Lambda - 1), h^2A_5 = 2 - \Lambda$.
(x) $\phi_n = (..., a, 0, -a, a, 0, -a, 0, ...)$ in case $\Lambda = 2, a^2h^2A_3 = -2$.
(xi) $\phi_n = (..., a, -a, 0, -a, a, 0, ...)$ in case

$$a^2 = \frac{2(\Lambda-2)}{\Lambda}\,,\ h^2 A_3 = \frac{\Lambda(\Lambda-3)}{\Lambda-2}\,,\quad h^2 A_5 = \frac{\Lambda}{\Lambda-2}\,. \tag{208}$$

Thus this solution is valid provided either $\Lambda > 2$ or $\Lambda < 0$.
(xii) $\phi_n = (..., a, -a, a, -a, 0, ...)$ in case

$$a^2 = \frac{3\Lambda-8}{\Lambda}\,,\ h^2 A_3 = \frac{2\Lambda(\Lambda-3)}{3\Lambda-8}\,,\quad h^2 A_5 = \frac{\Lambda(\Lambda-2)}{3\Lambda-8}\,. \tag{209}$$

Thus this solution is valid provided either $\Lambda > 8/3$ or $\Lambda < 0$.
(xiii) $\phi_n = (..., a, a, -a, ...)\,,\ (..., a, a, a, -a, ...)\,,\ (..., a, a, a, -a, -a, ...)\,,$ in case $a^2 = 1, h^2A_3 = \Lambda - 2, h^2a^2A_5 = 2$.
(xiv) In addition, there are a large number of solutions which are admitted in case $A_3 = A_5, \Lambda = 4, a^2 = 1$. These allowed solutions are

$$\phi_n = (..., a, -a, 0, a, ...)\,,\ (..., a, a, -a, 0, -a, ...)\,,\ (..., a, -a, a, a, -a, 0, ...)\,,\ (..., a, a, -a, 0, -a, a, ...)\,,\\ (..., a, 0, -a, a, a, a, -a, ...)\,,\ (..., a, a, -a, a, a, -a, 0, a, ...)\,. \tag{210}$$

F. SW Model 7: $A_1 = \frac{2\lambda}{9}, A_2 = A_3 = \frac{\lambda}{3}, A_4 = A_5 = 0, A_6 = \frac{\lambda}{9}$

In this case the admitted solutions are
(i) $\phi_n = (..., a, -a, ...)$ in case $a^2 = \frac{9(\Lambda-4)}{\lambda}$. Thus such a solution is valid if either $\Lambda > 4$ or $\Lambda < 0$.
(ii) $\phi_n = (..., a, -a, 0, ...)$ in case $a^2 = \frac{6(\Lambda-3)}{\lambda}$. Thus such a solution is valid if either $\Lambda > 3$ or $\Lambda < 0$.
(iii) $\phi_n = (..., a, a, -a, -a, ...)$ in case $a^2 = \frac{9(\Lambda-2)}{5\lambda}$. Thus such a solution is valid if either $\Lambda > 2$ or $\Lambda < 0$.
(iv) $\phi_n = (..., a, 0, -a, 0, ...)$ in case $a^2 = \frac{9(\Lambda-2)}{2\lambda}$. Thus such a solution is valid if either $\Lambda > 2$ or $\Lambda < 0$.
(v) $\phi_n = (..., a, a, 0, -a, -a, 0, ...)$ in case $a^2 = \frac{18(\Lambda-1)}{11\lambda}$. Thus such a solution is valid if either $\Lambda > 1$ or $\Lambda < 0$.
(vi) $\phi_n = (..., a, a, 0, -a, 0, ...), (..., a, a, 0, -a, 0, a, ...)$ in case $a^2 = 1, \Lambda = 18/7$.
(vii) $\phi_n = (..., a, a, 0, ...), (..., a, a, -a, -a, 0, ...)$ in case $a^2 = 3/2, \Lambda = 12$.
(viii) $\phi_n = (..., a, a, -a, ...), (..., a, a, a, -a, -a, ...)$ in case $a^2 = 1, \Lambda = 9/2$.
(ix) $\phi_n = (..., a, a, 0, -a, -a, 0, a, -a, 0, ...)$ in case $a^2 = 6/5, \Lambda = 15/4$.
(x) In addition, there are a large number of solutions which are admitted in case $\Lambda = 6$ and $a^2 = 3$. These allowed solutions are

$$\phi_n = (..., a, 0, ...)\,,\ (..., a, -a, a, -a, 0, ...)\,,\ (..., a, 0, a, 0, -a, ...)\,,\ (..., a, -a, 0, -a, a, 0, ...)\,,\\ (..., a, 0, a, 0, a, -a, ...)\,, (..., a, 0, -a, a, 0, -a, 0, ...)\,. \tag{211}$$

