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Connecting Jacobi elliptic functions with different modulus parameters

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Abstract. The simplest formulas connecting Jacobi elliptic functions with different modulus parameters were first obtained over two hundred years ago by John Landen. His approach was to change integration variables in elliptic integrals. We show that Landen's formulas and their subsequent generalizations can also be obtained from a different approach, using which we also obtain several new Landen transformations. Our new method is based on recently obtained periodic solutions of physically interesting nonlinear differential equations and remarkable new cyclic identities involving Jacobi elliptic functions.

Keywords. Landen transformations; Jacobi elliptic functions; cyclic identities.

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1. Introduction

The properties of Jacobi elliptic functions have been studied for over two hundred years and are listed in many texts [1,2] as well as standard mathematics reference books [3,4]. The three Jacobi elliptic functions $\mathrm{sn}(x,m)$, $\mathrm{cn}(x,m)$, $\mathrm{dn}(x,m)$ are functions of the variable x and the elliptic modulus parameter $m \equiv k^2$ $(0 \le m \le 1)$. The inverse functions are most easily defined in terms of elliptic integrals. For example,

$$\mathrm{sn}^{-1}(x,m) = \int_0^x \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-mz^2)}}\ , \tag{1}$$

with similar relations for $\mathrm{cn}^{-1}(x,m)$ and $\mathrm{dn}^{-1}(x,m)$. The Jacobi elliptic functions are all doubly periodic functions – the periods of $\mathrm{sn}(x,m)$, $\mathrm{cn}(x,m)$, $\mathrm{dn}(x,m)$ are $[4K(m), i2K'(m)]$, $[4K(m), 2K(m)+i2K'(m)]$, $[2K(m), i4K'(m)]$ respectively, where $K(m) \equiv \int_0^{\pi/2} \mathrm{d}\theta [1-m\sin^2\theta]^{-1/2}$ denotes the complete elliptic integral of the first kind, and $K'(m) \equiv K(1-m)$. Most properties of Jacobi elliptic functions found in the literature do not involve any change in the modulus parameter m. For example, the quadratic relations

$$\mathrm{cn}^2(x,m) = 1 - \mathrm{sn}^2(x,m), \quad \mathrm{dn}^2(x,m) = 1 - m\mathrm{sn}^2(x,m), \tag{2}$$

and the addition formulas for $\mathrm{sn}(x+y,m)$, $\mathrm{cn}(x+y,m)$, $\mathrm{dn}(x+y,m)$ just involve Jacobi elliptic functions with the same modulus parameter [3,4]. The exceptions are the Landen transformation formulas [2,5]

$$\mathrm{dn}\left[(1+k')u, \left(\frac{1-k'}{1+k'}\right)^2\right] = \frac{1-(1-k')\,\mathrm{sn}^2(u,m)}{\mathrm{dn}(u,m)}, \tag{3}$$

$$\mathrm{cn}\left[(1+k')u, \left(\frac{1-k'}{1+k'}\right)^2\right] = \frac{1-(1+k')\,\mathrm{sn}^2(u,m)}{\mathrm{dn}(u,m)}, \tag{4}$$

$$\mathrm{sn}\left[(1+k')u, \left(\frac{1-k'}{1+k'}\right)^2\right] = \frac{(1+k')\,\mathrm{sn}(u,m)\,\mathrm{cn}(u,m)}{\mathrm{dn}(u,m)}, \tag{5}$$

where $k' \equiv \sqrt{1-m}$. These results are obtained by making a change of integration variables [2] in the elliptic integrals in eqs (1)–(3). In the Landen formulas, the right-hand side involves Jacobi elliptic functions with modulus parameter m, whereas the Jacobi functions on the left-hand side have a smaller modulus parameter $\tilde{m} \equiv (1-\sqrt{1-m})^2/(1+\sqrt{1-m})^2$ which also lies in the range $0 \le \tilde{m} \le 1$. There are also similar formulas, known as the Gauss transformations [3,4], where the transformed parameter $\tilde{m}_{\mathrm{G}} = 4\sqrt{m}/(1+\sqrt{m})^2$ is always greater than m, and which are obtained by a different change of integration variables [2].

The Landen transformation formulas given above are said to be of order two, since there are two terms on the right-hand side. They show no obvious pattern, but a detailed study of modular transformations does permit generalizations of these formulas to arbitrary order p [1]. In this paper, we obtain these generalized transformations by an entirely different method. In particular, we have recently shown [6,7] that a specific limited superposition principle works for several non-linear problems like $\lambda\phi^4$ theory and for numerous non-linear equations of physical interest (Korteweg de Vries equation, modified Korteweg de Vries equation, non-linear Schrödinger equation, sine-Gordon equation). Using the idea of superposition, we have obtained seemingly new periodic solutions of these non-linear problems in terms of Jacobi elliptic functions. Here, we will argue on both mathematical and physical grounds that these solutions cannot really be new, but must be re-expressions of known periodic solutions [8]. In the process of proving this, we discover and derive the generalized Landen transformations and also obtain some new Landen formulas.

