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Local identities involving Jacobi elliptic functions

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Abstract. We derive a number of local identities involving Jacobi elliptic functions and use them to obtain several new results. First, we present an alternative, simpler derivation of the cyclic identities discovered by us recently, along with an extension to several new cyclic identities. Second, we obtain a generalization to cyclic identities in which successive terms have a multiplicative phase factor $\exp(2i\pi/s)$, where s is any integer. Third, we systematize the local identities by deriving four local 'master identities' analogous to the master identities for the cyclic sums discussed by us previously. Fourth, we point out that many of the local identities can be thought of as exact discretizations of standard nonlinear differential equations satisfied by the Jacobi elliptic functions. Finally, we obtain explicit answers for a number of definite integrals and simpler forms for several indefinite integrals involving Jacobi elliptic functions.

Keywords. Jacobi elliptic functions; cyclic identities; local identities.

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1. Introduction

In a recent paper [1] (henceforth referred to as I), we have given many new mathematical identities involving the Jacobi elliptic functions $\mathrm{sn}\,(x,m)$, $\mathrm{cn}\,(x,m)$, $\mathrm{dn}\,(x,m)$, where m is the elliptic modulus parameter ($0 \le m \le 1$). The functions $\mathrm{sn}\,(x,m)$, $\mathrm{cn}\,(x,m)$, $\mathrm{dn}\,(x,m)$ are doubly periodic functions with periods $(4K(m), i2K'(m))$, $(4K(m),\ 2K(m)+i2K'(m))$, $(2K(m), i4K'(m))$, respectively [2,3]. Here, $K(m) \equiv \int_0^{\pi/2} \mathrm{d}\theta [1-m\sin^2\theta]^{-1/2}$ denotes the complete elliptic integral of the first kind, and $K'(m) \equiv K(1-m)$. The $m=0$ limit gives $K(0)=\pi/2$ and trigonometric functions: $\mathrm{sn}(x,0)=\sin x$, $\mathrm{cn}(x,0)=\cos x$, $\mathrm{dn}(x,0)=1$. The $m \to 1$ limit gives $K(1) \to \infty$ and hyperbolic functions: $\mathrm{sn}(x,1) \to \tanh x$, $\mathrm{cn}(x,1) \to \mathrm{sech}\, x$, $\mathrm{dn}(x,1) \to \mathrm{sech}\, x$. For simplicity, from now on we will not explicitly display the modulus parameter m as an argument of the Jacobi elliptic functions.

The identities discussed in ref. I are all cyclic with the arguments of the Jacobi functions in successive terms separated by either $2K(m)/p$ or $4K(m)/p$, where p is an integer. Each p-point identity of rank R involves on its left-hand side a cyclic homogeneous polynomial in Jacobi elliptic functions of degree R with p equally spaced arguments. The separation is $2K(m)/p$ or $4K(m)/p$ depending on whether the period of any term on the left-hand side is $2K(m)$ or $4K(m)$. In another recent publication [4] (referred to as II), we presented rigorous mathematical proofs valid for arbitrary p and R even though, for simplicity, we only presented identities of low rank. In ref. II, we classified the identities into four types, each with its own 'master identity' which we proved using a combination of the Poisson summation formula and the special properties of elliptic functions [4,5]. We also provided a rigorous derivation of cyclic identities with successive terms having alternating signs.

In this paper, we provide several generalizations of previous results. Here, our approach is different and involves the use of 'local' identities which focus on any one term in a cyclic identity. This term involves a product of Jacobi elliptic functions and is expressed via the local identity as the sum of many terms of lower rank. The purpose of this paper is to derive and make use of a number of local identities for Jacobi elliptic functions. These local identities form the building blocks for cyclic as well as much more general identities. For instance, adding p local identities with equally spaced arguments permits us to re-derive cyclic identities. More generally, taking p local identities with a phase $(-1)^{(j-1)} = \mathrm{e}^{(j-1)i\pi}$ and summing over the index j gives previously derived identities in which successive terms have alternative signs. Finally, as discussed below, the generalization to taking p local identities with an even more general phase $\exp(2i(j-1)\pi/s)$, where s is any integer and summing over the index j yields interesting new identities in which successive terms have different weights. Also, while in principle we were able to prove the general form for identities of arbitrary rank in refs I and II, in practice it was very difficult to obtain the explicit coefficients in these identities. The use of local identities permits evaluation of these coefficients. As a byproduct, a number of definite integrals involving Jacobi elliptic functions can be explicitly evaluated and a number of indefinite integrals can be expressed in simpler form.

To clarify the above ideas, consider as an example, a specific basic local identity:

$$\begin{aligned}\mathrm{dn}^2(y)\mathrm{dn}(y+a) = &-\mathrm{cs}^2(a)\mathrm{dn}(y+a) + \mathrm{ds}(a)\mathrm{ns}(a)\mathrm{dn}(y)\\ &-m\,\mathrm{cs}(a)\mathrm{cn}(y)\mathrm{sn}(y).\end{aligned} \tag{1}$$

Choosing $y = x + (j-1)2K(m)/p$ with $j = 1, 2, \ldots, p$ actually corresponds to p identities, one for each value of the integer j. Taking $a = r2K(m)/p$, where r is an integer less than p and coprime to it, and summing over j yields the cyclic identity

$$\sum_{j=1}^{p} d_j^2 d_{j+r} = \sum_{j=1}^{p}\left[\frac{A}{2}d_j - m\,\mathrm{cs}(a)s_j c_j\right], \tag{2}$$

where the coefficient A is given by $A = 2[\mathrm{ds}(a)\mathrm{ns}(a) - \mathrm{cs}^2(a)]$ and we have used the notation

$$\begin{aligned}&d_j \equiv \mathrm{dn}(x+(j-1)2K(m)/p, m),\;\; s_j \equiv \mathrm{sn}(x+(j-1)2K(m)/p, m),\\ &c_j \equiv \mathrm{cn}(x+(j-1)2K(m)/p, m).\end{aligned} \tag{3}$$

Similar manipulations using $a = -r2K(m)/p$ yield cyclic identities for expressions like $\sum_{j=1}^{p} d_j^2[d_{j+r} \pm d_{j-r}]$. The result with the negative sign is new and will be discussed later in this paper. The result with the positive sign is

$$\sum_{j=1}^{p} d_j^2 \left[d_{j+r} + d_{j-r}\right] = A \sum_{j=1}^{p} d_j, \tag{4}$$

and is one of the cyclic identities derived in ref. II by more complicated techniques.

Further, while it was clear from refs I and II that the identities of arbitrary (odd) rank which are generalizations of eq. (4) must have the structure

$$\sum_{j=1}^{p} d_j^{2n} \left[d_{j+r} + d_{j-r}\right] = A_1 \sum_{j=1}^{p} d_j^{2n-1} + \cdots + A_n \sum_{j=1}^{p} d_j, \tag{5}$$

we were unable to obtain the coefficients $A_1, \ldots, A_n$. Here, we will obtain explicit expressions for the coefficients. In ref. II, we were able to obtain the analogue of identity (4) with alternating signs given by

$$\sum_{j=1}^{p} (-1)^{j-1} d_j^2 \left[d_{j+r} + d_{j-r}\right] = A \sum_{j=1}^{p} (-1)^{j-1} d_j,$$
$$A = 2\left[\mathrm{ds}(a)\mathrm{ns}(a) + \mathrm{cs}^2(a)\right], \quad a = \frac{r2K}{p}. \tag{6}$$

Here, we will obtain identities with more general weights like

$$\sum_{j=1}^{p} \omega^{j-1} d_j^2 [d_{j+r} + d_{j-r}], \quad \omega = \exp\left(\frac{2i\pi}{s}\right), \tag{7}$$

where ω is the sth root of unity, with s being any integer ($< p$) and p being 0 mod s. Finally, in ref. II we had obtained MI-II (class II master identity, also see §4 below) identities like

$$\sum_{j=1}^{p} d_j^2 d_{j+r}^2 = -2\mathrm{cs}^2(a) \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left(\int_0^{2K} \mathrm{dn}^2(t)\mathrm{dn}^2(t+a)\mathrm{d}t + 4E\, \mathrm{cs}^2(a) \right),$$
$$a = \frac{r2K}{p}, \tag{8}$$

where E is the complete elliptic integral of the second kind [3]. The approach in this paper will permit an evaluation of the definite integral on the right-hand side.

The paper is organized as follows: In §2, we state several local identities and indicate how they are derived. Each identity has an integer label R indicating the rank of the identity, i.e. the left-hand side of the identity is a homogeneous polynomial of degree R. We also show here that linear combinations of cyclic identities often yield simpler results. In §3, we use the local identities of rank 2, 3, 4 recursively to obtain local identities of arbitrary odd and even rank, using which one can immediately obtain the corresponding cyclic identities with weight factors ω. In §4, we provide a unified framework for the local identities by deriving four master local identities, from which all the identities can be derived in an alternative manner without using addition formulas. In §5, we concentrate on those identities of ref. II in which one of the terms on the right-hand side is a definite integral (which we were previously unable to evaluate). Using our local identities, we show that one can obtain cyclic identities where all the terms on the right-hand side are now explicitly known. In fact, we show that by starting from any given local identity, the indefinite integral of the left-hand side of this identity can be analytically obtained in terms of the well-known integrals of Jacobi elliptic functions and indefinite elliptic integrals of the first, second and third kind [3,6]. We would like to re-emphasize that most of these integrals do not seem to be known in the literature. In §6, we discuss continuum limits of the local and cyclic identities, showing that these degenerate to standard differential equations or integral formulas. Section 7 contains conclusions and a discussion of some open problems. All local identities of rank 2 and 3 are presented in Appendices A and B respectively. A few local identities of rank 4, 5 and arbitrary rank are presented in Appendix C. Some simple results obtained by taking suitable linear combinations of cyclic identities are given in Appendix D. Many new definite and indefinite integrals are given in Appendices E and F.

2. The basic local identities

In this section we shall obtain several basic local identities. These identities are easily derived using the well-known addition formulas for the sn, cn, dn functions [2,3]:

$$\mathrm{dn}(a+b) = \frac{\mathrm{dn}(a)\mathrm{dn}(b) - m\,\mathrm{cn}(a)\mathrm{cn}(b)\mathrm{sn}(a)\mathrm{sn}(b)}{1 - m\,\mathrm{sn}^2(a)\mathrm{sn}^2(b)}, \tag{9}$$

$$\mathrm{cn}(a+b) = \frac{\mathrm{cn}(a)\mathrm{cn}(b) - \mathrm{dn}(a)\mathrm{sn}(a)\mathrm{dn}(b)\mathrm{sn}(b)}{1 - m\,\mathrm{sn}^2(a)\mathrm{sn}^2(b)}, \tag{10}$$

$$\mathrm{sn}(a+b) = \frac{\mathrm{sn}(a)\mathrm{cn}(b)\mathrm{dn}(b) + \mathrm{cn}(a)\mathrm{dn}(a)\mathrm{sn}(b)}{1 - m\,\mathrm{sn}^2(a)\mathrm{sn}^2(b)}. \tag{11}$$

We shall also use the addition formula for the Jacobi zeta function given by

$$Z(a+b) = Z(a) + Z(b) - m\,\mathrm{sn}(a)\mathrm{sn}(b)\mathrm{sn}(a+b). \tag{12}$$

One of the simplest, local, rank 2 identities is

$$\mathrm{dn}(x)\mathrm{dn}(x+a) = \mathrm{dn}(a) + \mathrm{cs}(a)[Z(x+a) - Z(x) - Z(a)], \tag{13}$$

which is easily proved by algebraic simplification after using the addition formulas (9) to (12). The power of this local identity can be appreciated by the fact that we can immediately derive the cyclic identity

$$\sum_{j=1}^{p} d_j d_{j+1} = p\left[\mathrm{dn}\left(\frac{2K}{p}\right) - \mathrm{cs}\left(\frac{2K}{p}\right) Z\left(\frac{2K}{p}\right)\right], \tag{14}$$

which was obtained in ref. II. This is accomplished by adding local identities like (13) with x being replaced by $x+a,\ x+2a,\dots,\ x+(p-1)a$ and choosing $a = 2K/p$. Here p denotes the number of subdivisions of the period at which Jacobi elliptic functions $\mathrm{dn}(x)$ are evaluated. A generalization of this identity to rth neighbours is immediate, i.e., on choosing $a = r2K/p$ (where r is coprime to and less than p), we obtain the more general identity

$$\sum_{j=1}^{p} d_j d_{j+r} = p[\mathrm{dn}(a) - \mathrm{cs}(a)Z(a)], \quad a = \frac{r2K}{p}. \tag{15}$$