G. Model 2 with $\delta = 0, \gamma = 1/4$: Only A_5, A_6 nonzero with $A_5h^2 = A_6h^2 = \Lambda/2$

This is same as the Model 4 due to Kevrekidis. In this case the admitted solutions are
(i) $\phi_n = (..., a, -a, ...)$ in case $a^2 = \frac{4-\Lambda}{\lambda}$. Thus such a solution is valid if $0 < \Lambda < 4$.
(ii) $\phi_n = (..., a, -a, 0, ...)$ in case $a^2 = \frac{4(3-\Lambda)}{\lambda}$. Thus such a solution is valid if $0 < \Lambda < 3$.
(iii) $\phi_n = (..., a, a, -a, -a, ...)$ and $\phi_n = (..., a, 0, -a, 0, ...)$ in case $\Lambda = 2$ while a is *any* real number.
(iv) $\phi_n = (..., a, b, -a, -b, ...)$ in case $\Lambda = 2$ while a, b both are *any* arbitrary real numbers.
(v) $\phi_n = (..., a, a, 0, -a, -a, 0, ...)$ in case $a^2 = \frac{4(\Lambda-1)}{\lambda}$. Thus such a solution is valid if $\Lambda > 1$ or if $\Lambda < 0$.
(vi) In addition, there are a large number of solutions which are admitted in case $\Lambda = 2$ and $a^2 = 1$. These allowed solutions are

$$\begin{aligned}
&\phi_n = (..., a, a, -a, ...)\,,\ (..., a, a, a, -a, ...)\,,\ (..., a, 0, -a, a, ...)\,,\ (..., a, a, a, -a, -a, ...)\,, \\
&(..., a, -a, a, -a, 0, ...)\,,\ (..., a, a, -a, -a, 0, ...)\,,\ (..., a, a, 0, -a, 0, ...)\,,\ (..., a, 0, -a, a, a, ...)\,, \\
&(..., a, a, a, 0, -a, -a, ...)\,,\ (..., a, a, 0, -a, 0, a, ...)\,,\ (..., a, -a, a, a, -a, 0, ...)\,,\ (..., a, a, 0, -a, -a, 0, a, ...)\,, \\
&(..., a, 0, -a, a, 0, -a, 0, ...)\,,\ (..., a, -a, 0, a, a, 0-, a, ...)\,,\ (..., a, a, -a, a, a, -a, 0, ...)\,,\ (..., a, 0, -a, a, a, a, -a, ...)\,, \\
&(..., a, -a, 0, a, a, a, 0, -a, ...)\,,\ (..., a, a, 0, -a, -a, 0, a, -a, 0, ...)\,, \\
&(..., a, a, 0, -a, -a, 0, a, -a, 0, a, ...)\,,\ (..., a, -a, 0, a, a, a, 0, -a, a, a, -a, ...)\,,
\end{aligned} \tag{212}$$

H. Model 2 with $\delta = 1/4, \gamma = 0$: Only A_1, A_3, A_4 nonzero with $A_1h^2 = A_4h^2 = A_3h^2/2 = \Lambda/4$

In this case the admitted solutions are
(i) $\phi_n = (..., a, -a, ...)$ in case $a^2 = \frac{\Lambda-4}{\lambda}$. Thus such a solution is valid if either $\Lambda > 4$ or $\lambda < 0$.
(ii) $\phi_n = (..., a, -a, 0, ...)$ in case $a^2 = \frac{2(\Lambda-3)}{\lambda}$. Thus such a solution is valid if either $\Lambda > 3$ or $\Lambda < 0$.
(iii) $\phi_n = (..., a, b, -a, -b, ...)$ in case $(a^2 + b^2) = \frac{2(\Lambda-2)}{\Lambda} = 2$.
(iv) $\phi_n = (..., a, a, a, -a, ...)$ in case $a^2 = 1, \Lambda = 4$.
(v) $\phi_n = (..., a, a, 0, -a, 0, ...)$ in case $a^2 = 4/3, \Lambda = 3$.
(vi) $\phi_n = (..., a, a, 0, -a, -a, 0, a, ...)$ in case $a^2 = 1, \Lambda = 2$.
(vii) In addition, there are a large number of solutions which are admitted in case $a^2 = \frac{2(\Lambda-1)}{\Lambda}$. These allowed solutions are

$$\phi_n = (..., a, a, -a, ...)\,,\ (..., a, a, -a, -a, ...)\,,\ (..., a, 0, -a, 0, ...)\,,\ (..., a, a, 0, -a, -a, 0, ...)\,. \tag{213}$$