As an illustration, let us first focus on eq. (3). Using the identity (2), and changing variables to $x = (1+k')u$, one can re-write the order two 'dn' Landen formula (3) in the alternative form

$$\mathrm{dn}\left[x, \left(\frac{1-k'}{1+k'}\right)^2\right] = \alpha \left\{\mathrm{dn}\left[\alpha x, m\right] + \mathrm{dn}\left[\alpha x + K(m), m\right]\right\},$$
$$\alpha \equiv \frac{1}{(1+k')}. \tag{6}$$

Here, the right-hand side contains the sum of two terms with arguments separated by $K(m)$ [3,4]. As mentioned above, the generalized Landen formulas will have not two but p terms on the right-hand side. We will show that the generalization of eq. (3) [or equivalently eq. (6)], valid for any integer p is given by

$$\mathrm{dn}(x, \tilde{m}) = \alpha \sum_{j=1}^{p} \mathrm{dn}[\alpha x + 2(j-1)K(m)/p, m] \ , \tag{7}$$

where

$$\alpha \equiv \left\{\sum_{j=1}^{p} \mathrm{dn}[2(j-1)K(m)/p, m]\right\}^{-1} , \tag{8}$$

and

$$\tilde{m} = (m-2)\alpha^2 + 2\alpha^3 \sum_{j=1}^{p} \mathrm{dn}^3[2(j-1)K(m)/p, m] \ . \tag{9}$$

However, if one starts from the 'cn' or 'sn' Landen formulas of order 2 as given by eqs (4) and (5), the generalization to arbitrary order p is different depending on whether p is even or odd. The main results which we obtain are

$$\mathrm{cn}(x, \tilde{m}) \propto \sum_{j=1}^{p} \mathrm{cn}(\tilde{x}_j, m), \quad p \text{ odd},$$
$$\propto \sum_{j=1}^{p} (-1)^{j-1} \mathrm{dn}(x_j, m), \quad p \text{ even}, \tag{10}$$

$$\mathrm{sn}(x, \tilde{m}) \propto \sum_{j=1}^{p} \mathrm{sn}(\tilde{x}_j, m), \quad p \text{ odd},$$
$$\propto \prod_{j=1}^{p} \mathrm{sn}(x_j, m), \quad p \text{ even}, \tag{11}$$

where $\tilde{m}$ is as given in eq. (9), and we have used the notation

$$x_j \equiv \alpha x + 2(j-1)K(m)/p, \quad \tilde{x}_j \equiv \alpha x + 4(j-1)K(m)/p. \tag{12}$$

Note that all the above formulas have the same non-trivial scaling factor α of the argument x, as well as real shifts which are fractions of the periods of Jacobi elliptic functions. The richness of the generalized results is noteworthy and reflects the many different forms of periodic solutions for non-linear equations which we have recently obtained [6,7]. Some formulas involve the sum of p terms, the even p 'cn' formula involves alternating + and − signs, and the even p 'sn' formula has a product of p terms. In fact, there are also several interesting additional alternative forms for the above results which follow from use of several identities involving Jacobi elliptic functions which we have recently discovered [9–11]. For instance, we will later show that for odd p, the Landen formulas for dn, cn, sn which have been written above as the sum of p terms, can also be written as the product of the same p terms!

The plan of this paper is as follows: In §2 we obtain solutions of the sine-Gordon equation in two different ways and by requiring these solutions to be the same, thereby obtain the Landen transformations for the basic Jacobi elliptic functions $\mathrm{dn}, \mathrm{cn}, \mathrm{sn}$ for both odd and even p. In §3 we show that these Landen transformations can be expressed in other alternative forms. Further, we also show that for a given p, the relationship between $\tilde{m}$ and m is the same, regardless of the type of Landen formula that one is considering. Finally, concluding remarks are given in §4.

2. Generalized Landen formulas

Given the diversity of the generalized Landen formulas for Jacobi elliptic functions, it is necessary to establish them one at a time.

2.1 *'dn' Landen formulas*

To get an idea of our general approach, let us first focus on the proof of eq. (7) in detail. Consider the periodic solutions of the static sine-Gordon field theory in one space and one time dimension, that is, consider the periodic solutions of the second-order differential equation

$$\phi_{xx} = \sin\phi. \tag{13}$$

Note that the time dependent solutions are easily obtained from the static ones by Lorentz boosting. One of the simplest periodic solutions of eq. (13) is given by

$$\sin[\phi(x)/2] = \mathrm{dn}(x + x_0, \tilde{m}), \tag{14}$$

where x_0 is an arbitrary constant and this is immediately verifiable by direct substitution. It was shown in refs [6,7] that a kind of linear superposition principle works even for such non-linear equations as a consequence of several highly non-trivial, new identities satisfied by Jacobi elliptic functions [9–11]. In particular, one can show [7] that for any integer p, one has static periodic solutions of eq. (13) given by

$$\sin[\phi(x)/2] = \alpha\sum_{j=1}^{p}\mathrm{dn}[\alpha x + 2(j-1)K(m)/p, m]\ , \tag{15}$$

where α is as given by eq. (8).

