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Periodic Solutions of Nonlinear Equations Obtained by Linear Superposition

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Abstract

We show that a type of linear superposition principle works for several nonlinear differential equations. Using this approach, we find periodic solutions of the Kadomtsev-Petviashvili (KP) equation, the nonlinear Schrödinger (NLS) equation, the $\lambda\phi^4$ model, the sine-Gordon equation and the Boussinesq equation by making appropriate linear superpositions of known periodic solutions. This unusual procedure for generating solutions is successful as a consequence of some powerful, recently discovered, cyclic identities satisfied by the Jacobi elliptic functions.

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1 Introduction

The fact that Jacobi elliptic functions arise naturally as traveling wave solutions of many nonlinear systems has been known for quite some time (see for example [1]). Although for the solitary wave solutions of these nonlinear equations, there is no superposition principle (except when the solitary waves are far apart), for the periodic solutions the situation turns out to be quite different. It has recently been shown [2] that certain specific linear combinations of known periodic solutions of the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations as well as $\lambda\phi^4$ theory, also satisfy these equations. This unexpected result is a consequence of some remarkable, recently established, identities involving Jacobi elliptic functions [3]. Basically, the identities take the cross terms generated by the nonlinear terms in the differential equations and convert them into a manageable form. The purpose of this article is to show that such a procedure also works for other well-known nonlinear equations, namely the Kadomtsev-Petviashvili (KP) equation, the nonlinear Schrödinger equation(NLSE), the sine-Gordon equation and the Boussinesq equation. It should be noted that the above list includes both integrable as well as nonintegrable systems. These equations are of interest in several diverse areas of physics. The NLSE governs the propagation of an electromagnetic wave in a glass fiber, or the spatial evolution of an electromagnetic field in a planar waveguide. Temporal solitons described by the NLSE were first observed in 1980 [4], and the first confirmation and studies of spatial solitons in planar waveguides were reported in 1988 [5, 6]. Similarly, the $\lambda\phi^4$ and the sine-Gordon equations arise in several condensed matter physics applications.

2 The Kadomtsev-Petviashvili (KP) Equation

The KP equation is a two-dimensional generalization of the KdV equation and is given by

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \ . \tag{1}$$

Properties of the KP equation are discussed in many texts [1]. In particular, the simplest, periodic, cnoidal traveling wave solution is

$$u_1(x,y,t) = -2\alpha^2 \mathrm{dn}^2(\xi_1, m) + \beta\alpha^2 \ , \quad \xi_1 \equiv \alpha(x + \gamma\alpha y - b_1\alpha^2 t) \ , \tag{2}$$

where α, γ, m and β are constants, and the "velocity" b_1 is given by

$$b_1 = 8 - 4m - 6\beta + 3\gamma^2 \ . \tag{3}$$

In this article, for Jacobi elliptic functions, we use the standard notation $\mathrm{dn}\,(\xi, m)$, $\mathrm{sn}\,(\xi, m)$, $\mathrm{cn}\,(\xi, m)$, where m is the elliptic modulus parameter ($0 \le m \le 1$). The solution (2) remains unchanged when x is increased by $2K(m)/\alpha$, where $K(m)$ is the complete elliptic integral of the first kind [7]. In the limiting case $m = 1$ (and $\beta = 0$), one recovers the familiar single soliton form $-2\alpha^2 \mathrm{sech}^2(\alpha(x + \gamma\alpha y - b_1\alpha^2 t))$.

We will make suitable linear combinations of solution (2) and show that the result is also a periodic solution of the KP equation. Our procedure consists of adding terms of the kind given in (2) but centered at p equally spaced points along the period $2K(m)/\alpha$, where p is any integer. The p-point solution is

$$u_p(x, y, t) = -2\alpha^2 \sum_{i=1}^{p} d_i^2 + \beta\alpha^2 \; ; \; d_i \equiv \mathrm{dn}[\xi_p + \frac{2(i-1)K(m)}{p}, m] \; , \; \xi_p \equiv \alpha(x + \gamma\alpha y - b_p\alpha^2 t) \; . \quad (4)$$

Clearly, $p = 1$ is the original solution, but for any other p, we have new expressions which, as we shall show, also solve the KP equation. For convenience, we define the quantities s_i and c_i in analogy to the quantity d_i defined above:

$$s_i \equiv \mathrm{sn}[\xi_p + \frac{2(i-1)K(m)}{p}, m] \;\; , \; c_i \equiv \mathrm{cn}[\xi_p + \frac{2(i-1)K(m)}{p}, m] \;\; . \quad (5)$$

The KP equation contains the KdV operator $u_t - 6uu_x + u_{xxx}$. It has been shown in detail in ref. [2] that eq. (4) with $\gamma = 0$ is a solution of the KdV equation. The proof is based on the identity

$$\sum_{i<j}^{p} d_i^2 d_j^2 = A_1(p, m) \sum_{i=1}^{p} d_i^2 + A_2(p, m) \; . \quad (6)$$

This is one of many powerful new identities [3] which reduce by 2 (or a larger even number) the degree of cyclic homogeneous polynomials in Jacobi elliptic functions. The constants $A_1(p, m)$ and $A_2(p, m)$ in identity (6) can be evaluated in general by choosing any specific convenient value of the argument ξ of the Jacobi elliptic functions. The results for $A_1(p, m)$ for small values of p are:

$$A_1(p = 2, m) = 0 \;\; , \;\; A_1(p = 3, m) = \frac{-2(m - 1 + q^2)}{1 - q^2} \;\; , \;\; A_1(p = 4, m) = -2\sqrt{1 - m} \;\; , \quad (7)$$

where

$$q \equiv \mathrm{dn}(2K(m)/3, m) \; . \quad (8)$$

The limiting values at $m = 0, 1$ are also particularly simple:

$$A_1(p, m = 0) = -\frac{1}{3}(p - 1)(p - 2) \; ; \; A_1(p, m = 1) = 0 \; . \quad (9)$$