We can immediately obtain a local identity for $\mathrm{dn}(x)\mathrm{dn}(x-a)$ by changing a to $-a$ and recognizing the fact that while $\mathrm{cn}(a), \mathrm{dn}(a)$ are even functions of a, the functions $\mathrm{sn}(a), Z(a)$ are odd:

$$\mathrm{dn}(x)\mathrm{dn}(x-a) = \mathrm{dn}(a) - \mathrm{cs}(a)[Z(x-a) - Z(x) + Z(a)]. \tag{16}$$

Adding and subtracting eqs (13) and (16) yields alternative simple expressions:

$$\begin{aligned}&\mathrm{dn}(x)[\mathrm{dn}(x+a) + \mathrm{dn}(x-a)]\\ &= 2\mathrm{dn}(a) + \mathrm{cs}(a)[Z(x+a) - Z(x-a) - 2Z(a)],\end{aligned} \tag{17}$$

$$\mathrm{dn}(x)[\mathrm{dn}(x+a) - \mathrm{dn}(x-a)] = \mathrm{cs}(a)[Z(x+a) + Z(x-a) - 2Z(x)]. \tag{18}$$

If we now consider the local identities analogous to (13) with x being replaced by $x+a,\ x+2a,\dots,\ x+(p-1)a$, multiply them in turn by $\omega,\ \omega^2,\dots,\ \omega^{p-2}$, ω^{p-1} respectively and add to the local identity (13), then we obtain the remarkable identity

$$\sum_{j=1}^{p} \omega^{j-1} d_j d_{j+r} = p[\mathrm{dn}(a) - \mathrm{cs}(a)Z(a)]\delta_{s1} - \left(1 - \frac{1}{\omega}\right)\mathrm{cs}(a)\sum_{j=1}^{p}\omega^{j-1}Z_j, \tag{19}$$

where $a = r2K/p$. The phase ω is as given by eq. (7) with $s < p$ and p being 0 mod s. For the special case $s = 1$ we recover the cyclic identity (15) with all terms on the left-hand side having positive signs. For $s = 2$, eq. (19) gives the cyclic identity with terms having alternating signs as obtained in ref. II. Thus the local identities are very basic in the sense that once they are known, then the corresponding cyclic identities with and without arbitrary weight ω are immediately obtainable. It is worth emphasizing here that the cyclic identities with arbitrary weight are new.

Proceeding in the same way, we have derived all possible local identities of rank 2 and 3. They are given in Appendices A and B respectively. Some examples of local identities of rank 4, 5 and arbitrary rank are given in Appendix C. By following the procedure explained above, in each case it is easy to obtain the corresponding cyclic identities with weights ω.

One advantage of the local identities approach is that for the MI-II type of cyclic identities, the right-hand side is explicitly known. In this context, it is worth mentioning that in ref. II (also see [5]) we had obtained several MI-II cyclic identities in which one of the terms on the right-hand side is a definite integral. For example, one of the cyclic MI-II identities obtained in ref. II is given by eq. (8). However, if we take the local identity (C4) given in Appendix C, and use the procedure described above, we find a simpler, more elegant form for this cyclic identity

$$\sum_{j=1}^{p} d_j^2 d_{j+r}^2 = -2\mathrm{cs}^2(a) \sum_{j=1}^{p} d_j^2 + p[\mathrm{cs}^2(a) + \mathrm{ds}^2(a) - 2\mathrm{cs}(a)\mathrm{ds}(a)\mathrm{ns}(a)Z(a)],$$
$$a = \frac{r2K}{p}. \tag{20}$$

Using various local identities obtained in this paper, the corresponding MI-II cyclic identities are easily written where the constant on the right-hand side is now explicitly known and not just formally expressed as an unevaluated definite integral.

Let us now consider a specific local identity of rank 2

$$m\,\mathrm{cn}(x)[\mathrm{sn}(x+a) - \mathrm{sn}(x-a)]$$
$$= 2\mathrm{ns}(a)\mathrm{dn}(x) - \mathrm{ds}(a)[\mathrm{dn}(x+a) + \mathrm{dn}(x-a)], \tag{21}$$

obtained using eqs (9)–(12). This leads to the following cyclic identity with weighted terms:

$$\sum_{j=1}^{p} m\omega^{j-1} c_j [s_{j+r} - s_{j-r}] = 2\left[\mathrm{ns}(a) - \cos\left(\frac{2\pi}{s}\right)\mathrm{ds}(a)\right] \sum_{j=1}^{p} \omega^{j-1} d_j, \tag{22}$$

where $a = r2K/p$. Other examples of new cyclic identities of rank 3 and 4 with weighted terms are

$$m \sum_{j=1}^{p} \omega^{j-1} d_j [c_{j+r} s_{j+r} - c_{j-r} s_{j-r}]$$
$$= 2p[\mathrm{cs}(a) - \mathrm{ds}(a)\mathrm{ns}(a)Z(a)]\delta_{s1}$$
$$- 2i \sin\left(\frac{2\pi}{s}\right) \mathrm{ds}(a)\mathrm{ns}(a) \sum_{j=1}^{p} \omega^{j-1} Z_j - 2\cos\left(\frac{2\pi}{s}\right)\mathrm{cs}(a) \sum_{j=1}^{p} \omega^{j-1} d_j^2, \tag{23}$$

$$\sum_{j=1}^{p} \omega^{j-1} d_j [c_{j+r} d_{j+r} - c_{j-r} d_{j-r}] = 2\mathrm{cs}(b) \cos\left(\frac{2\pi}{s}\right) \sum_{j=1}^{p} \omega^{j-1} s_j d_j$$
$$-2i \sin\left(\frac{2\pi}{s}\right) \mathrm{ds}(b)\mathrm{ns}(b) \sum_{j=1}^{p} \omega^{j-1} c_j, \qquad (24)$$

$$m \sum_{j=1}^{p} \omega^{j-1} c_j [c_{j+r} s_{j+r} d_{j+r} - c_{j-r} s_{j-r} d_{j-r}]$$
$$= -2i \sin\left(\frac{2\pi}{s}\right) \mathrm{cs}(b)\mathrm{ns}(b) \sum_{j=1}^{p} \omega^{j-1} s_j d_j$$
$$- 2m \cos\left(\frac{2\pi}{s}\right) \mathrm{ds}(b) \sum_{j=1}^{p} \omega^{j-1} c_j^3$$
$$+ 2\mathrm{ds}(b) \left[(m + \mathrm{cs}^2(b)) \cos\left(\frac{2\pi}{s}\right) - \mathrm{cs}(b)\mathrm{ns}(b)\right] \sum_{j=1}^{p} \omega^{j-1} c_j,$$
$$b = \frac{r4K}{p}. \qquad (25)$$

Identities (24) and (25) are of type MI-IV and hence $c_j \equiv \mathrm{cn}(x + (j-1)4K/p, m)$. Proceeding in the same way, corresponding to most of the cyclic identities discussed in I and II, we can obtain cyclic identities with weighted terms. Some examples of such cyclic identities are given in Appendix D.

3. Local identities of arbitrary rank

Now that we have obtained local and hence cyclic identities of low rank, we will proceed to generalize the results to arbitrary rank. In particular, for each local low rank identity, we will obtain a local, and hence cyclic, identity of arbitrary rank with explicitly known coefficients. As an illustration, let us start from the local identity

$$\mathrm{dn}^2(x)[\mathrm{dn}(x+a) + \mathrm{dn}(x-a)] = A\mathrm{dn}(x) + B[\mathrm{dn}(x+a) + \mathrm{dn}(x-a)], \qquad (26)$$

where the constants A, B are defined by

$$A = 2\mathrm{ds}(a)\mathrm{ns}(a), \quad B = -\mathrm{cs}^2(a). \qquad (27)$$

This identity can be easily derived using the addition formula (9). On repeatedly multiplying both sides of identity (26) by $\mathrm{dn}^2(x)$ and simplifying, we obtain the following local identity of arbitrary odd rank:

$$\mathrm{dn}^{2n}(x)[\mathrm{dn}(x+a)+\mathrm{dn}(x-a)] = A\sum_{k=1}^{n} B^{k-1}\mathrm{dn}^{2(n-k)+1}(x) + B^n[\mathrm{dn}(x+a)+\mathrm{dn}(x-a)]. \quad (28)$$

The corresponding cyclic identities with arbitrary weight ω are immediately obtained:

$$\sum_{j=1}^{p} \omega^{j-1} d_j^{2n}[d_{j+r}+d_{j-r}] = A\sum_{j=1}^{p}\sum_{k=1}^{n} \omega^{j-1}B^{k-1}d_j^{2(n-k)+1} + 2B^n\cos\left(\frac{2\pi}{s}\right)\sum_{j=1}^{p}\omega^{j-1}d_j, \quad (29)$$

where $a = 2rK/p$ and $p = 0 \bmod s$.

In order to obtain the corresponding local identity of arbitrary even rank, we start from identity (28), multiply both sides by $\mathrm{dn}(x)$, and use the local identity (17) to obtain

$$\begin{aligned}&\mathrm{dn}^{2n+1}(x)[\mathrm{dn}(x+a)+\mathrm{dn}(x-a)]\\ &= A\sum_{k=1}^{n} B^{k-1}\mathrm{dn}^{2(n-k+1)}(x) + 2B^n\mathrm{dn}(a)\\ &\quad + B^n\mathrm{cs}(a)[Z(x+a)-Z(x-a)-2Z(a)]. \end{aligned} \quad (30)$$

The corresponding cyclic identities with arbitrary weight are then immediately obtainable.

Proceeding in the same way, but starting from the identity

$$\begin{aligned}&\mathrm{dn}^2(x)[\mathrm{dn}(x+a)-\mathrm{dn}(x-a)]\\ &= D\mathrm{cn}(x)\mathrm{sn}(x) + B[\mathrm{dn}(x+a)-\mathrm{dn}(x-a)],\\ &D = -2m\,\mathrm{cs}(a), \end{aligned} \quad (31)$$

multiplying recursively by $\mathrm{dn}^2(x)$, and using the identity (31), we obtain the following identities of arbitrary odd and even rank:

$$\begin{aligned}&\mathrm{dn}^{2n}(x)[\mathrm{dn}(x+a)-\mathrm{dn}(x-a)]\\ &= D\sum_{k=1}^{n} B^{k-1}\mathrm{cn}(x)\mathrm{sn}(x)\mathrm{dn}^{2(n-k)}(x)\\ &\quad + B^n[\mathrm{dn}(x+a)-\mathrm{dn}(x-a)], \end{aligned} \quad (32)$$

$$\begin{aligned}&\mathrm{dn}^{2n+1}(x)[\mathrm{dn}(x+a)-\mathrm{dn}(x-a)]\\ &= D\sum_{k=1}^{n} B^{k-1}\mathrm{cn}(x)\mathrm{sn}(x)\mathrm{dn}^{2(n-k)+1}(x)\\ &\quad + B^n\mathrm{cs}(a)[Z(x+a)+Z(x-a)-2Z(x)]. \end{aligned} \quad (33)$$

The corresponding cyclic identities are then immediately written down. Further, by adding identities (28) and (32), we obtain the basic local cyclic identity of any odd rank:

$$\mathrm{dn}^{2n}(x)\mathrm{dn}(x+a) = \frac{D}{2}\sum_{k=1}^{n} B^{k-1}\mathrm{cn}(x)\mathrm{sn}(x)\mathrm{dn}^{2(n-k)}(x) + B^{n}\mathrm{dn}(x+a) + \frac{A}{2}\sum_{k=1}^{n} B^{k-1}\mathrm{dn}^{2(n-k)+1}(x), \quad (34)$$

where A, B, D are given by eqs (27) and (31). It is worth pointing out that using this identity, we can immediately write down the local identity for the combination $\mathrm{dn}(x)\mathrm{dn}^{2n}(x+a)$. This is done by replacing x by $x-a$ followed by changing a to $-a$ in eq. (34). In this way we obtain

$$\mathrm{dn}(x)\mathrm{dn}^{2n}(x+a) = \frac{A}{2}\sum_{k=1}^{n} B^{k-1}\mathrm{dn}^{2(n-k)+1}(x+a) - \frac{D}{2}\sum_{k=1}^{n} B^{k-1}\mathrm{cn}(x+a)\mathrm{sn}(x+a)\mathrm{dn}^{2(n-k)}(x+a) + B^{n}\mathrm{dn}(x). \quad (35)$$

Now $\mathrm{dn}(x)\mathrm{dn}^{2n}(x-a)$ can be immediately obtained from here by replacing a by $-a$. We find that

$$\sum_{j=1}^{p} d_j^{2n}\left[d_{j+r} \pm d_{j-r}\right] = \pm\sum_{j=1}^{p} d_j\left[d_{j+r}^{2n} \pm d_{j-r}^{2n}\right], \quad (36)$$

$$\sum_{j=1}^{p}(-1)^{j-1} d_j^{2n}\left[d_{j+r} \pm d_{j-r}\right] = \mp\sum_{j=1}^{p}(-1)^{j-1} d_j\left[d_{j+r}^{2n} \pm d_{j-r}^{2n}\right]. \quad (37)$$

In the next section we shall see that similar relations are in fact true in general for any such combinations of Jacobi elliptic functions.