I. Model 2 with arbitrary δ, γ: $A_1 = A_4 = A_3/2 = \delta\lambda, A_5 = A_6 = 2\gamma\lambda, A_2 = (1 - 4\delta - 4\gamma)\lambda$

This is the full Model 2. Note that we have already obtained solutions in three special cases. In this general case, many more solutions are admitted. The admitted solutions are
(i) $\phi_n = (..., a, -a, ...)$ in case $a^2 = \frac{\Lambda-4}{(8\delta-1)\Lambda}$. Thus such a solution is valid if either $\Lambda > 4$ or $\lambda < 0$.
(ii) $\phi_n = (..., a, 0, ...)$ in case

$$a^2 = \frac{\Lambda - 2}{\delta\Lambda}\,, \quad 2\gamma\Lambda a^2 = 1\,. \tag{214}$$

(iii) $\phi_n = (..., a, -a, 0, ...)$ in case

$$a^2 = \frac{2(\Lambda - 3)}{(8\delta + 2\gamma - 1)\Lambda}\,. \tag{215}$$

(iv) $\phi_n = (..., a, a, -a, ...)$ in case

$$a^2 = \frac{3\Lambda - 4}{\Lambda}\,, \quad \delta = \frac{\Lambda - 2}{2(3\Lambda - 4)}\,. \tag{216}$$

(v) $\phi_n = (..., a, a, 0, ...)$ in case

$$a^2 = \frac{1}{2\gamma\Lambda} = \frac{2}{1 + 2\delta}\,. \tag{217}$$

(vi) $\phi_n = (..., a, a, a, 0, ...)$ in case

$$a^2 = 1\,, \quad \delta = \frac{1}{2}\,, \gamma = \frac{1}{2\Lambda}\,. \tag{218}$$

(vii) $\phi_n = (..., a, a, a, -a, ...)$ in case

$$a^2 = 1\,, \quad \delta = 0\,, \Lambda = 2\,. \tag{219}$$

(viii) $\phi_n = (..., a, b, -a, -b, ...)$ in case

$$(a^2 + b^2) = \frac{\Lambda - 2}{\delta\Lambda}\,. \tag{220}$$

(ix) $\phi_n = (..., a, a, -a, -a, ...)$ in case

$$a^2 = \frac{\Lambda - 2}{2\delta\Lambda}\,. \tag{221}$$

(x) $\phi_n = (..., a, 0, -a, 0, ...)$ in case

$$a^2 = \frac{\Lambda - 2}{\delta\Lambda}\,. \tag{222}$$

(xi) $\phi_n = (..., a, -a, 0, a, ...)$ in case

$$a^2 = \frac{1}{6\gamma}\,, \quad \Lambda = 3\,, \quad 8\delta + 2\gamma = 1\,. \tag{223}$$

(xii) $\phi_n = (..., a, a, -a, -a, ...)$ in case

$$a^2 = 1\,, \quad \delta = \frac{\Lambda - 2}{2\Lambda}\,. \tag{224}$$

(xiii) $\phi_n = (..., a, -a, 0, a, ...)$ in case

$$a^2 = \frac{1}{6\gamma}\,, \quad \Lambda = 3\,, \quad 8\delta + 2\gamma = 1\,. \tag{225}$$

(xiv) $\phi_n = (..., 0, a, 0, a, a, ...)$ in case

$$a^2 = \frac{\Lambda + 2}{\Lambda}\,, \quad \delta = \frac{\Lambda - 2}{2(\Lambda + 2)}\,, \quad \gamma = \frac{1}{2(\Lambda + 2)}\,. \tag{226}$$

(xv) $\phi_n = (..., a, -a, a, -a, 0, ...)$ in case

$$a^2 = \frac{\Lambda - 2}{2\gamma\Lambda}\,, \quad (8\delta - 1)(\Lambda - 2) = 2\gamma(\Lambda - 4)\,. \tag{227}$$