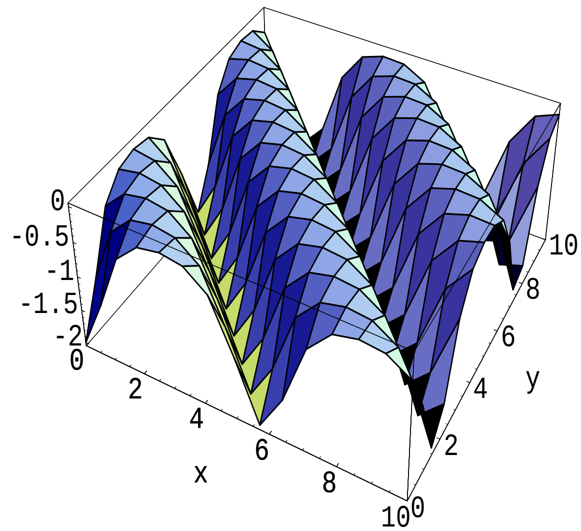

Figure 1: $u_1(x,y,t=0)$ vs. (x,y) for $m=0.95$, $\alpha=\gamma=1$, $\beta=0$.

Taking expression (4) and using identity (6), the left side of the KP equation (1) becomes

$$4m\alpha^5\{8-4m-6\beta-b_p+12A_1(p,m)\}\frac{d}{dx}\sum_{i=1}^{p}s_ic_id_i+12m\gamma\alpha^4\frac{d}{dy}\sum_{i=1}^{p}s_ic_id_i\ . \tag{10}$$

Clearly, this vanishes if the velocity is given by

$$b_p=8-4m-6\beta+12A_1(p,m)+3\gamma^2\ . \tag{11}$$

Thus for this choice of velocity, the KP equation is solved by our p-point expression (4). Effectively, our solutions of the KP equation and the corresponding solutions of the KdV equation have a difference of $3\gamma^2$ in their velocities b_p. Note that as in the KdV case, the results for b_p can be positive or negative depending on the values of the parameters [2]. The behavior of u_1 for the parameters $m=0.95$, $\alpha=\gamma=1$, $\beta=0$ is shown in fig. 1.

In addition to the solution (2), another well known periodic solution of the KP eq. (1) of period $4K(m)$ is

$$v_1(x,y,t)=\alpha^2\left[m\mathrm{sn}^2(\eta_1,m)\pm\sqrt{m}\mathrm{cn}(\eta_1,m)\mathrm{dn}(\eta_1,m)\right]\ ,\quad \eta_1\equiv\alpha(x+\gamma\alpha y-q_1\alpha^2t) \tag{12}$$

with velocity $q_1=(-1-m+3\gamma^2)$.

Starting from this solution, we can again obtain many additional periodic solutions of the KP equation of period $4K(m)/p$ in case p is an odd integer. The general p-point solution is given by

$$v_p(x,y,t) = \alpha^2 \sum_{i=1}^{p} [m\tilde{s}_i^2 \pm \sqrt{m}\,\tilde{c}_i\tilde{d}_i] \ , \quad p \ \text{ odd} \ , \tag{13}$$

where we define

$$\tilde{s}_i \equiv \mathrm{sn}[\eta_p + \frac{4(i-1)K(m)}{p}, m] \ , \ \tilde{c}_i \equiv \mathrm{cn}[\eta_p + \frac{4(i-1)K(m)}{p}, m] \ , \ \tilde{d}_i \equiv \mathrm{dn}[\eta_p + \frac{4(i-1)K(m)}{p}, m] \ . \tag{14}$$

As has been shown in detail in ref. [2], eq. (13) with $\gamma = 0$ is a solution of the KdV equation with velocity

$$q_p = -(1+m) - 6[B_1(p,m) - C_1(p,m)] \ , \tag{15}$$

where the quantities $B_1(p,m)$ and $C_1(p,m)$ come from the following identities:

$$m\sum_{i<j}^{p} \tilde{s}_i\tilde{s}_j = B_1(p,m) \ , \quad m\sum_{i<j<k}^{p} \tilde{s}_i\tilde{s}_j\tilde{s}_k = C_1(p,m) \sum_{i=1}^{p} \tilde{s}_i \ . \tag{16}$$

It is easily checked that even in the KP case ($\gamma \neq 0$), eq. (13) is an exact solution, the only difference being that the velocity in the KP case is larger by $3\gamma^2$. As an illustration, for the $p = 3$ case, it is easily shown that [3]

$$B_1(3,m) = -(1-q^2) \ , \quad C_1(3,m) = -m/(1-q^2) \ , \tag{17}$$

so that the velocity of the KP soliton is given by

$$q_3 = -1 - m + 6(1-q^2) - \frac{6m}{1-q^2} + 3\gamma^2 \ , \tag{18}$$

where q has been defined in eq. (8). Note that for p even, we do not obtain any new solutions. To illustrate our results, in fig. 2 we plot $v_3(x,y)$ at time $t = 0$ for the choice $\alpha = \gamma = 1$ and $m = 0.95$.

3 The Nonlinear Schrödinger Equation

The NLSE with both attractive and repulsive nonlinearity has found many physical applications in several diverse areas including fiber optics, Bose-Einstein condensates and waveguides [8].