Proceeding in the same way, by starting from each of the lower rank identities given in Appendices A, B, C we can write down the corresponding local and hence cyclic identities of arbitrary rank. Some illustrative examples with arbitrary even powers of $\mathrm{dn}(x)$ (or $\mathrm{sn}(x)$ or $\mathrm{cn}(x)$) are given in Appendix C. The results for arbitrary odd powers of $\mathrm{dn}(x)$ (or $\mathrm{sn}(x)$ or $\mathrm{cn}(x)$) are easily obtainable from the even rank ones via the identities of Appendix A.

4. Master local identities

In ref. II, we derived four master identities from which all the cyclic identities could be derived as special cases. In this section we show how a similar procedure works at the level of the local identities, thereby systematizing the identities and providing a unified framework for them. Besides, rather than using the addition formulas for

Jacobi elliptic functions, the master identities (MI) provide an alternative way to derive the local, and hence also cyclic, identities. We may note here the differences that arise in the two approaches: addition formulas do not lead to unevaluated constants in the form of integrals on the right-hand side, while the MI, in particular one of the four classes of MI do; on the other hand MI reduces the right-hand side maximally to standard forms, while the addition formulas approach needs considerable algebraic manipulation to attain the simplest final form. In any case, the two approaches are of course compatible and either of them may be used. In this section and in §6, we will use z instead of x as the variable to emphasize that there is no restriction to the real numbers.

The classification in ref. I and in the above sections has been in terms of the polynomial order of the elliptic functions appearing in the left-hand side, called the rank of the identities. In contrast, the master identities first use symmetries to identify four classes. Within each class, the identities are characterized by a number which is the highest order of the singularities in the fundamental domain of the left-hand side. Thus, the analytic structure of the functions appearing on the left-hand side of any identity determines the constants and the form of the functions that appear on the right-hand side. It is then easy to write symbolic manipulation programs that turn any given form of the left-hand side into a local identity.

We first recall essential details of the analytic properties of the Jacobi elliptic functions [6]. The function $\mathrm{dn}(z)$ is an even function of order two; there are two simple poles inside the period parallelogram $(0, 2K, 2K + 4iK', 4iK')$ situated at iK' and $3iK'$ with residues of $-i$ and i respectively. The function $\mathrm{sn}(z)$ is an odd function of order two; with two simple poles situated at iK' and $iK' + 2K$, with residues $1/\sqrt{m}$ and $-1/\sqrt{m}$, inside the fundamental period parallelogram $(0, 4K, 4K + 2iK', 2iK')$. The function $\mathrm{cn}(z)$ is an even function of order two; with two simple poles situated at iK' and $2K + iK'$, with residues $-i/\sqrt{m}$ and $i/\sqrt{m}$, inside the fundamental parallelogram $(-2K, 2K, 4K + 2iK', 2iK')$. We note that the lattice of the poles in the complex plane is identical for all these three functions. However these functions have the following important distinguishing properties: $\mathrm{dn}(z + 2iK') = -\mathrm{dn}(z)$, $\mathrm{sn}(z + 2iK') = \mathrm{sn}(z)$, $\mathrm{cn}(z + 2iK') = -\mathrm{cn}(z)$, $\mathrm{dn}(z + 2K) = \mathrm{dn}(z)$, $\mathrm{sn}(z + 2K) = -\mathrm{sn}(z)$, $\mathrm{cn}(z + 2K) = -\mathrm{cn}(z)$.

The symmetry and periodicity properties allow us to concentrate on the region $(0, 2K, 2K+2iK', 2iK')$ uniformly for all the functions, and consider only *one* simple pole at iK'. We supplement these possible symmetries with one additional one, for which we describe the properties of the elliptic function $\mathrm{dn}^2(z)$. Equivalently one may choose $\mathrm{cn}^2(z)$ or $\mathrm{sn}^2(z)$. The function $\mathrm{dn}^2(z)$ has the fundamental domain $(0, 2K, 2K + 2iK', 2iK')$ and consequently is completely periodic with respect to translations of $2K$ and $2iK'$. It is also of order two, with one double pole at iK', with a residue of 0. Thus we classify functions $f(z)$ constructed from the Jacobi elliptic functions into four symmetry classes. We define the quantities P, Q by

$$f(z + 2iK') = (-1)^P f(z), \quad f(z + 2K) = (-1)^Q f(z), \quad P, Q = 0, 1. \tag{38}$$

We denote the four possibilities by $(-,+)$, $(+,+)$, $(+,-)$ and $(-,-)$, where the first sign refers to the sign of $(-1)^P$ and the second to that of $(-1)^Q$. We note that as far as periodicity is concerned these functions are identical to $\mathrm{dn}(z)$, $\mathrm{dn}^2(z)$, $\mathrm{sn}(z)$ and $\mathrm{cn}(z)$ respectively. We also note that repeated differentiation does not change the symmetry class to which the functions belong, while this creates functions with arbitrarily high order of poles. This then allows us to tailor suitable combinations of derivatives of these four functions such that not only the periodicity, but also the singular parts match with the given function $f(z)$. Thus the difference between the function $f(z)$ and the tailored combination is an elliptic function with no poles anywhere including at infinity. We then use Liouville's theorem (which states that an analytic function with no poles anywhere including at infinity, must be a constant) to show that the constant is zero in all cases except the second, where it can be evaluated as a definite integral.

4.1 Master identities of Types I, III and IV

Let $f(z)$ be an elliptic function with the symmetry properties corresponding to $P=1$, $Q=0$, (Type-I) and having n_p poles at positions a_r $(r=1,\ldots,n_p)$ within the region $(0,2K,2K+2iK',2iK')$ which we will call $ABCD$. Let the principal part around the pole a_r be

$$\sum_{l_r=1}^{L_r}\frac{\alpha_{l_r}^{(r)}}{(z-a_r)^{l_r}}. \tag{39}$$

We note that the principal part of $\mathrm{dn}(z)$ around the pole iK' is $-i/(z-iK')$. Therefore if we consider the function $g(z)$:

$$g(z)=\sum_{r=1}^{n_p}\sum_{l_r=1}^{L_r}\frac{i(-1)^{l_r-1}}{(l_r-1)!}\alpha_{l_r}^{(r)}\frac{d^{l_r-1}\,\mathrm{dn}(z)}{dz^{l_r-1}}\bigg|_{z-a_r+iK'}, \tag{40}$$

this has identical poles as $f(z)$ and at these poles also has identical principal parts. Due to the symmetry requirements, the functions $f(z)$ and $g(z)$ also have identical periods and hence by Liouville's theorem they can differ utmost by a constant that is independent of z. However, integrating both these functions from 0 to $4iK'$, we see from the antisymmetry that these must vanish, implying that the constant is zero; and hence $f(z)=g(z)$. This is our 'master' local identity of Type-I; often the evaluated function $g(z)$ is of a simpler form than $f(z)$.

Consider as an illustration the identity that results when $f(z)=\mathrm{dn}^2(z)[\mathrm{dn}(z+a)+\mathrm{dn}(z-a)]$. This function has three poles, within $ABCD$ at $a_1=iK'$, $a_2=-a+iK'$, and $a_3=a+iK'$. This function has $P=1, Q=0$, and hence is of Type-I. At $a_1\equiv iK'$ the principal part is $-2\,i\mathrm{ds}(a)\mathrm{ns}(a)/(z-iK')$. We note that although $\mathrm{dn}^2(z)$ has a double pole at iK' it gets 'softened' by one, because $\mathrm{dn}(z+a)+\mathrm{dn}(z-a)$ has a zero at iK' for all a. This is the reason why we expect that the RHS of such identities are simpler than the LHS. Thus $\alpha_1^{(1)}=-2i\,\mathrm{ds}(a)\mathrm{ns}(a)$ and $L_1=1$. Similarly we get: $\alpha_1^{(2)}=i\,\mathrm{cs}^2(a)$, $L_2=1$ and $\alpha_1^{(3)}=i\,\mathrm{cs}^2(a)$, $L_3=1$. Hence this master identity yields the already stated result in eq. (26), which was alternatively derived using addition formulas.

We note that in the case of cyclic identities further simplification occurs and the α at the various poles can be summed up, while in the case of local identities they are left as they are. We note that the structure of the LHS yielded to simplification because roughly a zero cancelled a pole. We can look at the 'parts' of this identity where this does not happen fully. Thus when we take $f(z) = \mathrm{dn}^2(z)\mathrm{dn}(z-a)$ we get the identity

$$\mathrm{dn}^2(z)\,\mathrm{dn}(z-a) = \mathrm{ds}(a)\,\mathrm{ns}(a)\,\mathrm{dn}(z) + m\,\mathrm{cs}(a)\,\mathrm{cn}(z)\,\mathrm{sn}(z) \\ -\mathrm{cs}^2(a)\,\mathrm{dn}(z-a). \tag{41}$$

We note that the rank of the RHS is one less than that of the LHS, and further reduction by one occurs when a is changed to $-a$ and identities for $\mathrm{dn}^2(z)\mathrm{dn}(z+a)$ and $\mathrm{dn}^2(z)\mathrm{dn}(z-a)$ are added which results in the already quoted identity of eq. (26).

Similarly we derive the master identity for functions belonging to Type-III ($P = 0, Q = 1$):

$$f(z) = \sum_{r=1}^{n_p} \sum_{l_r=1}^{L_r} \frac{\sqrt{m}(-1)^{l_r-1}}{(l_r-1)!} \alpha_{l_r}^{(r)} \frac{d^{l_r-1}\,\mathrm{sn}(z)}{dz^{l_r-1}}\bigg|_{z-a_r+iK'}, \tag{42}$$

and Type-IV ($P = 1, Q = 1$):

$$f(z) = \sum_{r=1}^{n_p} \sum_{l_r=1}^{L_r} \frac{i\sqrt{m}(-1)^{l_r-1}}{(l_r-1)!} \alpha_{l_r}^{(r)} \frac{d^{l_r-1}\,\mathrm{cn}(z)}{dz^{l_r-1}}\bigg|_{z-a_r+iK'}. \tag{43}$$

That the constant is zero in the case of Type-III and Type-IV identities can be seen by integrating both sides from 0 to $4K$. It may be noted that symbolic manipulation packages that calculate series expansions can be effectively used to generate these identities.