(xvi) $\phi_n = (..., a, 0, a, 0, -a, ...)$ in case

$$a^2 = \frac{3(2\Lambda - 3)}{\Lambda}\,, \quad \delta = \frac{\Lambda - 2}{3(2\Lambda - 3)}\,, \quad \gamma = \frac{1}{6(2\Lambda - 3)}\,. \tag{228}$$

(xvii) $\phi_n = (..., a, a, -a, -a, 0, ...)$ in case

$$a^2 = \frac{(\Lambda - 2)}{2\delta\Lambda} = \frac{1}{1 - 4\gamma}\,. \tag{229}$$

(xviii) $\phi_n = (..., a, a, 0, -a, 0, ...)$ in case

$$a^2 = \frac{(\Lambda - 2)}{2\delta\Lambda} = \frac{2}{\Lambda(1 - 4\gamma)}\,. \tag{230}$$

(xix) $\phi_n = (..., a, -a, 0, -a, 0, ...)$ in case

$$a^2 = \frac{(2\Lambda - 1)}{\Lambda}\,, \quad \delta = \frac{\Lambda - 2}{2(2\Lambda - 1)}\,, \quad \gamma = \frac{1}{2(2\Lambda - 1)}\,. \tag{231}$$

(xx) $\phi_n = (..., a, a, 0, a, -a, ...)$ in case

$$a^2 = \frac{5}{3}\,, \quad \Lambda = 3\,, \quad \delta = \gamma = \frac{1}{10}\,. \tag{232}$$

(xxi) $\phi_n = (..., a, a, 0, -a, -a, 0, ...)$ in case

$$a^2 = \frac{2(\Lambda - 1)}{(1 + 2\delta - 4\gamma)\Lambda}\,. \tag{233}$$

(xxii) $\phi_n = (..., a, -a, 0, -a, a, 0, ...)$ in case

$$a^2 = \frac{1}{2\gamma\Lambda}\,, \quad (8\delta - 1) = 2\gamma(2\Lambda - 7)\,. \tag{234}$$

(xxiii) $\phi_n = (..., a, a, a, 0, -a, -a, ...)$ in case

$$a^2 = 1\,, \quad \gamma = 0\,, \quad \delta = \frac{\Lambda - 2}{\Lambda}\,. \tag{235}$$

(xxiv) $\phi_n = (..., a, a, 0, -a, 0, a, ...)$ in case

$$a^2 = 1\,, \quad \gamma = \frac{\Lambda - 2}{4\Lambda}\,, \quad \delta = \frac{\Lambda - 2}{\Lambda}\,. \tag{236}$$

(xxv) $\phi_n = (..., a, a, 0, a, -a, 0, ...)$ in case

$$a^2 = \frac{(6\Lambda + 7)}{5\Lambda}\,, \quad \delta = \frac{4\Lambda - 7}{2(6\Lambda + 7)}\,, \quad \gamma = \frac{5}{2(6\Lambda + 7)}\,. \tag{237}$$

(xxvi) $\phi_n = (..., a, a, -a, 0, -a, a, ...)$ in case

$$a^2 = 1\,, \quad \Lambda = 1\,, \quad \gamma = \frac{1}{2}\,, \quad \delta = \frac{-1}{2}\,. \tag{238}$$

(xxvii) $\phi_n = (..., a, 0, a, 0, a, -a, ...)$ in case

$$a^2 = 3\,, \quad \Lambda = 3\,, \quad \gamma = \frac{1}{18}\,, \quad \delta = \frac{1}{9}\,. \tag{239}$$

(xxviii) $\phi_n = (..., a, 0, a, 0, a, a, ...)$ in case

$$a^2 = 1\,, \quad \Lambda = 2\,, \quad \gamma = \frac{1}{4}\,, \quad \delta = 0\,. \tag{240}$$

(xxix) $\phi_n = (..., a, -a, a, a, 0, a, ...)$ and $(..., a, a, 0, a, a, 0, -a, ...)$ in case

$$a^2 = \frac{5}{3}\,, \quad \Lambda = 3\,, \quad \gamma = \delta = \frac{1}{10}\,. \tag{241}$$

(xxx) $\phi_n = (..., a, a, 0, -a, -a, 0, a, ...)$ in case

$$a^2 = 1\,, \; \delta - 2\gamma = \frac{\Lambda - 2}{2\Lambda}\,. \tag{242}$$

(xxxi) $\phi_n = (..., a, 0, -a, a, 0, -a, 0, ...)$ in case

$$a^2 = \frac{\Lambda - 2}{\delta\Lambda}\,, \quad 2\delta(5 - 3\Lambda) = (2\gamma - 1)(\Lambda - 2)\,. \tag{243}$$