3.1 Case I: Attractive Nonlinearity

The NLSE with attractive nonlinearity is given by ($\hbar = 2m = 1$)

$$iu_t + u_{xx} + u \mid u \mid^2 = 0 \ , \tag{19}$$

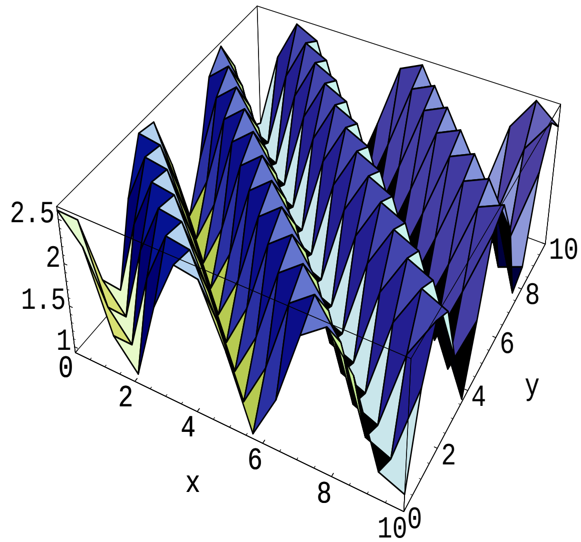

Figure 2: $v_3(x, y, t = 0)$ vs. (x, y) for $m = 0.95$, $\alpha = \gamma = 1$.

where without any loss of generality we have fixed the coefficient of the nonlinear term to be unity. As usual, one starts with the ansatz [1]

$$u(x,t) = r(\xi)e^{i(\theta(\xi)+nt)} \ , \quad \xi \equiv x - vt \ , \tag{20}$$

which on substituting in eq. (19) yields

$$\theta'(\xi) = \frac{1}{2}(v + \frac{A}{r^2}) \ , \tag{21}$$

$$r'^2(\xi) = -\frac{r^4}{2} + (n - \frac{v^2}{4})r^2 - \frac{B}{2} - \frac{A^2}{4r^2} \ , \tag{22}$$

where prime denotes a derivative with respect to the argument ξ and A, B are constants of integration. Thus the whole problem reduces to finding the solutions of eq. (22), after which θ is easily obtained by using eq. (21) and performing one integration.

The well known soliton solution of eq. (22) is

$$r(\xi) = \sqrt{2}\ \mathrm{sech}\ \xi \ , \quad \theta = \frac{v\xi}{2} \ , \quad A = B = 0 \ , \ v^2 = 4(n-1) \ , \tag{23}$$

which is valid only for $n \geq 1$. It may be noted that a somewhat more general solution with arbitrary amplitude α is easily obtained, since if $u(x,t)$ is a solution of the NLSE, then $\alpha u(\alpha x, \alpha^2 t)$ is also a solution of the same equation.

The two simplest, periodic, cnoidal traveling wave solutions of eq. (22) are ($\xi_1 = x - v_1 t$)

$$r_1(\xi) = \sqrt{2}\,\mathrm{dn}\;\xi_1 \;,\;\; \theta_1 = \frac{v_1\xi_1}{2} \;,\; A = 0 \;,\; B = 4(1-m) \;,\; v_1^2 = 4(n+m-2) \;, \tag{24}$$

$$r_1(\xi) = \sqrt{2m}\,\mathrm{cn}\;\xi_1 \;,\;\; \theta_1 = \frac{v_1\xi_1}{2} \;,\; A = 0 \;,\; B = -4m(1-m) \;,\; v_1^2 = 4(n+1-2m) \;. \tag{25}$$

In the limiting case $m = 1$, one recovers the familiar soliton solution (23).

We shall now show that suitable linear combinations of the solutions (24) and (25) are also solutions of eq. (22). Consider first the solution (24). Our solutions consist of adding terms of the kind given in this equation but centered at p equally spaced points along the period $2K(m)$, where p is any integer. The p-point solution is

$$r_p(x,t) = \sqrt{2}\sum_{i=1}^{p} d_i \;,\; d_i \equiv \mathrm{dn}[\xi_p + \frac{2(i-1)K(m)}{p}, m] \;,\; \xi_p \equiv (x - v_p t) \;. \tag{26}$$

Clearly, $p = 1$ is the original solution, but for any other p, we have solutions of period $2K(m)/p$.

In order to verify that expression (26) is indeed a solution of eq. (22), one needs the identities

$$\begin{aligned}
&(\sum_{i=1}^{p} d_i)^2 = \sum_{i=1}^{p} d_i^2 + A(p,m) \;, \\
&(\sum_{i=1}^{p} d_i)^4 = \sum_{i=1}^{p} d_i^4 + C(p,m)\sum_{i=1}^{p} d_i^2 + D(p,m) \;, \\
&m^2 \sum_{i<j}^{p} s_i c_i s_j c_j = E(p,m)\sum_{i=1}^{p} d_i^2 + F(p,m) \;,
\end{aligned} \tag{27}$$

which can be easily established by following the procedure discussed in ref. [3]. The general expression for the velocity v_p is

$$v_p^2 = 4[n + m - 2 - C(p,m) - 2E(p,m)] \;. \tag{28}$$

Some explicitly computed values of the constants $C(p,m)$ and $E(p,m)$ are

$$\begin{aligned}
&C(2,m) = 4E(2,m) = 4\sqrt{1-m} \;,\;\; C(3,m) = 4E(3,m) = \frac{8mq}{1-q^2} \;, \\
&C(4,m) = 4E(4,m) = 4\tilde{t}(2 + \tilde{t} + 2\tilde{t}^2) \;,
\end{aligned} \tag{29}$$

where q is given by eq. (8) and $\tilde{t}$ is given by

$$\tilde{t} \equiv (1-m)^{1/4} \;. \tag{30}$$

On the other hand, for any p at $m = 0$, $C(p,0) = 4E(p,0) = \frac{4(p^2-1)}{3}$ and at $m = 1$, $C(p,1) = E(p,1) = 0$. It then follows from eq. (28) that the solution r_p as given by eq. (26) is valid only if $n \ge 2p^2$, and in this case v^2 changes from $4(n - 2p^2)$ to $4(n-1)$ as m goes from 0 to 1.