4.2 Master identity of Type-II

This last type of identity deserves special mention; firstly the function archetype is $\mathrm{dn}^2(z)$ which has a double pole at iK', secondly it leads to identities with non-zero constants, and lastly the Jacobi zeta function appears in an essential way. A function belonging to this type is periodic with periods $2K$ and $2iK'$ (hence $P = 0, Q = 0$). Its principal part around iK' is $-1/(z-iK')^2$. Thus we write the master identity in this case as

$$f(z) = C + \sum_{r=1}^{n_p} \sum_{l_r=1}^{L_r} \frac{(-1)^{l_r-1}}{(l_r-1)!} \alpha_{l_r}^{(r)} \frac{d^{l_r-2}\,\mathrm{dn}^2(z)}{dz^{l_r-2}}\bigg|_{z-a_r+iK'}, \tag{44}$$

where C is a constant. Note that the derivative order starts from -1, which should be interpreted as an integral. Functions of this type can also have simple poles. $\mathrm{dn}^2(z)$ and its derivatives are not sufficient to construct these. If we include its integral the master identity is complete. Therefore we note the standard result [6] which can be taken to be the definition of the Jacobi zeta function $Z(z)$:

$$Z(z) = \int_0^z \left[\mathrm{dn}^2(u) - \frac{E}{K}\right] \mathrm{d}u = E(z) - \frac{E}{K}z, \tag{45}$$

where $E(z)$ is the incomplete elliptic integral of the second kind and E and K are the complete elliptic integrals of the second and first kinds respectively. Thus $Z(z)$ is closely related to the incomplete elliptic integral of the second kind, and is periodic with a period of $2K$, but is not elliptic. It is however *almost* elliptic due to the identity [6]: $Z(z + 2iK') = Z(z) - i\pi/K$. If we write the $l_r = 1$ part of this master equation, which has the only part with the Jacobi zeta function, it is

$$\sum_{r=1}^{n_p} \alpha_1^{(r)} Z(z - a_r + iK'). \tag{46}$$

We note that *this* is an elliptic function with the correct periods of $2K$ and $2iK'$, due to the fact that

$$\sum_{r=1}^{n_p} \alpha_1^{(r)} = 0. \tag{47}$$

This is the sum of the *residues* of the function at all the poles in $ABCD$. Making use of the double periodicity of $f(z)$ we find the integral around $ABCD$ vanishes, and hence from Cauchy's theorem it follows that the sum of the residues must also vanish, hence proving the above. Thus we are justified in using the zeta function in this type of master identity even if it is not elliptic: it will always appear in combinations that are elliptic functions.

Integrating both sides of the master identity from 0 to $2K$ we evaluate the constant C:

$$C = \frac{1}{2K}\int_0^{2K} f(z)\,\mathrm{d}z + \frac{\gamma_2 E}{K}, \tag{48}$$

where $\gamma_2 = \sum_{r=1}^{n_p} \alpha_2^{(r)}$. We can therefore evaluate the identity at some convenient z where there is no singularity (for instance perhaps $z = 0$) and then make use of the above to evaluate definite integrals. Assuming for instance that there is no pole at $z = 0$ we may write

$$\frac{1}{2K}\int_0^{2K} f(z)\,\mathrm{d}z = f(0) - \gamma_2 \frac{E}{K} - \sum_{r=1}^{n_p}\sum_{l_r=1}^{L_r} \frac{(-1)^{l_r-1}}{(l_r-1)!}\alpha_{l_r}^{(r)} \left.\frac{d^{l_r-2}\,\mathrm{dn}^2(z)}{dz^{l_r-2}}\right|_{iK'-a_r}. \tag{49}$$

Note that this integral cannot be evaluated by a direct application of Cauchy's theorem due to the vanishing of both the contour integral around $ABCD$ and its residue.

We also point out here something that is of relevance to the cyclic identities:

$$\sum_{j=1}^{p} g(z_j)[h(z_{j+1}) \pm h(z_{j-1})] = \pm \sum_{j=1}^{p} h(z_j)[g(z_{j+1}) \pm g(z_{j-1})], \tag{50}$$

$$\sum_{j=1}^{p}(-1)^j g(z_j)[h(z_{j+1}) \pm h(z_{j-1})] = \mp \sum_{j=1}^{p}(-1)^j h(z_j)[g(z_{j+1}) \pm g(z_{j-1})], \tag{51}$$

where $h(z)$ and $g(z)$ are combinations of Jacobi elliptic functions as above. These relate ordinary and alternating sums under an interchange of h and g and follow from quite general considerations related only to the periodicity of h and g. The RHS amounts to a rewriting of the LHS if we recall that either $h(z_{p+1}) = h(z_1)$ and $g(z_{p+1}) = g(z_1)$ or $h(z_{p+1}) = -h(z_1)$ and $g(z_{p+1}) = -g(z_1)$ as the whole function $g(z)[h(z+T/p) + h(z-T/p)]$ is periodic with period T.

5. Evaluation of several elliptic integrals

In ref. II, we obtained several cyclic identities of type MI-II in which the right-hand side contained a definite integral involving products of Jacobi elliptic functions. These integrals are not available in standard tables of integrals [3,6]. We now explicitly evaluate many such definite integrals using local identities.

As an illustration, we start from local identity (C4). On integrating both sides with respect to x over an interval $[0, 2K]$, one gets the definite integral

$$\int_0^{2K} \mathrm{dn}^2(x)\mathrm{dn}^2(x+a)\,\mathrm{d}x$$
$$= -4E\mathrm{cs}^2(a) + 2K[\mathrm{cs}^2(a) + \mathrm{ds}^2(a) - 2\mathrm{cs}(a)\mathrm{ds}(a)\mathrm{ns}(a)Z(a)]. \tag{52}$$

It may be noted that here a is any non-zero constant. Using this value of the integral in the MI-II cyclic identity (8) that we obtained in ref. II and choosing $a = 2rK/p$, immediately yields the cyclic identity (20) which we had directly obtained from the local identity.

The other definite integrals which we are now able to evaluate are related to cyclic identities containing an even number of dn or sn or cn. For example, in ref. II, we simply stated the identity

$$\frac{1}{p}\sum_{j=1}^{p} d_j d_{j+r} d_{j+s} d_{j+t} \equiv A$$
$$= \frac{1}{2K}\int_0^{2K} \mathrm{dn}(x)\mathrm{dn}(x+a)\mathrm{dn}(x+a')\mathrm{dn}(x+a'')\,\mathrm{d}x, \tag{53}$$

but were unable to evaluate the integral and hence find A. Here, $a = 2rK/p, a' = 2sK/p, a'' = 2tK/p$. We now use the local identities to evaluate this definite integral for arbitrary but unequal a, a', a''. The proof begins with the local identity (C1). Integrating both sides over the interval $[0, 2K]$ yields eq. (E7). Note that the special case $a = 2rK/p, a' = 2sK/p, a'' = 2tK/p$ yields the cyclic identity (53).

Finally, there are some MI-II cyclic identities and hence definite integrals which were not even discussed in ref. II. For example, consider the local identity (B10) from which by following the method explained in §2 we can deduce the cyclic identity

$$\frac{1}{p}\sum_{j=1}^{p} m d_j s_{j+r} c_{j+s} = \frac{1}{2K}\int_0^{2K} m\,\mathrm{dn}(x)\mathrm{sn}(x+a)\mathrm{cn}(x+a')\,\mathrm{d}x, \qquad (54)$$

where $a = 2rK/p, a' = 2sK/p$. By using the local identity (B10), we can in fact obtain this integral for arbitrary but unequal values of a, a'. In particular, integrating eq. (B10) over the interval $[0, 2K]$ yields

$$\frac{1}{2K}\int_0^{2K} m\,\mathrm{dn}(x)\mathrm{sn}(x+a)\mathrm{cn}(x+a')\,\mathrm{d}x$$
$$= -\mathrm{ds}(a-a')[\mathrm{dn}(a) - \mathrm{cs}(a)Z(a)] + \mathrm{ns}(a-a')[\mathrm{dn}(a') - \mathrm{cs}(a')Z(a')]. \qquad (55)$$

In the special case when $a = 2rK/p, a' = 2sK/p$, one recovers the cyclic identity (54).

Proceeding in the same way, we have obtained expressions for all the definite integrals which appear in the cyclic identities of the type MI-II. Some of these definite integrals are given in Appendix E.

We can even simplify several indefinite integrals. In fact, by starting from any local identity we can obtain the indefinite integral of its left-hand side in terms of the well-known integrals [3,6] of $\mathrm{sn}^n(x), \mathrm{dn}^n(x), \mathrm{cn}^n(x)$. The only exceptions are those MI-II local identities in which the right-hand side has a term proportional either to $[Z(x+a)+Z(x-a)-2Z(x)]$ or $[Z(x+a)-Z(x-a)]$. We show below that in that case the indefinite integral of the left-hand side also has terms containing indefinite elliptic integrals of the first, second and third kind [3,6].

We start from the local identity

$$\mathrm{dn}^2(x)\mathrm{dn}(x+a) = B\,\mathrm{dn}(x+a) + \mathrm{ds}(a)\mathrm{ns}(a)\mathrm{dn}(x) - m\,\mathrm{cs}(a)\mathrm{sn}(x)\mathrm{cn}(x),$$
$$B \equiv -\mathrm{cs}^2(a), \qquad (56)$$

which is obtained by adding the identities (26) and (31). On integrating both sides with respect to x and using the known integral of $\mathrm{dn}(x)$ [3] we then obtain

$$\int \mathrm{dn}^2(x)\mathrm{dn}(x+a)\,\mathrm{d}x = B\,\mathrm{am}(x+a) + \mathrm{ds}(a)\mathrm{ns}(a)\mathrm{am}(x) + \mathrm{cs}(a)\mathrm{dn}(x), \qquad (57)$$

where $\int \mathrm{dn}(x)\,\mathrm{d}x = \mathrm{am}(x) = \sin^{-1}(\mathrm{sn}(x)) = i\ln[\mathrm{cn}(x) - i\,\mathrm{sn}(x)]$. On multiplying both sides of eq. (56) by $\mathrm{dn}^2(x)$ and using eq. (57) we then find that

$$\int \mathrm{dn}^4(x)\mathrm{dn}(x+a)\,\mathrm{d}x = B^2\mathrm{am}(x+a)$$
$$+\mathrm{ds}(a)\mathrm{ns}(a)\left[1 + B - \frac{m}{2}\right]\mathrm{am}(x)$$
$$+B\,\mathrm{cs}(a)\mathrm{dn}(x) + \frac{\mathrm{cs}(a)}{3}\mathrm{dn}^3(x)$$
$$+\frac{m\,\mathrm{ds}(a)\mathrm{ns}(a)}{2}\mathrm{sn}(x)\mathrm{cn}(x). \qquad (58)$$

Proceeding in this way, one can find the indefinite integral of $\mathrm{dn}^{2n}(x)\mathrm{dn}(x+a)$. One can also derive a recursion relation relating the various integrals. In particular, using eq. (34) one can show that

$$\begin{aligned}\mathrm{dn}^{2n}(x)\mathrm{dn}(x+a) = {} & B\,\mathrm{dn}^{2n-2}(x)\mathrm{dn}(x+a)+\mathrm{ds}(a)\mathrm{ns}(a)\mathrm{dn}^{2n-1}(x)\\ & -m\,\mathrm{cs}(a)\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}^{2n-2}(x).\end{aligned} \tag{59}$$

On integrating both sides with respect to x, we find a recursion relation relating various integrals:

$$\begin{aligned} & I_n = BI_{n-1}+\frac{\mathrm{cs}(a)}{(2n-1)}\mathrm{dn}^{2n-1}(x)+\mathrm{ds}(a)\mathrm{ns}(a)\int \mathrm{dn}^{2n-1}(x)\,\mathrm{d}x;\\ & I_k \equiv \int \mathrm{dn}^{2k}(x)\mathrm{dn}(x+a)\,\mathrm{d}x.\end{aligned} \tag{60}$$

Note that the integral of any power of $\mathrm{dn}(x)$ (as well as $\mathrm{sn}(x), \mathrm{cn}(x)$) is known in principle [3]. We might also add that once the integral of say $\mathrm{dn}^{2n}(x)\mathrm{dn}(x+a)$ is known then the integral of $\mathrm{dn}(x)\mathrm{dn}^{2n}(x+a)$ is obtained from it by simply replacing x by $x-a$ followed by $a \to -a$.