(xxxii) $\phi_n = (..., a, a, 0, -a, -a, a, -a, ...)$ in case

$$a^2 = \frac{(7\Lambda - 12)}{\Lambda}\,, \quad \delta = \frac{\Lambda - 2}{2(7\Lambda - 12)}\,, \quad \gamma = \frac{7(\Lambda - 2)}{4(7\Lambda - 12)}\,. \tag{244}$$

(xxxiii) In case

$$a^2 = 1\,,\ \delta = \gamma = 0\,, \quad \Lambda = 2\,, \tag{245}$$

then the following solutions are allowed

$$\phi_n = (..., a, 0, -a, a, a, ...), (..., a, 0, -a, a, a, a, -a, ...), (..., a, -a, 0, a, a, a, 0, -a, ...), (..., a, 0, a, 0, a, a, a, -a, ...)\,,$$
$$(..., a, -a, 0, a, 0, a, a, 0, -a, ...), (..., a, -a, 0, a, 0, a, a, a, 0, -a, ...), (..., a, -a, 0, a, a, a, 0, -a, a, a, -a, ...)\,. \tag{246}$$

(xxxiv) $\phi_n = (..., a, a, 0, a, 0, -a, -a, ...)$ in case

$$a^2 = 2\,,\ \delta = 0\,, \quad \gamma = \frac{1}{4\Lambda}\,. \tag{247}$$

(xxxv) $\phi_n = (..., a, a, -a, 0, -a, 0, -a, ...)$ in case

$$a^2 = \frac{5 - 2\Lambda}{\Lambda}\,,\ \delta = 0\,, \quad \gamma = \frac{1}{2(5 - 2\Lambda)}\,. \tag{248}$$

(xxxvi) $\phi_n = (..., a, -a, a, a, a, 0, -a, ...)$ in case

$$a^2 = 1\,,\ \delta = 0\,, \quad \gamma = \frac{1}{4}\,, \quad \Lambda = 2\,. \tag{249}$$

(xxxvii) $\phi_n = (..., a, a, 0, -a, -a, 0, a, -a, 0, ...)$ in case

$$a^2 = \frac{2}{\Lambda(1 - 3\delta - 3\gamma)}\,, \quad 6(1 - 3\gamma) = (3\Lambda - 1)(1 - 3\delta - 3\gamma)\,. \tag{250}$$

(xxxviii) $\phi_n = (..., a, a, 0, -a, -a, 0, a, -a, 0, a, ...)$ in case

$$a^2 = 1\,,\ \delta = \frac{7(\Lambda - 2)}{18\Lambda}\,, \quad \gamma = \frac{1}{9\Lambda}\,. \tag{251}$$

(xxxiv) In case

$$a^2 = \frac{3\Lambda - 4}{\Lambda}\,,\ \delta = \gamma = \frac{\Lambda - 2}{2(3\Lambda - 4)}\,, \tag{252}$$

then the following solutions are allowed

$$\phi_n = (..., a, 0, -a, a, ...), (..., a, -a, a, a, -a, 0, ...), (..., a, -a, 0, a, a, 0, -a, ...), (..., a, a, -a, a, a, -a, 0, ...)\,. \tag{253}$$

J. Case (v): Only A_3, A_4, A_5 nonzero, $(A_3 + A_4 + A_5)h^2 = \Lambda$

In this case the admitted solutions are
(i) $\phi_n = (..., a, -a, ...)$ in case $h^2a^2(A_3 + A_4 - A_5) = \Lambda - 4$.
(ii) $\phi_n = (..., a, 0, ...)$ in case $a^2h^2A_5 = 2\,, \Lambda = 2$.
(iii) $\phi_n = (..., a, -a, 0, ...)$ in case $a^2h^2A_3 = 2(\Lambda - 3)$.
(iv) $\phi_n = (..., a, a, -a, ...)$ in case $a^2h^2(A_3 - A_4) = (\Lambda - 2), a^2h^2(H + F - J) = \Lambda - 4$.
(v) $\phi_n = (..., a, a, 0, ...)$ in case $a^2h^2A_3 = 2(\Lambda - 1), a^2h^2A_5 = 2$.
(vi) $\phi_n = (..., a, 0, -a, 0, ...)$ in case $\Lambda = 2$ and a is any real number.
(vii) $\phi_n = (..., a, a, -a, -a, ...)$ in case $a^2h^2A_3 = \Lambda - 2$.
(viii) $\phi_n = (..., a, a, a, 0, ...)$ in case $a^2 = 1, h^2A_3 = 2(\Lambda - 1)\,, h^2A_5 = 2, h^2A_4 = -\Lambda$.
(ix) $\phi_n = (..., a, -a, 0, a, ...)$ in case $a^2 = 1, h^2A_3 = 2(\Lambda - 3), h^2A_4 = 4 - \Lambda, h^2A_5 = 2$.
(x) $\phi_n = (..., a, b, -a, -b, ...)$ in case $A_3 = A_4, \Lambda = 2$ and a, b are arbitrary real numbers.