For odd p, using the solution (25), we obtain the following solution of the NLSE (22) by linear superposition

$$r_p(x,t) = \sqrt{2m}\sum_{i=1}^{p} \tilde{c}_i \ , \ \tilde{c}_i \equiv \mathrm{cn}[\eta_p + \frac{4(i-1)K(m)}{p}, m] \ , \ \eta_p \equiv (x - v_p t) \ . \tag{31}$$

In order to verify that (31) is indeed a solution to the NLSE eq. (22) one needs the identities

$$\begin{aligned}
(\sum_{i=1}^{p} \tilde{c}_i)^2 &= \sum_{i=1}^{p} \tilde{c}_i^2 + G(p,m) \ , \\
(\sum_{i=1}^{p} \tilde{c}_i)^4 &= \sum_{i=1}^{p} \tilde{c}_i^4 + H(p,m)\sum_{i=1}^{p} \tilde{c}_i^2 + I(p,m) \ , \\
m^2 \sum_{i<j}^{p} \tilde{s}_i \tilde{d}_i \tilde{s}_j \tilde{d}_j &= J(p,m)\sum_{i=1}^{p} \tilde{c}_i^2 + K(p,m) \ ,
\end{aligned} \tag{32}$$

which can be established following the procedure discussed in ref. [3]. The general expression for the velocity v_p is

$$v_p^2 = 4[n + 1 - 2m - mH(p,m) - 2J(p,m)] \ . \tag{33}$$

Some explicitly computed values of the constants $H(p,m)$ and $J(p,m)$ are

$$mH(3,m) = 4J(3,m) = -4q[\frac{q+2}{(1+q)^2} + \frac{q}{1-q^2}] \ , \tag{34}$$

where q is given by eq. (8). Thus v^2 varies from $4(n+9)$ to $4(n-1)$ as the elliptic modulus parameter m changes from 0 to 1.

On the other hand, for even p, we have obtained the following solution of the NLSE eq. (22):

$$r_p(x,t) = \sqrt{2}\sum_{i \text{ odd}}^{p} [d_i - d_{i+1}] \ . \tag{35}$$

In order to verify that eq. (35) is a solution of the NLSE eq. (22) one needs the identities

$$\begin{aligned}
\left(\sum_{i \text{ odd}}^{p} [d_i - d_{i+1}]\right)^2 &= \sum_{i=1}^{p} d_i^2 + P(p,m) \ , \\
\left(\sum_{i \text{ odd}}^{p} [d_i - d_{i+1}]\right)^4 &= \sum_{i=1}^{p} d_i^4 + L(p,m)\sum_{i=1}^{p} d_i^2 + M(p,m) \ , \\
m^2\left[\sum_{i+j \text{ even}}^{p} s_i c_i s_j c_j - \sum_{i+j \text{ odd}}^{p} s_i c_i s_j c_j\right] &= N(p,m)\sum_{i=1}^{p} d_i^2 + Q(p,m) \ ,
\end{aligned} \tag{36}$$

which can be established following the procedure discussed in ref. [3]. The general expression for the velocity v_p is

$$v_p^2 = 4\left[n + m - 2 - L(p,m) - 2N(p,m)\right] \ . \tag{37}$$

Some explicitly computed values of the constants $L(p,m)$ and $N(p,m)$ are

$$L(2,m) = 4N(2,m) = -4\sqrt{1-m} \ ; \quad L(4,m) = 4N(4,m) = -4\tilde{t}(2-\tilde{t}+2\tilde{t}^2) \ , \tag{38}$$

where $\tilde{t}$ is as given by eq. (30). Thus for $p = 2$ [4], v^2 varies from $4(n+4)$ $[4(n+16)]$ to $4(n-1)$ as m changes from 0 to 1. Generalizing the results in eqs. (31) or (35), one finds that v^2 varies from $4(n+p^2)$ to $4(n-1)$ as m changes from 0 to 1.

3.2 Case II: Repulsive Nonlinearity

The NLSE with repulsive nonlinearity is given by

$$iu_t + u_{xx} - u \mid u \mid^2 = 0 \ , \tag{39}$$

We again start with the ansatz given by eq. (20) and on following the same steps as given in eqs. (20) to (22) it is easily seen that the θ equation (eq. (21)) is the same as before while the r equation is almost the same except for the sign of the r^4 term. In particular, the r equation is now given by

$$r'^2(\xi) = \frac{r^4}{2} + (n - \frac{v^2}{4})r^2 - \frac{B}{2} - \frac{A^2}{4r^2} \ , \tag{40}$$

The well known soliton solution to this equation is

$$r(\xi) = \sqrt{2}\tanh \xi \ , \quad \theta = \frac{v\xi}{2} \ , \quad A = 0 \ , \quad B = -4 \ , \ v^2 = 4(n+2) \ . \tag{41}$$

The simplest, periodic, cnoidal traveling wave solution to eq. (40) is

$$r_1(\xi) = \sqrt{2m}\,\mathrm{sn}\ \xi_1 \ , \quad \theta_1 = \frac{v_1\xi_1}{2} \ , \quad A = 0 \ , \ B = -4m \ , \ v_1^2 = 4(n+1+m) \ . \tag{42}$$