Using the above procedure one can obtain indefinite integrals of the left-hand sides of all the local identities given in this paper except for those MI-II local identities in which the combination $[Z(x+a)-Z(x-a)]$ or $[Z(x+a)+Z(x-a)-2Z(x)]$ occurs on the right-hand side. To handle these integrals, we start from the local identity (17). On using eq. (9) it is easily shown that

$$\mathrm{dn}(x)[\mathrm{dn}(x+a)+\mathrm{dn}(x-a)] = \frac{2\mathrm{dn}(a)[1-m\,\mathrm{sn}^2(x)]}{1-m\,\mathrm{sn}^2(a)\mathrm{sn}^2(x)}\ . \tag{61}$$

On integrating both sides of eq. (17) over x, using eq. (61) and well-known integrals (see integrals 336.01 and 337.01 of [6]) we finally find that

$$\begin{aligned} & \int \mathrm{dn}(x)[\mathrm{dn}(x+a)+\mathrm{dn}(x-a)]\,\mathrm{d}x = 2[\mathrm{dn}(a)-\mathrm{cs}(a)Z(a)]x\\ & \quad +\mathrm{cs}(a)\int [Z(x+a)-Z(x-a)]\,\mathrm{d}x\\ & = 2\mathrm{ds}(a)\mathrm{ns}(a)F(\mathrm{am}\ x,k)-2\mathrm{dn}(a)\mathrm{cs}^2(a)\Pi(\mathrm{am}\ x,k^2\mathrm{sn}^2(a),k),\end{aligned} \tag{62}$$

where $k^2=m$. Here $F(\mathrm{am}\ x,k)$ and $\Pi(\mathrm{am}\ x,k^2\mathrm{sn}^2(a),k)$ are indefinite elliptic integrals of first and third kind respectively. Similarly, on using eqs (9) and (18) it is easy to show that

$$\begin{aligned} & \int \mathrm{dn}(x)[\mathrm{dn}(x+a)-\mathrm{dn}(x-a)]\,\mathrm{d}x\\ & = \mathrm{cs}(a)\int [Z(x+a)+Z(x-a)-2Z(x)]\,\mathrm{d}x\\ & = \mathrm{cs}(a)\ln[1-m\,\mathrm{sn}^2(a)\mathrm{sn}^2(x)],\end{aligned} \tag{63}$$

and hence

$$\int \mathrm{dn}(x)\mathrm{dn}(x+a)\,\mathrm{d}x = [\mathrm{dn}(a) - \mathrm{cs}(a)Z(a)]x$$
$$- \mathrm{cs}(a)\int [Z(x+a) - Z(x)]\,\mathrm{d}x$$
$$= \mathrm{ds}(a)\mathrm{ns}(a)F(\mathrm{am}\ x, k) - \mathrm{dn}(a)\mathrm{cs}^2(a)\Pi(\mathrm{am}\ x, k^2\mathrm{sn}^2(a), k)$$
$$+ (1/2)\mathrm{cs}(a)\ln[1 - m\,\mathrm{sn}^2(a)\mathrm{sn}^2(x)]. \quad (64)$$

We now show that using eq. (64) one can obtain the indefinite integral of the left-hand side of any local MI-II identity. As an illustration, consider the local identity (C15). It is easy to show from here that

$$\mathrm{dn}^{2n}(x)\mathrm{dn}^2(x+a)$$
$$= B\,\mathrm{dn}^{2n-2}(x)[\mathrm{dn}^2(x) + \mathrm{dn}^2(x+a)]$$
$$- (1-m)\mathrm{dn}^{2n-2}(x) + 2B^{n-1}\mathrm{ds}(a)\mathrm{ns}(a)\mathrm{dn}(x+a)\mathrm{dn}(x)$$
$$+ 2\mathrm{ds}(a)\mathrm{ns}(a)\,[\mathrm{ds}(a)\mathrm{ns}(a)\mathrm{dn}(x) - m\,\mathrm{cs}(a)\mathrm{sn}(x)\mathrm{cn}(x)]$$
$$\times \sum_{k=1}^{n-1} B^{k-1}[\mathrm{dn}(x)]^{2(n-k)-1}, \quad (65)$$

where $B \equiv -\mathrm{cs}^2(a)$ and $n \geq 2$. On integrating both sides of this equation, we get the recursion relation

$$I_n = BI_{n-1} + \int B\,\mathrm{dn}^{2n}(x)\,\mathrm{d}x - (1-m)\int \mathrm{dn}^{2n-2}(x)\,\mathrm{d}x$$
$$+2B^{n-1}\mathrm{ds}(a)\mathrm{ns}(a)\int \mathrm{dn}(x+a)\mathrm{dn}(x)\,\mathrm{d}x$$
$$+\mathrm{cs}(a)\mathrm{ds}(a)\mathrm{ns}(a)\sum_{k=1}^{n-1} B^{k-1}\frac{\mathrm{dn}^{2n-2k}(x)}{n-k}$$
$$+2\mathrm{ds}^2(a)\mathrm{ns}^2(a)\sum_{k=1}^{n-1}\int B^{k-1}\mathrm{dn}^{2(n-k)}(x)\,\mathrm{d}x, \quad (66)$$

where I_1 is easily obtained by using the integral (64) and the local identity (C4). We find

$$I_1 \equiv \int \mathrm{dn}^2(x)\mathrm{dn}^2(x+a)\,\mathrm{d}x$$
$$= -\mathrm{cs}^2(a)[E(\mathrm{am}\ x, k) + E(\mathrm{am}(x+a), k)]$$
$$-(1-m)x + 2\mathrm{ds}(a)\mathrm{ns}(a)\int \mathrm{dn}(x+a)\mathrm{dn}(x)\mathrm{d}x, \quad (67)$$

where $E(\mathrm{am}\ x, k)$ is the indefinite elliptic integral of the second kind.

Similar recursion relations for several other indefinite elliptic integrals are given in Appendix F.

6. Continuum limit of local and cyclic identities

We now study what happens to the local identities as $a \to 0$. Although the identities are not valid at $a = 0$, this limit leads to well-known non-linear ordinary differential equations satisfied by the Jacobi elliptic functions. Thus, the local identities may be viewed as an *exact* discretization of these differential equations. This provides the justification for calling these identities 'local'. This is to be contrasted with the cyclic identities that are exact discretizations of integral identities. Just as the differential equation can be integrated, the local identities are simply summed to produce cyclic identities.

Take the simple Type-I identity:

$$\begin{aligned} &\mathrm{dn}^2(z)[\mathrm{dn}(z+a) + \mathrm{dn}(z-a)] \\ &= 2\mathrm{ds}(a)\mathrm{ns}(a)\mathrm{dn}(z) - \mathrm{cs}^2(a)[\mathrm{dn}(z+a) + \mathrm{dn}(z-a)]. \end{aligned} \tag{68}$$

Since $\mathrm{cs}^2(a)$ has a pole of order two at $a = 0$, we expand to second order in a:

$$\mathrm{dn}(z+a) + \mathrm{dn}(z-a) = 2\mathrm{dn}(z) + a^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathrm{dn}(z) + O(a^3). \tag{69}$$

Using $\lim_{a\to 0}\, [\mathrm{ds}(a)\mathrm{ns}(a) - \mathrm{cs}^2(a)] = 1 - m/2$, then leads to the limiting differential equation:

$$(2-m)y - \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = 2y^3, \tag{70}$$

with $y = \mathrm{dn}(z)$. Of the many applications of such differential equations, the most straightforward one would be the interpretation of this as Newton's equation of motion for a particle in a one-dimensional double well potential, with z taking the role of time.

It is well-known that finite difference versions of such non-linear one-dimensional problems tend to exhibit chaos, and therefore the difference equation does not have analytical solutions in terms of the Jacobi elliptic functions. However, we precisely achieve this when converting this differential equation into a finite difference equation in the following manner:

$$\begin{aligned} &(2-m)y(z) - \frac{[y(z+\Delta) + y(z-\Delta) - 2y(z)]}{\Delta^2} \\ &= y^2(z)[y(z+\Delta) + y(z-\Delta)]. \end{aligned} \tag{71}$$

Of course depending on the smallness of Δ, the above discrete equation will only be an approximation to the actual solution $\mathrm{dn}(z)$. However, replacing $2-m$ in the difference scheme with $2[\mathrm{ds}(\Delta)\mathrm{ns}(\Delta) - \mathrm{cs}^2(\Delta)]$ and the $1/\Delta^2$ factor multiplying the second difference with $\mathrm{cs}^2(\Delta)$ leads to an *exact* difference scheme:

$$\begin{aligned} &2[\mathrm{ds}(\Delta)\mathrm{ns}(\Delta) - \mathrm{cs}^2(\Delta)]y_n - \mathrm{cs}^2(\Delta)[y_{n+1} + y_{n-1} - 2y_n] \\ &= y_n^2[y_{n+1} + y_{n-1}], \end{aligned} \tag{72}$$

where $y(n) \equiv y(z + n\Delta)$, which is identical to the local identity in eq. (68) when we put $\Delta \equiv a$. This is of course just the reverse of the small a limit.

The local identity implies that $y_n = \mathrm{dn}(an)$ is a solution of this difference equation with the initial condition $y_0 = 1$, $y_1 = \mathrm{dn}(a)$. The modulus parameter is m in all the elliptic functions involved. This is analogous to the differential equation (eq. (70)) whose solution is $y(z) = \mathrm{dn}(z)$, when the initial conditions are $y(0) = 1, y'(0) = 0$, where the prime denotes differentiation. It may be noted that the local identity in eq. (41) also has the same continuum limit as the above, and that the difference equation is therefore genuinely second order rather than first. All of the two-point local identities limit to well-known differential equations, or to those that can be derived from these.

Cyclic identities limit to definite integrals, but we can generalize to indefinite integrals. For instance, consider the Type-I local identity in eq. (68). Let $z_i = z_0 + A\,(i-1)/p$, with $i = 1,\ldots,p$. Here A is arbitrary and p is an integer, z_i are the sample points between z_0 and $z \equiv z_0 + A$. The local identities give

$$\begin{aligned}
&\sum_{i=1}^{p} \mathrm{dn}^2(z_i)[\mathrm{dn}(z_{i+1}) + \mathrm{dn}(z_{i-1})] \\
&= [2\mathrm{ds}(a)\mathrm{ns}(a) - \mathrm{cs}^2(a)] \sum_{i=1}^{p} \mathrm{dn}(z_i) \\
&\quad - \mathrm{cs}^2(a)[\mathrm{dn}(z_{p+1}) - \mathrm{dn}(z_1) + \mathrm{dn}(z_0) - \mathrm{dn}(z_p)],
\end{aligned} \tag{73}$$

with $a = A/p$. This becomes a cyclic identity when $A = 2K$ in which case the 'end correction' that is the last term in the above equation vanishes due to the periodicity of $\mathrm{dn}(z)$. Multiply both sides by A/p and then take the $a \to 0$ (or equivalently $p \to \infty$) limit. The end corrections are rewritten and evaluated as follows:

$$\begin{aligned}
&-\frac{A}{p}\mathrm{cs}^2(a)[\mathrm{dn}(z_{p+1}) - \mathrm{dn}(z_1) + \mathrm{dn}(z_0) - \mathrm{dn}(z_p)] \\
&\to -\frac{p}{A}[\mathrm{dn}(z_{p+1}) - \mathrm{dn}(z_p) + \mathrm{dn}(z_0) - \mathrm{dn}(z_1)] \\
&\to m\,\mathrm{sn}(z)\mathrm{cn}(z) - m\,\mathrm{sn}(z_0)\mathrm{cn}(z_0).
\end{aligned} \tag{74}$$

Here we have used that the derivative of $\mathrm{dn}(z)$ is $-m\,\mathrm{sn}(z)\mathrm{cn}(z)$. Thus the entire identity in eq. (73) limits effectively to the indefinite integral

$$\int \mathrm{dn}^3(z)\,\mathrm{d}z = \frac{1}{2}(2-m)\int \mathrm{dn}(z)\,\mathrm{d}z + \frac{m}{2}\mathrm{sn}(z)\mathrm{cn}(z), \tag{75}$$

which is a standard identity, for instance eq. (314.03) of Byrd and Friedman [6]. Thus the cyclic identities are exact Riemann sums of integral identities while local identities are discretizations of differential equations.

In the case of three-point local identities (those which have the elliptic functions evaluated at three points, z, $z + a$ and $z + a'$) there are more than one limiting cases. The case $a' \to a$ or $a' \to 0$ leads to two-point local identities involving z and $z + a$. For instance starting from the local identity in eq. (B1) and taking the $a \to a'$ limit or the $a' \to 0$ limit leads essentially to the local two-point identity in eq. (41).

7. Comments and discussion

In this paper, we have proved a wide class of local identities, using which we have obtained corresponding cyclic identities with arbitrary weights. These results are easily extended in several directions. For example, we could evaluate these identities at points separated by gaps of T/p with imaginary or complex period T thereby obtaining corresponding local identities for pure imaginary as well as complex shifts. Secondly, we can also obtain similar identities for all nine auxiliary Jacobi functions like $\mathrm{ns}(x,m)$ as well as six ratios of Jacobi functions like $\mathrm{cn}(x)\mathrm{dn}(x)/\mathrm{sn}(x)$. Further, by following the procedure in ref. II, we can readily write down the corresponding local as well as cyclic identities for Weierstrass functions, as well as for the ratio of any two of the four Jacobi theta functions.