The question one would like to address here is whether solution (15) is completely new, or if it can be re-expressed in terms of simpler solutions like (14), but where m and $\tilde{m}$ need not be the same. To that end, consider eq. (13) more generally. We note that on integrating once, we obtain

$$\phi_x^2 = C - 2\cos\phi, \tag{16}$$

where C is a constant of integration. Integrating again, one gets

$$\int \frac{\mathrm{d}\phi}{\sqrt{C - 2\cos\phi}} = x + x_0, \tag{17}$$

where x_0 is a second constant of integration, which we put equal to zero without loss of generality since it corresponds to a choice of the origin of coordinates. On substituting $\sin(\phi/2) = \psi$, eq. (17) takes the form

$$\int \frac{\mathrm{d}\psi}{\sqrt{1-\psi^2}\sqrt{\frac{C-2}{4} + \psi^2}} = x. \tag{18}$$

Now the important point to note is that if we perform the integral for different values of C then we will get all the solutions of eq. (13). Further, if two solutions have the same value of C, then they must necessarily be the same. As far as the integral (18) is concerned, it is easily checked that the three simplest solutions covering the entire allowed range of C are

$$\psi = \mathrm{sech}\ x, \quad C = 2, \tag{19}$$

$$\psi = \mathrm{dn}(x, \tilde{m}), \quad C = 4\tilde{m} - 2, \tag{20}$$

$$\psi = \mathrm{cn}(x/\sqrt{\tilde{m}}, \tilde{m}), \quad C = \frac{4}{\tilde{m}} - 2, \tag{21}$$

where $0 \le \tilde{m} \le 1$. Note that the constant C has been computed here by using eq. (16), which in terms of $\psi(x)$ takes the form

$$C = 2 - 4\psi^2 + \frac{4\psi_x^2}{1-\psi^2}. \tag{22}$$

Thus, whereas for solution (20), C lies in the range $-2 \le C \le 2$, for solution (21), C lies between 2 and ∞. Note that for $C < -2$, there is no real solution to eq. (18).

Now the strategy is clear. We will take solution (15) and compute C for it and thereby try to relate it to one of the basic solutions as given by eqs (19)–(21). One simple way of obtaining the constant C from eq. (22) is to evaluate it at a convenient value of x, say $x = 0$. In this way, we find that for solution (15), C is given by

$$C = -2 + 4\alpha^2(m-2) + 8\alpha^3 \sum_{j=1}^{p} \mathrm{dn}^3(2(j-1)K(m)/p, m). \tag{23}$$

Now, as $m \to 0$, $\alpha = 1/p$, $\mathrm{dn}(x, m=0) = 1$ and hence $C = -2$. On the other hand, as $m \to 1$, $K(m=1) = \infty$, $\mathrm{dn}(x, m=1) = \mathrm{sech}\, x$ and hence $\alpha = 1$ so that $C = 2$. Thus for solution (15), as m varies in the range $0 \le m \le 1$, the value of C varies in the range $-2 \le C \le 2$. Hence it is clear that solutions (15) and (20) must be same. On equating the two C values as given by eqs (20) and (23), we find that the two solutions are identical provided m and $\tilde{m}$ are related by eq. (9) and hence the appropriate Landen transformation valid for any integer p is given by eq. (7). Note that when $p = 2$, one recovers the Landen formula (6), since $\mathrm{dn}(K(m), m) = k'$, and $\tilde{m}$ simplifies to $(1-k')^2/(1+k')^2$.

2.2 *'cn' Landen formulas*

Unlike the dn case, it turns out that in this case the Landen transformation formulas for odd and even p have very different forms. We first derive the form for the odd p case and then consider the even p case.

As shown in ref. [7], another periodic solution of the static sine-Gordon eq. (13) is

$$\sin(\phi(x)/2) = \alpha_1 \sum_{j=1}^{p} \mathrm{cn}\left[\frac{\alpha_1 x}{\sqrt{m}} + 4(j-1)K(m)/p, m\right], \quad p \text{ odd}, \tag{24}$$

with α_1 being given by

$$\alpha_1 \equiv \left\{ \sum_{j=1}^{p} \mathrm{cn}[4(j-1)K(m)/p, m] \right\}^{-1} . \tag{25}$$

Using eq. (22), we can now compute the corresponding value of the constant C. We obtain

$$C = -2 + \frac{4\alpha_1^2(1-2m)}{m} + 8\alpha_1^3 \sum_{j=1}^{p} \mathrm{cn}^3[4(j-1)K(m)/p, m] . \tag{26}$$