(xi) $\phi_n = (..., a, a, 0, -a, 0, ...)$ in case $\Lambda = 2, h^2a^2A_3 = 2$.
(xii) $\phi_n = (..., a, -a, a, -a, 0, ...)$ in case $h^2a^2(A_3 + A_4 - A_5) = \Lambda - 4, h^2a^2A_3 = 2(\Lambda - 3)$.
(xiii) $\phi_n = (..., a, a, a, -a, -a, ...)$ in case $a^2 = 1, h^2(A_3 - A_4) = \Lambda - 2$.
(xiv) $\phi_n = (..., 0, a, 0, a, a, ...)$ in case $A_3 = A_5, \Lambda = 2, h^2a^2A_5 = 2$.
(xv) $\phi_n = (..., a, 0, a, 0, -a, ...)$ in case $\Lambda = 2, A_3 = -A_5, h^2A_4 = 2, h^2a^2A_5 = 2$.
(xvi) $\phi_n = (..., a, a, -a, -a, 0, ...)$ in case $h^2a^2A_4 = \Lambda, h^2a^2A_3 = 2(\Lambda - 1)$.
(xvii) $\phi_n = (..., a, -a, 0, -a, a, ...)$ in case

$$a^2 = \frac{3\Lambda - 8}{\Lambda}\,, \;\; h^2A_3 = \frac{2\Lambda(\Lambda - 3)}{3\Lambda - 8}\,, \quad h^2A_4 = \frac{\Lambda(\Lambda - 4)}{3\Lambda - 8}\,, \quad h^2A_5 = \frac{2\Lambda}{3\Lambda - 8}\,. \tag{254}$$

(xviii) $\phi_n = (..., a, a, 0, -a, -a, 0, ...)$ in case $a^2h^2A_3 = 2(\Lambda - 1)$.
(xix) $\phi_n = (..., a, -a, 0, -a, a, 0, ...)$ in case $a^2h^2A_3 = 2(\Lambda - 3), h^2a^2A_5 = 2$.
(xx) $\phi_n = (..., a, a, a, 0, -a, -a, ...)$ in case $a^2 = 1, A_3 = -A_5, h^2A_3 = 2(\Lambda - 1), h^2A_4 = \Lambda$.
(xxi) $\phi_n = (..., a, a, 0, -a, 0, a, ...)$ in case $a^2 = 1, \Lambda = 2, h^2A_3 = 2$.
(xxii) $\phi_n = (..., a, -a, a, a, -a, 0, ...)$ in case

$$a^2 = \frac{5\Lambda - 16}{\Lambda}\,, \quad A_3 = A_5\,, \quad h^2A_3 = \frac{2\Lambda(\Lambda - 3)}{5\Lambda - 16}\,, \quad h^2A_4 = \frac{\Lambda(\Lambda - 4)}{5\Lambda - 16}\,. \tag{255}$$

(xxiii) $\phi_n = (..., a, a, 0, -a, -a, 0, a, ...)$ in case $a^2 = 1, h^2A_3 = 2(\Lambda - 1)$.
(xxiv) $\phi_n = (..., a, 0, -a, a, 0, -a, 0, ...)$ in case $\Lambda = 2, a^2h^2A_3 = -2$.
(xxv) $\phi_n = (..., a, a, 0, -a, -a, a, -a, ...)$ in case $\Lambda = 2, h^2a^2A_3 = 2\,, h^2a^2(A_4 - A_5) = -4$.
(xxvi) $\phi_n = (..., a, a, 0, a, 0, -a, -a, ...)$ in case $\Lambda = 2, a^2 = 3, h^2A_3 = h^2A_4 = h^2A_5 = 2/3$.
(xxvii) $\phi_n = (..., a, -a, 0, a, 0, a, a, a, 0, -a, ...)$ in case $\Lambda = 2, h^2a^2A_3 = 2, A_5 = 2A_3 + A_4$.

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