Many of these identities have non-trivial $m = 0, 1$ limits. For example, a cyclic identity derived in ref. II, which is valid for any odd integer p, and $l < p$, is

$$\sum_{j=1}^{p} d_j d_{j+r} \dots d_{j+(l-1)r} = \left[\prod_{k=1}^{(l-1)/2} \mathrm{cs}^2(ka) + 2(-1)^{(l-1)/2} \sum_{k=1}^{(l-1)/2} \prod_{n=1, n\neq k}^{l} \mathrm{cs}([n-k]a) \right] \sum_{j=1}^{p} d_j, \tag{76}$$

while for $l = p$, we have the simpler identity

$$\prod_{j=1}^{p} d_j = \prod_{n=1}^{(p-1)/2} \mathrm{cs}^2\left(\frac{2Kn}{p}\right) \sum_{j=1}^{p} d_j, \tag{77}$$

where $a = r2K/p$. At $m = 0$, these identities reduce to interesting trigonometric identities

$$1 = \prod_{k=1}^{(l-1)/2} \cot^2\left(\frac{rk\pi}{p}\right) + 2(-1)^{(l-1)/2} \sum_{k=1}^{(l-1)/2} \prod_{n=1, n\neq k}^{l} \cot\left([n-k]\frac{r\pi}{p}\right), \tag{78}$$

$$\frac{1}{p} = \prod_{n=1}^{(p-1)/2} \cot^2\left(\frac{n\pi}{p}\right). \tag{79}$$

Actually, identity (78) is also valid for even p and $r = 1$, provided $l < (p+2)/2$.

For the special case of $l = 3$, one can write down cyclic identities for products of three dn's at arbitrary separation (in units of $2K/p$) for both even and odd p. In particular, using the local identity (B1), one can immediately write down cyclic identities for combinations like $\sum_{j=1}^{p} d_j d_{j+r} d_{j+s}$ with r and s being unequal but arbitrary otherwise. In the limit $m = 0$, summing of all such independent cyclic identities yields the following remarkable trigonometric identities:

$$\frac{(p-1)(p-2)}{3} = \sum_{j=1}^{p-1} \cot^2\left(\frac{j\pi}{p}\right), \quad p > 2. \tag{80}$$

Similarly, many of the other local identities we have derived in this paper also reduce to interesting trigonometric identities in the limit $m = 0$.

Finally, note that even though our local identities have been derived assuming that a is an arbitrary constant, the identities are also valid when a is any function of x. For example, on calling $x + a = b$, the local identities (A4) and (A5) give generalized addition theorems

$$\mathrm{dn}(a-b)\mathrm{sn}(a)\mathrm{sn}(b) + \mathrm{cn}(a)\mathrm{cn}(b) = \mathrm{cn}(a-b), \tag{81}$$

$$\mathrm{dn}(a-b)\mathrm{sn}(a)\mathrm{cn}(b) - \mathrm{cn}(a)\mathrm{sn}(b) = -\mathrm{dn}(a)\mathrm{sn}(a-b), \tag{82}$$

which in the limit $m = 0$ reduce to the well-known addition theorems for the trigonometric functions.

Appendix A: Local identities of rank 2

Rank 2 identities with 2 distinct arguments $(x, x+a)$

$$\mathrm{dn}(x)\mathrm{dn}(x+a) = \mathrm{dn}(a) + \mathrm{cs}(a)\left[Z(x+a) - Z(x) - Z(a)\right], \tag{A1}$$

$$m\,\mathrm{sn}(x)\mathrm{sn}(x+a) = -\mathrm{ns}(a)\left[Z(x+a) - Z(x) - Z(a)\right], \tag{A2}$$

$$m\,\mathrm{cn}(x)\mathrm{cn}(x+a) = m\,\mathrm{cn}(a) + \mathrm{ds}(a)\left[Z(x+a) - Z(x) - Z(a)\right], \tag{A3}$$

$$\mathrm{dn}(x)\mathrm{sn}(x+a) = \mathrm{ns}(a)\mathrm{cn}(x) - \mathrm{cs}(a)\mathrm{cn}(x+a), \tag{A4}$$

$$\mathrm{dn}(x)\mathrm{cn}(x+a) = -\mathrm{ds}(a)\mathrm{sn}(x) + \mathrm{cs}(a)\mathrm{sn}(x+a), \tag{A5}$$

$$m\,\mathrm{sn}(x)\mathrm{cn}(x+a) = \mathrm{ds}(a)\mathrm{dn}(x) - \mathrm{ns}(a)\mathrm{dn}(x+a). \tag{A6}$$

Appendix B: Local identities of rank 3

Rank 3 identities with 3 distinct arguments $(x, x+a, x+a')$

$$\mathrm{dn}(x)\mathrm{dn}(x+a)\mathrm{dn}(x+a') = -\mathrm{cs}(a)\mathrm{cs}(a')\mathrm{dn}(x)$$
$$-\mathrm{cs}(a)\mathrm{cs}(a-a')\mathrm{dn}(x+a) + \mathrm{cs}(a')\mathrm{cs}(a-a')\mathrm{dn}(x+a'), \tag{B1}$$

$$m\,\mathrm{sn}(x)\mathrm{sn}(x+a)\mathrm{sn}(x+a') = \mathrm{ns}(a)\mathrm{ns}(a')\mathrm{sn}(x)$$
$$+\mathrm{ns}(a)\mathrm{ns}(a-a')\mathrm{sn}(x+a) - \mathrm{ns}(a')\mathrm{ns}(a-a')\mathrm{sn}(x+a'), \tag{B2}$$

$$m\,\mathrm{cn}(x)\mathrm{cn}(x+a)\mathrm{cn}(x+a') = -\mathrm{ds}(a)\mathrm{ds}(a')\mathrm{cn}(x)$$
$$-\mathrm{ds}(a)\mathrm{ds}(a-a')\mathrm{cn}(x+a) + \mathrm{ds}(a')\mathrm{ds}(a-a')\mathrm{cn}(x+a'), \tag{B3}$$
$$\mathrm{dn}(x)\mathrm{dn}(x+a)\mathrm{sn}(x+a') = -\mathrm{cs}(a)\mathrm{ns}(a')\mathrm{sn}(x)$$
$$-\mathrm{cs}(a)\mathrm{ns}(a-a')\mathrm{sn}(x+a) + \mathrm{cs}(a')\mathrm{cs}(a-a')\mathrm{sn}(x+a'), \tag{B4}$$
$$\mathrm{dn}(x)\mathrm{dn}(x+a)\mathrm{cn}(x+a') = -\mathrm{cs}(a)\mathrm{ds}(a')\mathrm{cn}(x)$$
$$-\mathrm{cs}(a)\mathrm{ds}(a-a')\mathrm{cn}(x+a) + \mathrm{cs}(a')\mathrm{cs}(a-a')\mathrm{cn}(x+a'), \tag{B5}$$
$$m\,\mathrm{sn}(x)\mathrm{sn}(x+a)\mathrm{dn}(x+a') = \mathrm{ns}(a)\mathrm{cs}(a')\mathrm{dn}(x)$$
$$+\mathrm{ns}(a)\mathrm{cs}(a-a')\mathrm{dn}(x+a) - \mathrm{ns}(a')\mathrm{ns}(a-a')\mathrm{dn}(x+a'), \tag{B6}$$
$$m\,\mathrm{sn}(x)\mathrm{sn}(x+a)\mathrm{cn}(x+a') = \mathrm{ns}(a)\mathrm{ds}(a')\mathrm{cn}(x)$$
$$+\mathrm{ns}(a)\mathrm{ds}(a-a')\mathrm{cn}(x+a) - \mathrm{ns}(a')\mathrm{ns}(a-a')\mathrm{cn}(x+a'), \tag{B7}$$
$$m\,\mathrm{cn}(x)\mathrm{cn}(x+a)\mathrm{dn}(x+a') = -\mathrm{ds}(a)\mathrm{cs}(a')\mathrm{dn}(x)$$
$$-\mathrm{ds}(a)\mathrm{cs}(a-a')\mathrm{dn}(x+a) + \mathrm{ds}(a')\mathrm{ds}(a-a')\mathrm{dn}(x+a'), \tag{B8}$$
$$m\,\mathrm{cn}(x)\mathrm{cn}(x+a)\mathrm{sn}(x+a') = -\mathrm{ds}(a)\mathrm{ns}(a')\mathrm{sn}(x)$$
$$-\mathrm{ds}(a)\mathrm{ns}(a-a')\mathrm{sn}(x+a) + \mathrm{ds}(a')\mathrm{ds}(a-a')\mathrm{sn}(x+a'), \tag{B9}$$
$$m\,\mathrm{dn}(x)\mathrm{sn}(x+a)\mathrm{cn}(x+a') = -\mathrm{ds}(a-a')$$
$$\times\{\mathrm{dn}(a) + \mathrm{cs}(a)\,[Z(x+a) - Z(x) - Z(a)]\}$$
$$+\mathrm{ns}(a-a')\{\mathrm{dn}(a') + \mathrm{cs}(a')\,[Z(x+a') - Z(x) - Z(a')]\}. \tag{B10}$$

Rank 3 identities with two distinct arguments $(x, x+a)$

As explained in the text, local identities for say $m\,\mathrm{cn}(x)\mathrm{sn}(x+a)\mathrm{dn}(x+a)$ and $m\,\mathrm{sn}(x)\mathrm{dn}(x)\mathrm{cn}(x+a)$ are related to each other by $x \to x-a$ followed by $a \to -a$ and hence only one of these identities is given below.

$$\mathrm{dn}^2(x)\mathrm{dn}(x+a) = -\mathrm{cs}^2(a)\mathrm{dn}(x+a) + \mathrm{ds}(a)\mathrm{ns}(a)\mathrm{dn}(x)$$
$$-m\,\mathrm{cs}(a)\mathrm{cn}(x)\mathrm{sn}(x), \tag{B11}$$
$$m\,\mathrm{sn}^2(x)\mathrm{sn}(x+a) = \mathrm{ns}^2(a)\mathrm{sn}(x+a) - \mathrm{cs}(a)\mathrm{ds}(a)\mathrm{sn}(x)$$
$$-\mathrm{ns}(a)\mathrm{cn}(x)\mathrm{dn}(x), \tag{B12}$$
$$m\,\mathrm{cn}^2(x)\mathrm{cn}(x+a) = -\mathrm{ds}^2(a)\mathrm{cn}(x+a) + \mathrm{cs}(a)\mathrm{ns}(a)\mathrm{cn}(x)$$
$$-\mathrm{ds}(a)\mathrm{sn}(x)\mathrm{dn}(x), \tag{B13}$$
$$\mathrm{dn}(x)\mathrm{sn}(x)\mathrm{dn}(x+a) = -\mathrm{cs}(a)\mathrm{ns}(a)\mathrm{sn}(x+a) + \mathrm{ds}(a)\mathrm{ns}(a)\mathrm{sn}(x)$$
$$+\mathrm{cs}(a)\mathrm{cn}(x)\mathrm{dn}(x), \tag{B14}$$

$$\mathrm{dn}(x)\mathrm{cn}(x)\mathrm{dn}(x+a) = -\mathrm{cs}(a)\mathrm{ds}(a)\mathrm{cn}(x+a) + \mathrm{ds}(a)\mathrm{ns}(a)\mathrm{cn}(x) - \mathrm{cs}(a)\mathrm{sn}(x)\mathrm{dn}(x), \tag{B15}$$

$$m\,\mathrm{dn}(x)\mathrm{sn}(x)\mathrm{sn}(x+a) = \mathrm{cs}(a)\mathrm{ns}(a)\mathrm{dn}(x+a) - \mathrm{cs}(a)\mathrm{ds}(a)\mathrm{dn}(x) + m\,\mathrm{ns}(a)\mathrm{cn}(x)\mathrm{sn}(x), \tag{B16}$$

$$m\,\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{sn}(x+a) = \mathrm{ds}(a)\mathrm{ns}(a)\mathrm{cn}(x+a) - \mathrm{cs}(a)\mathrm{ds}(a)\mathrm{cn}(x) + \mathrm{ns}(a)\mathrm{sn}(x)\mathrm{dn}(x), \tag{B17}$$

$$m\,\mathrm{dn}(x)\mathrm{cn}(x)\mathrm{cn}(x+a) = -\mathrm{cs}(a)\mathrm{ds}(a)\mathrm{dn}(x+a) + \mathrm{cs}(a)\mathrm{ns}(a)\mathrm{dn}(x) - m\,\mathrm{ds}(a)\mathrm{cn}(x)\mathrm{sn}(x), \tag{B18}$$

$$m\,\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{cn}(x+a) = -\mathrm{ds}(a)\mathrm{ns}(a)\mathrm{sn}(x+a) + \mathrm{cs}(a)\mathrm{ns}(a)\mathrm{sn}(x) + \mathrm{ds}(a)\mathrm{cn}(x)\mathrm{dn}(x), \tag{B19}$$

$$m\,\mathrm{dn}(x)\mathrm{sn}(x)\mathrm{cn}(x+a) = -\mathrm{ds}(a) - \mathrm{cs}(a)\mathrm{ns}(a)[Z(x+a) - Z(x) - Z(a)] + \mathrm{ds}(a)\mathrm{dn}^2(x), \tag{B20}$$

$$m\,\mathrm{cn}(x)\mathrm{dn}(x)\mathrm{sn}(x+a) = -\mathrm{ds}(a)\mathrm{dn}(a) - \mathrm{cs}(a)\mathrm{ds}(a) \times [Z(x+a) - Z(x) - Z(a)] + \mathrm{ns}(a)\mathrm{dn}^2(x), \tag{B21}$$

$$m\,\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x+a) = -\mathrm{cs}(a) - \mathrm{ds}(a)\mathrm{ns}(a)\,[Z(x+a) - Z(x) - Z(a)] + \mathrm{cs}(a)\mathrm{dn}^2(x). \tag{B22}$$