For odd p, using the solution eq. (42), we obtain the following solutions of the NLSE eq. (22) by linear superposition

$$r_p(x,t) = \sqrt{2m}\sum_{i=1}^{p} \tilde{s}_i \ , \ \tilde{s}_i \equiv \mathrm{sn}[\eta_p + \frac{4(i-1)K(m)}{p}, m] \ , \ \eta_p \equiv (x - v_p t) \ . \tag{43}$$

In order to verify that eq. (43) is a solution to the NLSE eq. (40) one needs the identities

$$\begin{aligned}
&(\sum_{i=1}^{p} \tilde{s}_i)^2 = \sum_{i=1}^{p} \tilde{s}_i^2 + R(p,m) \ , \\
&(\sum_{i=1}^{p} \tilde{s}_i)^4 = \sum_{i=1}^{p} \tilde{s}_i^4 + S(p,m)\sum_{i=1}^{p} \tilde{s}_i^2 + T(p,m) \ , \\
&\sum_{i<j}^{p} \tilde{c}_i\tilde{d}_i\tilde{c}_j\tilde{d}_j = U(p,m)\sum_{i=1}^{p} \tilde{s}_i^2 + Y(p,m) \ ,
\end{aligned} \tag{44}$$

which can be established by following [3]. The general expression for the velocity v_p is

$$v_p^2 = 4[n + 1 + m + mS(p,m) - 2U(p,m)] \ . \tag{45}$$

Some explicitly computed values of the constants $H(p,m)$ and $J(p,m)$ are

$$mS(3,m) = -4U(3,m) = 4m[\frac{1}{1-q^2} - \frac{1-q^2}{m}] \ , \tag{46}$$

where q is given by eq. (8). Thus v^2 changes from $4(n+9)$ to $4(n+2)$ as m changes from 0 to 1.

For even integer p, the linear superposition of elementary solutions does not work. However, remarkably enough we find that the products of elementary solutions are also solutions. For example, the solution for $p = 2$ is

$$r_2(x,t) = \sqrt{2} m s_1 s_2 \ , \tag{47}$$

and the corresponding velocity is given by $v_2^2 = 4(n + 4 - 2m)$. Generalization to higher even values of p is straightforward.

3.3 Solutions With $A \neq 0$

It may be noted that since $A = 0$ for all the solutions discussed so far, the expressions for θ were rather trivial. One way of obtaining a solution with $A \neq 0$ is to start with the ansatz

$$r^2(\xi) = 2\mathrm{dn}^2\ \xi + \alpha \ , \tag{48}$$

where α is a constant. It is easily checked that (48) is a solution to the NLSE (22) provided

$$\alpha = \frac{2}{3}(n - 2 + m - \frac{c^2}{4}) \ , A^2 = 4\alpha[\alpha^2 - (n - \frac{c^2}{4})\alpha + 2(1-m)] \ ,$$
$$B = -3\alpha^2 + 4(n - \frac{c^2}{4})\alpha - 4(1-m) \ . \tag{49}$$

Starting from the solution (48) we can obtain a class of solutions by an appropriate linear superposition. For example, the 2-point solution is

$$r^2(\xi) = 2(d_1^2 + d_2^2) + \alpha \ , \tag{50}$$

It is easily checked that this is indeed a solution provided

$$\alpha = \frac{2}{3}(n - 2 + m - \frac{c^2}{4}) \ , \quad B = -3\alpha^2 + 4(n - \frac{c^2}{4})\alpha - 16(1-m) \ ,$$
$$A^2 = 4\Big[\alpha^3 - (n - \frac{c^2}{4}\alpha^2 + 8(1-m)\alpha + 8(1-m)(n + 2 - m - \frac{c^2}{4})\Big] \ . \tag{51}$$

Generalization to arbitrary p is straightforward.

4 The $\lambda\phi^4$ Model

The kink (domain wall) solutions to the $\lambda\phi^4$ field theory in $(1+1)$ dimensions

$$\phi_{xx} - \phi_{tt} = \lambda\phi(\phi^2 - a^2) \ , \tag{52}$$

have been widely discussed in the literature [9]. The famous static kink solution is

$$\phi(x) = a\tanh(\sqrt{\lambda/2}\, ax) \ , \tag{53}$$

from which the time-dependent solution

$$\phi(x,t) = a\tanh\left[\sqrt{\frac{\lambda}{2(1-v^2)}}\, a(x-vt)\right] \ , \tag{54}$$

is immediately obtained by Lorentz boosting. Therefore, to begin with, we shall discuss only the static periodic kink solutions in case $v^2 < 1$ (and $\lambda > 1$). Later, we shall discuss time dependent solutions with $v^2 > 1$ (or $\lambda < 1$).

4.1 Static Periodic Kink Solutions

It is well known that the static periodic kink solution to the field eq. (52) is

$$\phi_1(x) = \sqrt{\frac{2m}{1+m}} a \ \mathrm{sn}(\eta_1, m) \ , \ \eta_1 \equiv \sqrt{\frac{\lambda}{1+m}}\, ax \ . \tag{55}$$

For any odd integer p, we find the following static kink solutions of the $\lambda\phi^4$ field theory by a specific linear superposition of the basic solution eq. (55):

$$\phi_p(x) = \sqrt{2m}\, \alpha a \sum_{i=1}^{p} \tilde{s}_i \ , \ p \ \text{ odd} \ , \tag{56}$$

where $\tilde{s}_i, \tilde{c}_i, \tilde{d}_i$ are as given in eq. (14) with $\eta_p \equiv \sqrt{\lambda}\,\alpha a x$. In order to verify that eq. (56) is a static periodic kink solution to the $\lambda\phi^4$ theory field eq. (52), one needs the identity [3]

$$(\sum_{i=1}^{p} \tilde{s}_i)^3 = \sum_{i=1}^{p} \tilde{s}_i^3 + V(p,m) \sum_{i=1}^{p} \tilde{s}_i \ . \tag{57}$$