It is easily checked that since $0 \le m \le 1$, C varies from 2 to ∞, and hence the solutions (21) and (24) must be identical. On equating the two values of C as given by eqs (21) and (26), we then find that for odd p, the Landen transformation is

$$\mathrm{cn}(x, \tilde{m}_1) = \alpha_1 \sum_{j=1}^{p} \mathrm{cn}\left[\frac{\alpha_1 \sqrt{\tilde{m}_1} x}{\sqrt{m}} + 4(j-1)K(m)/p, m\right], \tag{27}$$

where $\tilde{m}_1$ is given by

$$\tilde{m}_1 = \frac{m}{\alpha_1^2} \left\{ (1-2m) + 2m\alpha_1 \sum_{j=1}^{p} \mathrm{cn}^3[4(j-1)K(m)/p, m] \right\}^{-1} . \tag{28}$$

What happens if p is an even integer? As shown in [7], in that case another periodic solution of the static sine-Gordon eq. (13) is

$$\sin(\phi(x)/2) = \alpha_2 \sum_{j=1}^{p} (-1)^{j-1} \mathrm{dn}[\alpha_2 x + 2(j-1)K(m)/p, m], \quad p \text{ even}, \tag{29}$$

where α_2 is given by

$$\alpha_2 \equiv \left\{ \sum_{j=1}^{p} (-1)^{j-1} \mathrm{dn}[2(j-1)K(m)/p, m] \right\}^{-1}. \tag{30}$$

Using eq. (22), the value of C for this solution is easily computed and we find that $2 \le C \le \infty$. On comparing with solution (21) we find that in this case the Landen formula is

$$\mathrm{cn}(x, \tilde{m}_2) = \alpha_2 \sum_{j=1}^{p} (-1)^{j-1} \mathrm{dn}[\alpha_2 \sqrt{\tilde{m}_2}\, x + 2(j-1)K(m)/p, m], \tag{31}$$

where $\tilde{m}_2$ is given by

$$\tilde{m}_2 = \frac{1}{\alpha_2^2} \left\{ (m-2) + 2\alpha_2 \sum_{j=1}^{p} (-1)^{j-1} \mathrm{dn}^3[2(j-1)K(m)/p, m] \right\}^{-1}. \tag{32}$$

As expected, in the special case of $p = 2$, we immediately recover the Landen formula (4).

2.3 *'sn' Landen formulas*

As in the cn case, here too the Landen formulas for even and odd values of p have very different forms and so we consider them separately.

We start from the sine-Gordon field equation

$$\phi_{xx} - \phi_{tt} = \sin\phi, \tag{33}$$

and look for time-dependent, traveling wave solutions with velocity $v > 1$ (which are called optical soliton solutions in the context of condensed matter physics). In terms of the variable

$$\eta \equiv \frac{x - vt}{\sqrt{v^2 - 1}}, \tag{34}$$

eq. (33) takes the simpler form

$$\phi_{\eta\eta} = -\sin\phi. \tag{35}$$

Note that the only change from eq. (13) is an additional negative sign on the right-hand side. On integrating this equation once, we obtain

$$\phi_\eta^2 = C + 2\cos\phi. \tag{36}$$

On integrating further, we get

$$\int \frac{\mathrm{d}\phi}{\sqrt{C + 2\cos\phi}} = \eta + \eta_0, \tag{37}$$

where η_0 is a constant of integration which we put equal to zero without loss of generality. Substituting $\sin(\phi/2) = \psi$, yields

$$\int \frac{\mathrm{d}\psi}{\sqrt{1-\psi^2}\sqrt{\frac{C+2}{4} - \psi^2}} = \eta. \tag{38}$$

If we now perform the integral for different values of C, then we get all the solutions. It is easily checked that the three simplest solutions of eq. (38) covering the entire allowed range of C are

$$\psi = \tanh\eta, \quad C = 2, \tag{39}$$

$$\psi = \sqrt{\tilde{m}}\,\mathrm{sn}\,(\eta, \tilde{m}), \quad C = 4\tilde{m} - 2, \tag{40}$$

$$\psi = \mathrm{sn}\left(\frac{\eta}{\sqrt{\tilde{m}}}, \tilde{m}\right), \quad C = \frac{4}{\tilde{m}} - 2, \tag{41}$$

where $0 \le \tilde{m} \le 1$. Note that the constant C has been computed here using eq. (36), which in terms of ψ takes the form

$$C = -2 + 4\psi^2 + \frac{4\psi_\eta^2}{1-\psi^2}. \tag{42}$$

Thus, for solution (40), C is in the range $-2 \le C \le 2$, whereas for solution (41), C lies between 2 and ∞. Note that for $C < -2$, there is no real solution to eq. (38).