Appendix C: Some examples of local identities of rank > 3

$$\begin{aligned}
&\mathrm{dn}(x)\mathrm{dn}(x+a)\mathrm{dn}(x+a')\mathrm{dn}(x+a'') \\
&= \mathrm{dn}(a)\mathrm{dn}(a')\mathrm{dn}(a'') + \mathrm{cs}(a)\mathrm{cs}(a')\mathrm{cs}(a'')Z(x) \\
&\quad - \mathrm{cs}(a)\mathrm{cs}(a'-a)\mathrm{cs}(a''-a)[Z(x+a) - Z(a)] \\
&\quad + \mathrm{cs}(a')\mathrm{cs}(a'-a)\mathrm{cs}(a''-a')[Z(x+a') - Z(a')] \\
&\quad - \mathrm{cs}(a'')\mathrm{cs}(a''-a)\mathrm{cs}(a''-a')[Z(x+a'') - Z(a'')],
\end{aligned} \tag{C1}$$

$$\begin{aligned}
&\mathrm{dn}^2(x)\mathrm{dn}(x+a)\mathrm{dn}(x+a') \\
&= -\mathrm{cs}(a)\mathrm{cs}(a-a')\{\mathrm{dn}(a) + \mathrm{cs}(a)\big[Z(x+a) - Z(x) - Z(a)\big]\} \\
&\quad + \mathrm{cs}(a')\mathrm{cs}(a-a')\{\mathrm{dn}(a') + \mathrm{cs}(a')\big[Z(x+a') - Z(x) - Z(a')\big]\} \\
&\quad - \mathrm{cs}(a)\mathrm{cs}(a')\mathrm{dn}^2(x),
\end{aligned} \tag{C2}$$

$$\mathrm{dn}^3(x)\mathrm{dn}(x+a) = -m\,\mathrm{cs}(a)\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x) + \mathrm{ds}(a)\mathrm{ns}(a)\mathrm{dn}^2(x) -\mathrm{cs}^2(a)\mathrm{dn}(a) - \mathrm{cs}^3(a)[Z(x+a) - Z(x) - Z(a)], \tag{C3}$$

$$\mathrm{dn}^2(x)\mathrm{dn}^2(x+a) = -\mathrm{cs}^2(a)[\mathrm{dn}^2(x) + \mathrm{dn}^2(x+a)] + [\mathrm{ds}^2(a) + \mathrm{cs}^2(a)] +2\mathrm{cs}(a)\mathrm{ds}(a)\mathrm{ns}(a)\left[Z(x+a) - Z(x) - Z(a)\right], \tag{C4}$$

$$\begin{aligned}&\mathrm{dn}(x)\mathrm{dn}(x+a)\mathrm{dn}(x+a')\mathrm{sn}(x+a'')\\ &= -\mathrm{cs}(a)\mathrm{cs}(a')\big[\mathrm{ns}(a'')\mathrm{cn}(x) - \mathrm{cs}(a'')\mathrm{cn}(x+a'')\big] - \mathrm{cs}(a)\mathrm{cs}(a-a')\\ &\quad\times\big[\mathrm{ns}(a''-a)\mathrm{cn}(x+a) - \mathrm{cs}(a''-a)\mathrm{cn}(x+a'')\big]\\ &\quad + \mathrm{cs}(a')\mathrm{cs}(a-a')\big[\mathrm{ns}(a''-a')\mathrm{cn}(x+a') - \mathrm{cs}(a''-a')\mathrm{cn}(x+a'')\big],\end{aligned} \tag{C5}$$

$$\begin{aligned}&m^2\mathrm{cn}(x)\mathrm{sn}(x+a)\mathrm{cn}(x+a')\mathrm{sn}(x+a'')\\ &= m^2\mathrm{sn}(a)\mathrm{cn}(a')\mathrm{sn}(a'') - \mathrm{ns}(a)\mathrm{ds}(a')\mathrm{ns}(a'')Z(x)\\ &\quad + \mathrm{ds}(a)\mathrm{ds}(a'-a)\mathrm{ns}(a''-a)[Z(x+a) - Z(a)]\\ &\quad - \mathrm{ds}(a')\mathrm{ns}(a'-a)\mathrm{ns}(a''-a')[Z(x+a') - Z(a')]\\ &\quad + \mathrm{ds}(a'')\mathrm{ns}(a''-a)\mathrm{ds}(a''-a')[Z(x+a'') - Z(a'')],\end{aligned} \tag{C6}$$

$$\begin{aligned}&m\,\mathrm{dn}(x)\mathrm{cn}(x)\mathrm{sn}(x+a)\mathrm{cn}(x+a')\\ &= -\mathrm{ds}(a)\mathrm{ds}(a-a')\big[\mathrm{ns}(a)\mathrm{cn}(x) - \mathrm{cs}(a)\mathrm{cn}(x+a)\big]\\ &\quad + \mathrm{ds}(a')\mathrm{ns}(a-a')\big[\mathrm{ns}(a')\mathrm{cn}(x) - \mathrm{cs}(a')\mathrm{cn}(x+a')\big]\\ &\quad - \mathrm{ns}(a)\mathrm{ds}(a')\mathrm{sn}(x)\mathrm{dn}(x),\end{aligned} \tag{C7}$$

$$\begin{aligned}&m\,\mathrm{dn}(x)\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x+a)\mathrm{sn}(x+a)\\ &= -\mathrm{cs}(a)\mathrm{ns}(a)[\mathrm{cs}^2(a) + \mathrm{ds}^2(a) + \mathrm{ns}^2(a)]\mathrm{cn}(x)\\ &\quad + \left[\mathrm{ns}^2(a)(\mathrm{ds}^2(a) + \mathrm{cs}^2(a)) + \mathrm{cs}^2(a)\mathrm{ds}^2(a)\right]\mathrm{cn}(x+a)\\ &\quad + \mathrm{cs}(a)\mathrm{ds}(a)\mathrm{ns}(a)\mathrm{sn}(x+a)\mathrm{dn}(x+a)\\ &\quad + \mathrm{ds}(a)[\mathrm{cs}^2(a) + \mathrm{ns}^2(a)]\mathrm{sn}(x)\mathrm{dn}(x) + m\,\mathrm{cs}(a)\mathrm{ns}(a)\mathrm{cn}^3(x).\end{aligned} \tag{C8}$$

In the following identities, $B \equiv -\mathrm{cs}^2(a)$, $B_1 \equiv \mathrm{ns}^2(a)$, $B_2 \equiv -\mathrm{ds}^2(a)$. We only give identities involving arbitrary even powers $2n$ of Jacobi elliptic functions. The identities for odd powers $2n+1$ are easily obtained by multiplication of the even power identity by one additional Jacobi elliptic function and using the rank 2 identities of Appendix A for simplification – as an illustration, see the derivation of identity (30) from identity (28) in the text.

$$\begin{aligned}\mathrm{dn}^{2n}(x)\mathrm{dn}(x+a) = {}& B^n\mathrm{dn}(x+a) \\ &+ [\mathrm{ds}(a)\mathrm{ns}(a)\mathrm{dn}(x) - m\,\mathrm{cs}(a)\mathrm{cn}(x)\mathrm{sn}(x)] \\ &\times \sum_{k=1}^{n} B^{k-1}[\mathrm{dn}(x)]^{2(n-k)}, \end{aligned} \tag{C9}$$

$$\begin{aligned}m^n\,\mathrm{sn}^{2n}(x)\mathrm{sn}(x+a) = {}& B_1^n\mathrm{sn}(x+a) \\ &- [\mathrm{cs}(a)\mathrm{ds}(a)\mathrm{sn}(x) + \mathrm{ns}(a)\mathrm{cn}(x)\mathrm{dn}(x)] \\ &\times \sum_{k=1}^{n} m^{n-k}B_1^{k-1}[\mathrm{sn}(x)]^{2(n-k)}, \end{aligned} \tag{C10}$$

$$\begin{aligned}m^n\mathrm{cn}^{2n}(x)\mathrm{cn}(x+a) = {}& B_2^n\mathrm{cn}(x+a) \\ &+ [\mathrm{cs}(a)\mathrm{ns}(a)\mathrm{cn}(x) - \mathrm{ds}(a)\mathrm{sn}(x)\mathrm{dn}(x)] \\ &\times \sum_{k=1}^{n} m^{n-k}B_2^{k-1}[\mathrm{cn}(x)]^{2(n-k)}, \end{aligned} \tag{C11}$$

$$\begin{aligned}m^n\,\mathrm{cn}^{2n}(x)\mathrm{sn}(x)\mathrm{dn}(x+a) = {}& -B_2^n\mathrm{ns}(a)\mathrm{cn}(x+a) \\ &+\mathrm{cs}(a)m^n\mathrm{cn}^{2n+1}(x) - \mathrm{ns}(a) \\ &\times [\mathrm{cs}(a)\mathrm{ns}(a)\mathrm{cn}(x) - \mathrm{ds}(a)\mathrm{sn}(x)\mathrm{dn}(x)] \\ &\times \sum_{k=1}^{n} m^{n-k}B_2^{k-1}[\mathrm{cn}(x)]^{2(n-k)}, \end{aligned} \tag{C12}$$

$$\begin{aligned}m^n\,\mathrm{sn}^{2n}(x)\mathrm{cn}(x)\mathrm{dn}(x+a) = {}& B_1^n\mathrm{ds}(a)\mathrm{sn}(x+a) \\ &-\mathrm{cs}(a)m^n\mathrm{sn}^{2n+1}(x) - \mathrm{ds}(a) \\ &\times [\mathrm{cs}(a)\mathrm{ds}(a)\mathrm{sn}(x) + \mathrm{ns}(a)\mathrm{cn}(x)\mathrm{dn}(x)] \\ &\times \sum_{k=1}^{n} m^{n-k}B_1^{k-1}[\mathrm{sn}(x)]^{2(n-k)}, \end{aligned} \tag{C13}$$

$$\begin{aligned}m\,\mathrm{dn}^{2n}(x)\mathrm{cn}(x)\mathrm{sn}(x+a) = {}& -B^n\mathrm{ds}(a)\mathrm{dn}(x+a) \\ &+\mathrm{ns}(a)\mathrm{dn}^{2n+1}(x) - \mathrm{ds}(a) \\ &\times [\mathrm{ds}(a)\mathrm{ns}(a)\mathrm{dn}(x) - m\,\mathrm{cs}(a)\mathrm{cn}(x)\mathrm{sn}(x)] \\ &\times \sum_{k=1}^{n} B^{k-1}[\mathrm{dn}(x)]^{2(n-k)}, \end{aligned} \tag{C14}$$

$$\begin{aligned}\mathrm{dn}^{2n}(x)\mathrm{dn}^2(x+a) = {} & 2nB^{n-1}\mathrm{ds}(a)\mathrm{ns}(a) \\ & \times[\mathrm{dn}(a)+\mathrm{cs}(a)(Z(x+a)-Z(x)-Z(a))] \\ & -(1-m)B^{n-1}+\sum_{k=1}^{n-1}B^{k-1} \\ & \times[B^2-(1-m)+2k\mathrm{ds}^2(a)\mathrm{ns}^2(a)][\mathrm{dn}(x)]^{2(n-k)} \\ & +B^n\mathrm{dn}^2(x+a)+B\,\mathrm{dn}^{2n}(x) \\ & -2m\,\mathrm{cs}(a)\mathrm{ds}(a)\mathrm{ns}(a)\mathrm{sn}(x)\mathrm{cn}(x) \\ & \times\sum_{k=1}^{n-1}kB^{k-1}[\mathrm{dn}(x)]^{2(n-k)-1}. \qquad \text{(C15)}\end{aligned}$$