The constant α is given by

$$\alpha = \frac{1}{\sqrt{1+m+2mV(p,m)}} \ . \tag{58}$$

As an illustration, consider $p = 3$. $V(3,m)$ is given by

$$mV(3,m) = 3\left[\frac{m}{1-q^2} - (1-q^2)\right] \ , \tag{59}$$

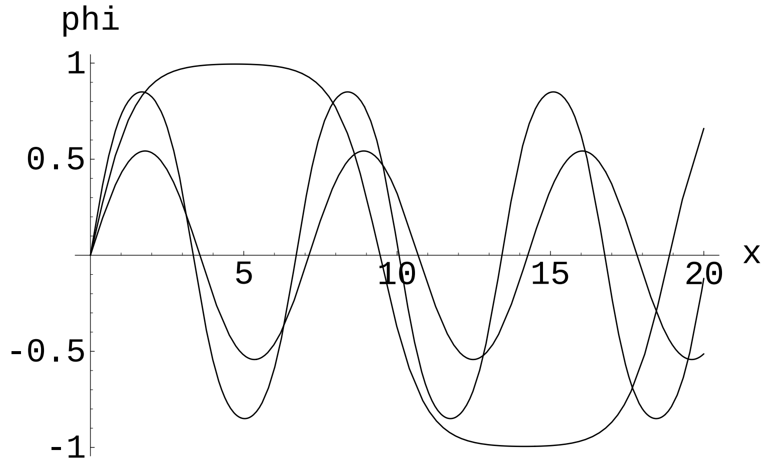

Figure 3: $\phi_1(x)$ (largest amplitude curve), $\phi_2(x)$ and $\phi_3(x)$ (smallest amplitude curve) for $m = 0.98$, $\lambda = a = 1$.

where q is defined in eq. (8). Note that α varies from 1/3 to $1/\sqrt{2}$ as m varies from 0 to 1.

For even integer p, the linear superposition of elementary solutions does not work. However, remarkably enough we find that even in this case, the products of elementary solutions are solutions. For example, the solution for $p = 2$ is

$$\phi_2(x) = \sqrt{2}\, m\alpha a s_1 s_2 \ , \tag{60}$$

where $s_{1,2}$ are as defined in eq. (5) with $\xi \equiv \sqrt{\lambda}\,\alpha a x$. Using the identities derived in ref. [3], it is easily shown that (60) is a static periodic kink solution to the field eq. (52) provided

$$\alpha = \frac{1}{\sqrt{2(2-m)}} \ . \tag{61}$$

Note that α varies from 1/2 to $1/\sqrt{2}$ as m varies from 0 to 1. Generalization to arbitrary even p is straightforward. It is thus clear that for arbitrary integer p, for static periodic kink solutions of $\lambda\phi^4$ theory, α will vary from $1/p$ to $1/\sqrt{2}$ as m varies from 0 to 1. For the values $m = 0.98$ and $\lambda = a = 1$, we plot $\phi_1(x)$ from eq. (55), $\phi_2(x)$ from eq. (60) and $\phi_3(x)$ from eq. (56) in fig. 3.

4.2 Periodic Time-Dependent Kink Solutions

While in the relativistic field theory context with $\lambda > 0$, eqs. (53) and (55) are the only solutions of $\lambda\phi^4$ field theory, in the condensed matter physics context, where velocity v can exceed velocity of sound (optical modes), or for relativistic case with $\lambda < 0$, one also has another soliton solution given by

$$\phi(x,t) = \sqrt{2}a\,\text{sech}(\beta(x-vt)) \ , \quad \beta = a\sqrt{\frac{\lambda}{(v^2-1)}} \ , \tag{62}$$

which is real if either $\lambda < 0$ (and $v^2 < 1$) or $v^2 > 1$ (and $\lambda > 0$).

The corresponding periodic soliton solutions to eq. (52) are well known and are given by [10]

$$\phi_1(\xi) = \sqrt{\frac{2}{2-m}}\, a\,\text{dn}(\xi_1, m) \ , \ \xi_1 = a\sqrt{\frac{\lambda}{(2-m)(v_1^2-1)}}(x-vt) \ , \tag{63}$$

$$\phi_1(\eta) = \sqrt{\frac{2m}{2m-1}} a\,\text{cn}(\eta_1, m) \ , \ \eta_1 = a\sqrt{\frac{\lambda}{(2m-1)(v_1^2-1)}}(x-vt) \ . \tag{64}$$

Notice that the solution (64) is valid only for $1/2 < m < 1$ and both solutions (63) and (64) are only valid for $v^2 > 1$ (and $\lambda > 0$) or $\lambda < 0$ (and $v^2 < 1$).

Appropriate linear superposition of solutions (63) and (64) are also periodic time-dependent kink solutions. For example, by using the linear superposition of solutions (63), we have the following solution to the field eq. (52), which is valid for any integer p:

$$\phi_p(\xi) = \sqrt{2}\, a\alpha \sum_{i=1}^{p} d_i \ , \tag{65}$$

where d_i is as defined in eq. (5) with $\xi_p = a\sqrt{\frac{\lambda}{v_p^2-1}}\alpha(x-v_pt)$. In order to prove that this is a solution, one needs the identity

$$(\sum_{i=1}^{p} d_i)^3 = \sum_{i}^{p} d_i^3 + W(p,m)\sum_{i} d_i \ , \tag{66}$$

which is easily proved following ref. [3]. Using this identity one finds that (65) is a solution provided

$$\alpha^2 = \frac{1}{[2-m+W(p,m)]} \ . \tag{67}$$

Some explicitly computed values of $W(p,m)$ are

$$W(2,m) = 3\sqrt{1-m} \ , \ W(3,m) = \frac{6mq}{1-q^2} \ , \ W(4,m) = 3\tilde{t}[2+\tilde{t}+2\tilde{t}^2] \ , \tag{68}$$

where $\tilde{t}$ is as defined in eq. (30) and q is given by eq. (8). Further, $W(p,0) = p^2-1$, while $W(p,1) = 0$.