Using appropriate linear superposition, it was shown in ref. [7] that for odd p one of the solutions of eq. (35) is given by

$$\sin(\phi(\eta)/2) = \sqrt{m}\alpha \sum_{j=1}^{p} \mathrm{sn}[\alpha\eta + 4(j-1)K(m)/p, m], \quad p \text{ odd}, \tag{43}$$

with α being given by eq. (8). Using eq. (42), we can now compute the corresponding value of C. We find

$$C = -2 + \frac{4m\alpha^2}{\alpha_1^2}, \tag{44}$$

where α, α_1 are given by eqs (8) and (25) respectively. It is easily checked that since $0 \leq m \leq 1$, C has values between -2 and 2. Hence solutions (40) and (43) must be identical. On equating the two values of C as given by eqs (40) and (44), we then find that the 'sn' Landen transformation formula for odd p is given by

$$\mathrm{sn}(x, \tilde{m}_3) = \alpha_1 \sum_{j=1}^{p} \mathrm{sn}[\alpha x + 4(j-1)K(m)/p, m], \tag{45}$$

with $\tilde{m}_3$ and m being related by

$$\tilde{m_3} = m \frac{\alpha^2}{\alpha_1^2} \ . \tag{46}$$

Finally, we turn to the 'sn' Landen transformation formula for the case when p is an even integer. One can show [7] that in this case, a solution to eq. (35) is given by

$$\sin(\phi(\eta)/2) = m^{p/2} \alpha A_0 \prod_{j=1}^{p} \mathrm{sn}[\alpha\eta + 2(j-1)K(m)/p, m], \quad p \text{ even}, \tag{47}$$

with α being given by eq. (8) and A_0 defined by

$$A_0 = \prod_{j=1}^{p-1} \mathrm{sn}(2jK(m)/p, m). \tag{48}$$

Using eq. (42), we can now compute the corresponding value of C. We obtain

$$C = -2 + 4m^p \alpha^4 A_0^4. \tag{49}$$

It is easily checked that since $0 \leq m \leq 1$, the value of C varies between -2 and 2 and hence solutions (40) and (47) must be identical. On equating the two values of C as given by eqs (40) and (49), we find that for even p, the Landen transformation formula is

$$A_0 \, \alpha \, \mathrm{sn}(x, \tilde{m}_4) = \prod_{j=1}^{p} \mathrm{sn}[\alpha x + 2(j-1)K(m)/p, m], \tag{50}$$

with $\tilde{m}_4$ given by

$$\tilde{m}_4 = m^p \alpha^4 A_0^4. \tag{51}$$

Not surprisingly, for $p = 2$ we recover the Landen transformation formula (5). It is amusing to notice that as $m \to 0$,

$$A_0(p, m = 0) = \prod_{j=1}^{p-1} \sin(j\pi/p) = \frac{p}{2^{p-1}}. \tag{52}$$

At this point, we have generalized all three of the celebrated two hundred year old $p = 2$ Landen formulas [eqs (3)–(5)] to arbitrary values of p, the generalization being different depending on whether p is an even or odd integer. In the next section, we re-cast the formulas in even simpler form. Although several of these Landen formulas are already known [1], to our knowledge, they have never been derived via the novel approach of this article which makes use of solutions of non-linear field equations.

3. Alternative forms for Landen transformations

3.1 *Relation between the transformed modulus parameters $\tilde{m}_i$ and m*

So far, we have obtained several seemingly different relationships between the transformed modulus parameters $\tilde{m}, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3, \tilde{m}_4$ and m. Let us recall that while relation (7) for $\tilde{m}$ is valid for all p, relations (27) and (45) for $\tilde{m}_1$ and $\tilde{m}_3$ are valid only for odd p and relations (31) and (50) for $\tilde{m}_2$ and $\tilde{m}_4$ are valid only for even p. One would like to know if for any given p, the different Landen transformations give the same relationship between $\tilde{m}_i$ and m or not. For $p = 2$, it is known that $\tilde{m} = \tilde{m}_2 = \tilde{m}_4$ [see eqs (3)–(5)]. Similarly, with a little algebraic manipulation, one can show that for $p = 3$ the corresponding relations are

$$\tilde{m} = \tilde{m}_1 = \tilde{m}_3 = m\frac{(1-q)^2}{(1+q)^2(1+2q)^2}\ , \tag{53}$$

where $q \equiv \mathrm{dn}(2K(m)/3, m)$. Note that while deriving this result, use has been made of the fact that $\mathrm{cn}(4K(m)/3, m) = -\frac{q}{1+q}$ and that q satisfies the identity $q^4 + 2q^3 - 2(1-m)q - (1-m) = 0$. Similarly, using the relations $\mathrm{dn}(K(m)/2, m) = \mathrm{dn}(3K(m)/2, m) = (1-m)^{1/4} \equiv t$ and $\mathrm{dn}(K(m), m) = t^2$, it is easily proved that for $p = 4$, the relations are $\tilde{m} = \tilde{m}_2 = \tilde{m}_4 = (1-t)^4/(1+t)^4$.