Appendix D: Examples of identities with weighted terms and their linear combinations

In this appendix $a=2rK/p, a'=2sK/p, b=4rK/p$.

$$m\sum_{j=1}^{p}c_j[s_{j+r}-s_{j-r}]=2[\mathrm{ns}(a)-\mathrm{ds}(a)]\sum_{j=1}^{p}d_j, \qquad \text{(D1)}$$

$$\sum_{j=1}^{p}d_j^2[d_{j+r}-d_{j-r}]=-2m\,\mathrm{cs}(a)\sum_{j=1}^{p}c_js_j, \qquad \text{(D2)}$$

$$m\sum_{j=1}^{p}c_js_j[s_{j+r}-s_{j-r}]=2\mathrm{ns}(b)\sum_{j=1}^{p}s_jd_j, \qquad \text{(D3)}$$

$$m\sum_{j=1}^{p}s_jd_j[c_{j+r}-c_{j-r}]=2\mathrm{ds}(a)\sum_{j=1}^{p}d_j^2-2p\mathrm{ns}(a)[\mathrm{dn}(a)-\mathrm{cs}(a)Z(a)], \qquad \text{(D4)}$$

$$m\sum_{j=1}^{p}c_j[c_{j+r}^3-c_{j-r}^3]=2\mathrm{ds}(a)\sum_{j=1}^{p}c_js_jd_j, \qquad \text{(D5)}$$

$$\begin{aligned}& m\sum_{j=1}^{p}c_js_jd_j[s_{j+r}-s_{j-r}] \\ & =-2\mathrm{ns}(b)[1-\mathrm{ds}^2(b)+\mathrm{cs}(b)\mathrm{ds}(b)]\sum_{j=1}^{p}s_j-2m\,\mathrm{ns}(b)\sum_{j=1}^{p}s_j^3. \qquad \text{(D6)}\end{aligned}$$

Appendix E: Some definite integrals

Here K, E correspond to the complete elliptic integrals of the first and second kind respectively.

$$\int_0^{2K} \mathrm{dn}^3(x)\mathrm{dn}(x+a)\,\mathrm{d}x = 2\mathrm{ds}(a)\mathrm{ns}(a)E - 2K\mathrm{cs}^2(a)[\mathrm{dn}(a) - \mathrm{cs}(a)Z(a)], \tag{E1}$$

$$\int_0^{2K} m^2\mathrm{sn}^3(x)\mathrm{sn}(x+a)\,\mathrm{d}x$$
$$= 2\mathrm{cs}(a)\mathrm{ds}(a)E - 2K[\mathrm{cs}(a)\mathrm{ds}(a) - \mathrm{ns}^3(a)Z(a)], \tag{E2}$$

$$\int_0^{2K} m^2\,\mathrm{cn}^3(x)\mathrm{cn}(x+a)\,\mathrm{d}x$$
$$= 2\mathrm{cs}(a)\mathrm{ns}(a)E + 2K[m^2\mathrm{cn}(a) - \mathrm{cs}(a)\mathrm{ns}(a) + \mathrm{ds}^3(a)Z(a)], \tag{E3}$$

$$\int_0^{2K} m\,\mathrm{dn}(x)\mathrm{sn}(x)\mathrm{dn}(x+a)\mathrm{sn}(x+a)\,\mathrm{d}x$$
$$= 4\mathrm{cs}(a)\mathrm{ns}(a)E - 2K\mathrm{ns}(a)$$
$$\times [\mathrm{cs}(a)(1+\mathrm{dn}^2(a)) - (1+\mathrm{cn}^2(a))\mathrm{ds}(a)\mathrm{ns}(a)Z(a)], \tag{E4}$$

$$\int_0^{2K} m\,\mathrm{dn}(x)\mathrm{cn}(x)\mathrm{dn}(x+a)\mathrm{cn}(x+a)\mathrm{d}n$$
$$= -4\mathrm{cs}(a)\mathrm{ds}(a)E + 2K[2\mathrm{cs}(a)\mathrm{ds}(a) - (\mathrm{cs}^2(a) + \mathrm{ds}^2(a))\mathrm{ns}(a)Z(a)], \tag{E5}$$

$$\int_0^{2K} m^2\,\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{sn}(x+a)\mathrm{cn}(x+a)\mathrm{d}x$$
$$= 4\mathrm{ds}(a)\mathrm{ns}(a)E + 2K\mathrm{ns}(a)(1+\mathrm{dn}^2(a))[\mathrm{cs}(a)\mathrm{ns}(a)Z(a) - \mathrm{ds}(a)], \tag{E6}$$

$$\frac{1}{2K}\int_0^{2K} \mathrm{dn}(x)\mathrm{dn}(x+a)\mathrm{dn}(x+a')\mathrm{dn}(x+a'')\mathrm{d}x$$
$$= \mathrm{dn}(a)\mathrm{dn}(a')\mathrm{dn}(a'') + \mathrm{cs}(a)\mathrm{cs}(a'-a)\mathrm{cs}(a''-a)Z(a)$$
$$- \mathrm{cs}(a')\mathrm{cs}(a'-a)\mathrm{cs}(a''-a')Z(a') + \mathrm{cs}(a'')\mathrm{cs}(a''-a)\mathrm{cs}(a''-a')Z(a''), \tag{E7}$$

$$\frac{1}{2K}\int_0^{2K} m^2\,\mathrm{sn}(x)\mathrm{sn}(x+a)\mathrm{sn}(x+a')\mathrm{sn}(x+a'')\mathrm{d}x$$
$$= \mathrm{ns}(a)\mathrm{ns}(a'-a)\mathrm{ns}(a''-a)Z(a) - \mathrm{ns}(a')\mathrm{ns}(a'-a)\mathrm{ns}(a''-a')Z(a')$$
$$+ \mathrm{ns}(a'')\mathrm{ns}(a''-a)\mathrm{ns}(a''-a')Z(a''), \tag{E8}$$

$$\frac{1}{2K}\int_0^{2K} m^2\,\mathrm{cn}(x)\mathrm{cn}(x+a)\mathrm{cn}(x+a')\mathrm{cn}(x+a'')\mathrm{d}x$$
$$= m^2\mathrm{cn}(a)\mathrm{cn}(a')\mathrm{cn}(a'') + \mathrm{ds}(a)\mathrm{ds}(a'-a)\mathrm{ds}(a''-a)Z(a)$$
$$- \mathrm{ds}(a')\mathrm{ds}(a'-a)\mathrm{ds}(a''-a')Z(a')$$
$$+ \mathrm{ds}(a'')\mathrm{ds}(a''-a)\mathrm{ds}(a''-a')Z(a''), \tag{E9}$$

$$\frac{1}{2K}\int_0^{2K} m^2\,\mathrm{cn}(x)\mathrm{sn}(x+a)\mathrm{cn}(x+a')\mathrm{sn}(x+a'')\mathrm{d}x$$
$$= m^2\mathrm{sn}(a)\mathrm{cn}(a')\mathrm{sn}(a'') - \mathrm{ds}(a)\mathrm{ds}(a'-a)\mathrm{ns}(a''-a)Z(a)$$
$$+ \mathrm{ds}(a')\mathrm{ns}(a'-a)\mathrm{ns}(a''-a')Z(a')$$
$$- \mathrm{ds}(a'')\mathrm{ns}(a''-a)\mathrm{ds}(a''-a')Z(a''), \tag{E10}$$

$$\frac{1}{2K}\int_0^{2K} m\,\mathrm{sn}(x)\mathrm{dn}(x+a)\mathrm{sn}(x+a')\mathrm{dn}(x+a'')\mathrm{d}x$$
$$= -\mathrm{ns}(a)\mathrm{ns}(a'-a)\mathrm{cs}(a''-a)Z(a)$$
$$+ \mathrm{ns}(a')\mathrm{cs}(a'-a)\mathrm{cs}(a''-a')Z(a')$$
$$- \mathrm{ns}(a'')\mathrm{cs}(a''-a)\mathrm{ns}(a''-a')Z(a''). \tag{E11}$$

Appendix F: Some indefinite integrals

We give below recursion relations expressing certain arbitrary order integrals in terms of lower order integrals, well-known integrals of $\mathrm{sn}^n(x)$, $\mathrm{dn}^n(x)$, $\mathrm{cn}^n(x)$ [3], and incomplete elliptic integrals of the first, second and third kind, which essentially occur due to the integral (64). It should be noted that in view of the identities (A2) and (A3), the integrals for $m\,\mathrm{sn}(x+a)\mathrm{sn}(x)$ and $m\,\mathrm{cn}(x+a)\mathrm{cn}(x)$ are related to the integral (64). In this appendix, $n \geq 1$.

$$I_n = \mathrm{ns}^2(a)I_{n-1} - \frac{m^{n-1}\mathrm{ns}(a)}{(2n-1)}\mathrm{sn}^{2n-1}(x)$$
$$-\mathrm{cs}(a)\mathrm{ds}(a)m^{n-1}\int \mathrm{sn}^{2n-1}(x)\mathrm{d}x, \tag{F1}$$

where $I_k \equiv \int m^k\mathrm{sn}^{2k}(x)\mathrm{sn}(x+a)\mathrm{d}x$.

$$I_n = \mathrm{ns}^2(a)I_{n-1} - \frac{m^{n-1}\mathrm{ds}(a)\mathrm{ns}(a)}{(2n-1)}\mathrm{sn}^{2n-1}(x)$$
$$+\mathrm{cs}(a)m^n\int \mathrm{sn}^{2n-1}(x)\mathrm{cn}^2(x)\mathrm{d}x, \tag{F2}$$

where $I_k \equiv \int m^k \mathrm{sn}^{2k}(x)\mathrm{cn}(x)\mathrm{dn}(x+a)\mathrm{d}x$, $I_0 \equiv \mathrm{ds}(a)\int \mathrm{sn}(x+a)\mathrm{d}x - \mathrm{cs}(a) \int \mathrm{sn}(x)\mathrm{d}x$.

$$\begin{aligned} I_n = BI_{n-1} &+ \frac{\mathrm{ds}(a)\mathrm{ns}(a)}{(2n-1)}\mathrm{dn}^{2n-1}(x) \\ &+2\mathrm{ds}(a)\mathrm{ns}(a)\sum_{k=1}^{n-1} B^k \frac{[\mathrm{dn}(x)]^{2(n-k)-1}}{2(n-k)-1} \\ &-2\mathrm{cs}(a)\mathrm{ds}(a)\mathrm{ns}(a)\int \mathrm{dn}(x+a)\mathrm{d}x \\ &+\mathrm{cs}(a)[m+2\mathrm{ds}^2(a)]\int \mathrm{dn}^{2n-1}(x)\mathrm{d}x \\ &-2\mathrm{cs}(a)\mathrm{ds}^2(a)\mathrm{ns}^2(a)\sum_{k=1}^{n-1} B^{k-1}\int [\mathrm{dn}(x)]^{2(n-k)-1}\mathrm{d}x, \end{aligned} \tag{F3}$$

where $I_k \equiv \int m\,\mathrm{dn}^{2k}(x)\mathrm{cn}(x+a)\mathrm{sn}(x+a)\mathrm{d}x$, $B \equiv -\mathrm{cs}^2(a)$, $I_0 = -\mathrm{dn}(x+a)$.

$$I_n = -\mathrm{cs}^2(a)I_{n-1} + \frac{\mathrm{cs}(a)}{(2n)}\mathrm{dn}^{2n}(x) + \mathrm{ds}(a)\mathrm{ns}(a)\int \mathrm{dn}^{2n}(x)\mathrm{d}x, \tag{F4}$$

where $I_k \equiv \int m\mathrm{dn}^{2k+1}(x)\mathrm{dn}(x+a)\mathrm{d}x$, $I_0 \equiv \int \mathrm{dn}(x+a)\mathrm{dn}(x)\mathrm{d}x$.

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