For any odd integer p, we also have the following solution to the field eq. (52) by linear superposition

$$\phi(\eta) = \sqrt{2m}\, a\alpha \sum_{i=1}^{p} \tilde{c}_i \ , \tag{69}$$

where $\tilde{c}_i$ is as defined in eq. (14) with $\eta_p = a\sqrt{\frac{\lambda}{v_p^2-1}}\alpha(x-v_p t)$. In order to prove that this is a solution, one needs the identity [3]

$$(\sum_{i=1}^{p} \tilde{c}_i)^3 = \sum_{i}^{p} \tilde{c}_i^3 + X(p,m) \sum_{i=1}^{p} \tilde{c}_i \ . \tag{70}$$

Using this identity one finds that (69) is a solution provided

$$\alpha^2 = \frac{1}{[2m-1+2mX(p,m)]} \ . \tag{71}$$

For example, one can check [3] that

$$X(3,m) = -6(1-m+q) + \frac{6q^2}{1-q^2} \ , \tag{72}$$

with q being given by eq. (8), so that unlike the $p = 1$ case, this is an acceptable solution for all values of m $(0 \le m \le 1)$.

Similarly, for even integer p, we have solutions of the form

$$\phi(\xi) = \sqrt{2}\, a\alpha \sum_{i\ odd}^{p} [d_i - d_{i+1}] \ , \tag{73}$$

where d_i is as defined in eq. (5) with $\xi_p = a\sqrt{\frac{\lambda}{v_p^2-1}}\alpha(x-v_p t)$. Unfortunately, it appears that all these solutions are only valid in a very narrow range of values of m corresponding to real values of α.

5 Sine-Gordon Field Theory

In recent years, both sine-Gordon and $\lambda\phi^4$ field theory have received considerable attention [1, 9]. In particular, sine-Gordon theory is the only relativistically invariant field theory having true soliton solutions. The equation under consideration is

$$\phi_{xx} - \phi_{tt} = \sin\phi \ . \tag{74}$$

5.1 Static Soliton Solution by Linear Superposition

The well known static one-soliton solution of this equation is given by

$$\phi(x) = 4\tan^{-1} e^{\pm x} \ . \tag{75}$$

The corresponding time-dependent solution is easily obtained by Lorentz boosting and hence without any loss of generality we shall restrict our discussion to the static solution only (except when $v^2 > 1$).

The solution (75) can also be written in the alternative form

$$\sin(\frac{\phi(x)}{2}) = \text{sech } x \quad . \tag{76}$$

The two corresponding periodic static soliton solutions are well known and are given by [1]

$$\sin(\frac{\phi(x)}{2}) = \text{dn}(x,m) \ , \tag{77}$$

$$\sin(\frac{\phi(x)}{2}) = \text{cn}(x/\sqrt{m},m) \ , \ m > 0 \ . \tag{78}$$

For any odd integer p, we obtain the following periodic static soliton solutions by linear superposition:

$$\sin(\frac{\phi(x)}{2}) = \alpha \sum_{i=1}^{p} \tilde{d}_i \ , \tag{79}$$

where $\tilde{d}_i$ is as defined in eq. (14) with $\eta_p \equiv \alpha x$, while α is given by

$$\alpha^2 = \frac{1}{[p + A(p,m) + mR(p,m)]} \ . \tag{80}$$

Here $A(p,m)$ and $R(p,m)$ are as defined by eq. (27) and (44) respectively, and with this choice of α, one gets

$$\cos(\frac{\phi(x)}{2}) = \sqrt{m}\,\alpha \sum_{i=1}^{p} \tilde{s}_i \ . \tag{81}$$

Note that use has been made of the identities [3]

$$\tilde{s}_1[\tilde{c}_2 + ... + \tilde{c}_p] + c.p. = 0 \ , \ \tilde{s}_1[\tilde{d}_2 + ... + \tilde{d}_p] + c.p. = 0 \ , \ \tilde{d}_1[\tilde{c}_2 + ... + \tilde{c}_p] + c.p. = 0 \ , \tag{82}$$

in proving that (79) is indeed a solution to the field eq. (74). For $p = 3$ the values of the constants are

$$A(3,m) = 2q(q+2) \ , \ mR(3,m) = 2(q^2 - 1) \ , \tag{83}$$

where q is given by eq. (8), so that $\alpha = 1/(1+2q)$ changes from 1/3 to 1 as m varies from 0 to 1.