As p increases, the algebra becomes messier and it is not easy to write the corresponding relations between $\tilde{m}_i$ and m in a neat closed form. However, we can still establish the equivalence of all $\tilde{m}_i$ for any given p. For this purpose we equate the periods of the left- and right-hand sides of the various Landen transformation relations [(7), (27), (31), (45), (50)]. We get

$$K(\tilde{m}) = \frac{K(m)}{p\alpha}, \quad K(\tilde{m}_3) = \frac{K(m)}{p\alpha}, \quad K(\tilde{m}_4) = \frac{K(m)}{p\alpha}, \tag{54}$$

$$K(\tilde{m}_1) = \frac{K(m)\sqrt{m}}{p\alpha_1\sqrt{\tilde{m}_1}}, \quad K(\tilde{m}_2) = \frac{K(m)}{p\alpha_2\sqrt{\tilde{m}_2}}. \tag{55}$$

From eq. (54) one immediately sees that $\tilde{m} = \tilde{m}_3$ for odd p and $\tilde{m} = \tilde{m}_4$ for even p. In order to establish that $\tilde{m}_1 = \tilde{m}_3$ for odd p, we suitably rescale, square and add the Landen transformations (27) and (45). This yields

$$\mathrm{sn}^2(x, \tilde{m}) + \mathrm{cn}^2\left(\frac{\alpha\sqrt{m}x}{\alpha_1\sqrt{\tilde{m}_1}}, \tilde{m}_1\right) = C, \tag{56}$$

where the constant C on the right-hand side comes from making use of cyclic identities [9–11]. Since eq. (56) is valid for all x, and in particular for $x = 0$, it follows that $C = 1$, and hence one must have

$$\mathrm{sn}^2(x, \tilde{m}) = \mathrm{sn}^2\left(\frac{\alpha\sqrt{m}x}{\alpha_1\sqrt{\tilde{m}_1}}, \tilde{m}_1\right). \tag{57}$$

This implies that $(\alpha\sqrt{m}/\alpha_1\sqrt{\tilde{m}_1}) = 1$ and hence $\tilde{m}_1 = \tilde{m}$ for odd p. Similar reasoning yields $(\alpha/\alpha_2\sqrt{\tilde{m}_2}) = 1$ and $\tilde{m}_2 = \tilde{m}$ for even p.

Consequently, for odd p, the three Landen transformations are

$$\mathrm{dn}(x, \tilde{m}) = \alpha \sum_{j=1}^{p} \mathrm{dn}(x_j, m), \tag{58}$$

$$\mathrm{cn}(x, \tilde{m}) = \alpha_1 \sum_{j=1}^{p} \mathrm{cn}(\tilde{x_j}, m), \quad \mathrm{sn}(x, \tilde{m}) = \alpha_1 \sum_{j=1}^{p} \mathrm{sn}(\tilde{x_j}, m), \tag{59}$$

where α, α_1 are given by eqs (8) and (25), $x_j, \tilde{x_j}$ are as given by eq. (12) while $\tilde{m}$ and m are related by eq. (9).

Likewise, for even p, the three Landen formulas are

$$\mathrm{dn}(x, \tilde{m}) = \alpha \sum_{j=1}^{p} \mathrm{dn}(x_j, m), \quad \mathrm{cn}(x, \tilde{m}) = \alpha_2 \sum_{j=1}^{p} (-1)^{j-1} \mathrm{dn}(x_j, m),$$

$$A_0\, \alpha\, \mathrm{sn}(x, \tilde{m}) = \prod_{j=1}^{p} \mathrm{sn}(x_j, m), \tag{60}$$

where α, α_2, A_0 are given by eqs (8), (30) and (48) respectively while $\tilde{m}$ and m are related by eq. (9).

We might add here that for any fixed value of p, as m increases from 0 to 1, $\tilde{m}$ also increases monotonically from 0 to 1 (but is always less than m). Further, for a given fixed value of m, $\tilde{m}$ decreases monotonically as p increases and is always less than m.

3.2 *Alternative forms of Landen transformation formulas*

Recently we [9–11] have obtained several new identities for Jacobi elliptic functions. Three of these, valid for odd p, are

$$\prod_{j=1}^{p} \mathrm{dn}(x_j, m) = \prod_{n=1}^{(p-1)/2} \mathrm{cs}^2\left(\frac{2Kn}{p}, m\right) \sum_{j=1}^{p} \mathrm{dn}(x_j, m), \tag{61}$$

$$\prod_{j=1}^{p} \mathrm{sn}(\tilde{x}_j, m) = (-1/m)^{(p-1)/2} \prod_{n=1}^{(p-1)/2} \mathrm{ns}^2\left(\frac{4Kn}{p}, m\right) \sum_{j=1}^{p} \mathrm{sn}(\tilde{x}_j, m), \tag{62}$$

$$\prod_{j=1}^{p} \mathrm{cn}(\tilde{x}_j, m) = (1/m)^{(p-1)/2} \prod_{n=1}^{(p-1)/2} \mathrm{ds}^2\left(\frac{4Kn}{p}, m\right) \sum_{j=1}^{p} \mathrm{cn}(\tilde{x}_j, m). \quad (63)$$

Here $\mathrm{ds}(x,m)$, etc. are defined by $\mathrm{ds}(x,m) = \mathrm{dn}(x,m)/\mathrm{sn}(x,m)$. Hence, for odd p, the three Landen transformation formulas can also be written as products (rather than sums) of p terms. In particular, for odd p the three Landen transformation formulas can also be written in the form

$$\mathrm{dn}(x, \tilde{m}) = \frac{\prod_{j=1}^{p} \mathrm{dn}(x_j, m)}{\prod_{n=1}^{p-1} \mathrm{dn}(2nK(m)/p, m)}, \quad (64)$$

$$\mathrm{cn}(x, \tilde{m}) = \frac{\prod_{j=1}^{p} \mathrm{cn}(\tilde{x}_j, m)}{\prod_{n=1}^{p-1} \mathrm{cn}(4nK(m)/p, m)}, \quad (65)$$

$$\mathrm{sn}(x, \tilde{m}) = \frac{\prod_{j=1}^{p} \mathrm{sn}(\tilde{x}_j, m)}{\alpha(m) \prod_{n=1}^{p-1} \mathrm{sn}(4nK(m)/p, m)}. \quad (66)$$

Similarly, for even p, on making use of the identity [9,10]

$$(m)^{p/2} \prod_{j=1}^{p} \mathrm{sn}(x_j, m) = \left[\prod_{n=1}^{\frac{p}{2}-1} \mathrm{ns}^2(2nK(m)/p, m)\right] \sum_{j=1}^{p} (-1)^{j-1} \mathrm{Z}(x_j, m), \quad (67)$$

where $\mathrm{Z}(u,m)$ is the Jacobi zeta function, we can rewrite the Landen formula (60) for sn in the form

$$\mathrm{sn}(x, \tilde{m}) = \alpha_2 \sum_{j=1}^{p} (-1)^{j-1} \mathrm{Z}(x_j, m). \quad (68)$$

We have not seen this particular form for the sn Landen transformation in the mathematics literature. Here the coefficient multiplying the right-hand side of eq. (68) has been fixed by demanding consistency.

On comparing eqs (60), (67) and (68) we obtain an interesting identity

$$m^{p/2} \alpha \alpha_2 \prod_{j=1}^{p-1} \mathrm{sn}(2jK(m)/p, m) = \prod_{n=1}^{\frac{p}{2}-1} \mathrm{ns}^2(2nK(m)/p, m). \quad (69)$$

In concluding this section, we note that many of the Landen transformation formulas derived above lead to interesting known trigonometric relations by taking the limiting case $m = \tilde{m} = 0$. For instance, eqs (60), (65) and (66) lead to [4]

$$\sin px = 2^{p-1} \prod_{j=1}^{p} \sin[x + (j-1)\pi/p], \quad p \text{ even}, \quad (70)$$

$$\cos px = 2^{p-1} \prod_{j=1}^{p} \cos[x + 2(j-1)\pi/p], \quad p \text{ odd}, \tag{71}$$

$$\sin px = (-4)^{\frac{p-1}{2}} \prod_{j=1}^{p} \sin[x + 2(j-1)\pi/p], \quad p \text{ odd}. \tag{72}$$

4. Concluding remarks

We have used a novel method to obtain generalized Landen transformation formulas involving changes of the modulus parameter. Analogous to the generalized Landen transformations, it is also possible to obtain generalized Gauss transformations in which the parameter $\tilde{m}_{\mathrm{G}}$ is greater than m. In particular, the generalized Gauss transformations can be systematically obtained [12] from the Landen transformations of arbitrary order p by considering shifts in the arguments by pure imaginary amounts and making use of the fact that the Jacobi elliptic functions are doubly periodic. In particular, one can start from any Landen transformation formula, use the relations [2,3]

$$\mathrm{dn}(x, m') = \mathrm{dc}(ix, m), \quad \mathrm{cn}(x, m') = \mathrm{nc}(ix, m),$$
$$\mathrm{sn}(x, m') = -i\mathrm{sc}(ix, m), \tag{73}$$

and then redefine $ix = u$. Here $m' = 1 - m$ while $\mathrm{dc}(x, m) = \mathrm{dn}(x, m)/\mathrm{cn}(x, m)$. As an illustration, the Landen transformation formula (58), which is valid for any p, takes the form

$$\mathrm{dc}(x, \tilde{m}) = \beta \sum_{j=1}^{p} \mathrm{dc}[\beta x + 2i(j-1)K'(m)/p, m]\,, \tag{74}$$

where

$$\beta \equiv \beta(m) = \alpha(m') = \left(\sum_{j=1}^{p} \mathrm{dn}[2(j-1)K'(m)/p, m'] \right)^{-1}. \tag{75}$$

We might add here that just as we have considered Landen formulas where the shifts are in units of $K(m)$ or $iK'(m)$ on the real or pure imaginary axis respectively, we can also consider [12] Landen formulas corresponding to shifts in units of $K(m)+iK'(m)$ in the complex plane. In particular, one can start from any of the Landen formula, change $m(\tilde{m})$ to $1/m(1/\tilde{m})$, change x to kx and obtain the corresponding Landen transformation in the complex plane by using the formulas

$$\mathrm{dn}\left(kx, \frac{1}{m}\right) = \mathrm{cn}(x, m), \quad \mathrm{cn}\left(kx, \frac{1}{m}\right) = \mathrm{dn}(x, m),$$
$$\mathrm{sn}\left(kx, \frac{1}{m}\right) = k\mathrm{sn}(x, m). \tag{76}$$

We would like to point out here that by multiplication, integration and differentiation of the Landen formulas for the three basic Jacobi elliptic functions sn, cn, dn, we can readily obtain useful Landen formulas for their products and various other combinations [12]. For example, squaring the Landen formula (58) and using the cyclic identity [9,10]

$$\sum_{j=1}^{p} \mathrm{dn}(x_j, m)\,\mathrm{dn}(x_{j+r}, m) = p[\mathrm{dn}(a, m) - \mathrm{cs}(a, m)\mathrm{Z}(a, m)],$$
$$a \equiv 2rK(m)/p\,, \tag{77}$$

yields the Landen formula for dn^2, i.e. we get

$$\mathrm{dn}^2(x, \tilde{m}) = \alpha^2\left[\sum_{j=1}^{p} \mathrm{dn}^2(x_j, m) + 2A_d\right], \tag{78}$$

where

$$\begin{aligned} A_d \equiv & \sum_{i<j=1}^{p} \mathrm{dn}(x_i, m)\mathrm{dn}(x_j, m) \\ &= p \sum_{j=1}^{(p-1)/2} [\mathrm{dn}(2jK(m)/p, m) \\ &\quad -\mathrm{cs}(2jK(m)/p, m)\mathrm{Z}(2jK(m)/p)]\,,\ p \text{ odd}\,, \\ &= (p/2)\sqrt{1-m} + p \sum_{j=1}^{(p-2)/2} [\mathrm{dn}(2jK(m)/p, m) \\ &\quad -\mathrm{cs}(2jK(m)/p, m)\mathrm{Z}(2jK(m)/p)]\,,\ p \text{ even.} \end{aligned} \tag{79}$$

If we now integrate both sides of eq. (78) with respect to x, then we simultaneously obtain Landen transformations for both the Jacobi zeta function as well as the complete elliptic integral of the second kind $E(m)$. In particular, on using the well-known formula [3,4]

$$\int \mathrm{dn}^2(x, m)\,\mathrm{d}x = \mathrm{Z}(x, m) + \frac{xE(m)}{K(m)}\,, \tag{80}$$

and integrating eq. (78) yields the following two remarkable identities

$$\mathrm{Z}(x, \tilde{m}) = \alpha \sum_{j=1}^{p} \mathrm{Z}(x_j, m)\,, \quad E(\tilde{m}) = \alpha\,[E(m) + 2A_d K(m)/p]\,. \tag{81}$$

While deriving this relation, use has been made of the fact that $\mathrm{Z}(x = 0, m) = 0$, that it is an odd function of its argument x and that it is a periodic function of x with period $2K(m)$.

Finally, it is worth pointing out that recently we [9–11] have obtained a large number of cyclic identities where combinations of Jacobi elliptic functions at different points are expressed in terms of sums like $\sum_{j=1}^{p} \mathrm{dn}(x_j, m)$. But, the remarkable thing is that, it is precisely these sums for which Landen formulas, mentioned above, have been obtained. Thus by combining the cyclic identities and the Landen formulas given in this paper, we can obtain a wide class of *generalized* Landen transformations for many combinations of the Jacobi elliptic functions [12]. For example, a simple cyclic identity valid for both even and odd integral p is

$$\sum_{j=1}^{p} \mathrm{dn}^2(x_j, m)[\mathrm{dn}(x_{j+r}, m) + \mathrm{dn}(x_{j-r}, m)]$$
$$= 2[\mathrm{ds}(a, m)\mathrm{ns}(a, m) - \mathrm{cs}^2(a, m)] \sum_{j=1}^{p} \mathrm{dn}(x_j, m), \tag{82}$$

where $a = 2rK(m)/p$. On combining it with relation (58), we obtain the Landen formula

$$\sum_{j=1}^{p} \mathrm{dn}^2(x_j, m)[\mathrm{dn}(x_{j+r}, m) + \mathrm{dn}(x_{j-r}, m)]$$
$$= (2/\alpha)[\mathrm{ds}(a, m)\mathrm{ns}(a, m) - \mathrm{cs}^2(a, m)]\mathrm{dn}(x, \tilde{m}). \tag{83}$$

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