Another solution valid for any odd integer p is

$$\sin(\frac{\phi(x)}{2}) = \alpha \sum_{i=1}^{p} \tilde{c}_i \ , \tag{84}$$

where $\tilde{c}_i$ is as defined in eq. (14) with $\eta_p \equiv \alpha x/\sqrt{m}$. This solution is strictly valid only if $m > 0$. It is easily checked that this is indeed a solution to the field eq. (74) provided

$$\alpha^2 = \frac{1}{[p + G(p,m) + R(p,m)]} \ , \tag{85}$$

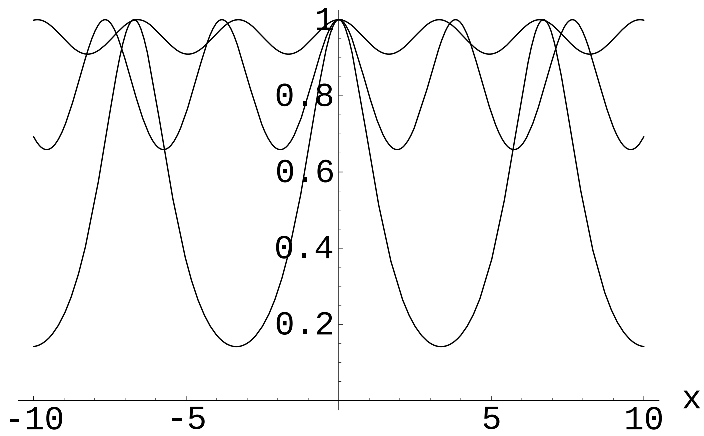

Figure 4: $\sin[\phi(x)/2]$ for $p = 1, 2, 3$. Increasing p decreases the amplitude and increases the frequency.

where $G(p,m)$ and $R(p,m)$ are as defined by eq. (23) and (40) respectively. Note that with this choice of α

$$\cos(\frac{\phi(x)}{2}) = \alpha \sum_{i=1}^{p} \tilde{s}_i \ . \tag{86}$$

For $p = 3$, $G(3,m)$ is given by

$$G(3,m) = -\frac{2q(q+2)}{(1+q)^2} \ , \tag{87}$$

where $R(3,m)$ is as given by eq. (83), so that $\alpha = \frac{1+q}{1-q}$ $(m > 0)$.

Similarly, for $p = 2$ we have the solution

$$\sin(\frac{\phi(x)}{2}) = \alpha \sum_{i=1}^{2} d_i \ , \tag{88}$$

where d_i is as defined in eq. (5) with $\xi_2 \equiv \alpha x$. It is easily checked that this is indeed a solution of the field eq. (74) provided

$$\alpha = \frac{1}{1+\sqrt{1-m}} \ , \tag{89}$$

so that α varies between 1/2 and 1 as m changes from 0 to 1. Note that with this choice of α

$$\cos(\frac{\phi(x)}{2}) = m\alpha s_1 s_2 \ . \tag{90}$$

In fig. 4, we have plotted $\sin[\phi/2]$ vs. x for $p = 1, 2, 3$ corresponding to the right hand side of eqs. (77), (88) and (79).

Another solution for $p = 2$ is

$$\sin(\frac{\phi(x)}{2}) = \alpha[d_1 - d_2] \ , \tag{91}$$

where d_i is as defined in eq. (5) with $\xi_2 \equiv \alpha x$. It is easily checked that this is indeed a solution to the field eq. (74) provided $0 < m \leq 1$ since α given by

$$\alpha = \frac{1}{1-\sqrt{1-m}} \ , \tag{92}$$

diverges at $m = 0$.

5.2 Periodic Time-Dependent Solutions

As in the $\lambda\phi^4$ field theory case, in this case also we get solutions by linear superposition, which are only valid for $v^2 > 1$. In particular, if p is any odd integer, the periodic time-dependent solution is given by

$$\sin(\frac{\phi(x)}{2}) = \sqrt{m}\alpha \sum_{i=1}^{p} \tilde{s}_i \ , \tag{93}$$

where $\tilde{s}_i$ is as defined in eq. (14) with $\eta_p \equiv \alpha(x-vt)/\sqrt{v^2-1}$. It is easily checked that this is indeed a solution of the field eq. (74) provided α^2 is again given by eq. (80). Note that with this choice of α

$$\cos(\frac{\phi(x)}{2}) = \alpha \sum_{i=1}^{p} \tilde{d}_i \ . \tag{94}$$

Another solution for any odd integer p is

$$\sin(\frac{\phi(x)}{2}) = \alpha \sum_{i=1}^{p} \tilde{s}_i \ , \tag{95}$$

where $\tilde{s}_i$ is as defined in eq. (14) with $\eta_p \equiv \alpha(x-vt)/\sqrt{m(v^2-1)}$. Thus, this solution is strictly valid only if $m > 0$. It is easily checked that this is indeed a solution of the field eq. (74) provided α^2 is as given by eq. (85). Note that with this choice of α,

$$\cos(\frac{\phi(x)}{2}) = \alpha \sum_{i=1}^{p} \tilde{c}_i \ . \tag{96}$$

Finally, for $p = 2$ we have the solution

$$\sin(\frac{\phi(x)}{2}) = m\alpha s_1 s_2 \ , \tag{97}$$

where $s_{1,2}$ is as defined in eq. (5) with $\xi_2 \equiv \alpha(x-vt)/\sqrt{v^2-1}$. It is easily checked that this is indeed a solution of the field eq. (74) provided α^2 satisfies eq. (88). Note that with this choice of α

$$\cos(\frac{\phi(x)}{2}) = \alpha \sum_{i=1}^{2} d_i \ . \tag{98}$$

6 Boussinesq Equation

The Boussinesq equation is given by

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0 \ . \tag{99}$$

The periodic one soliton solution of this equation is known to be

$$u(x,t) = -2\alpha^2 \mathrm{dn}^2(\alpha(x - vt)) + \beta\alpha^2 \ , \tag{100}$$

where

$$2(4 - 2m - 3\beta)\alpha^2 = v^2 - 1 \ . \tag{101}$$

Thus $v^2 < 1$ if $\beta \geq 4/3$ and $v^2 > 1$ if $\beta \leq 2/3$. For $2/3 < \beta < 4/3, v^2$ changes sign at some value of m $(0 \leq m \leq 1)$. Note that in the limit $m \to 1$ and $\beta = 2$, this solution goes over to the one-soliton solution

$$u(x,t) = 2\alpha^2 \tanh^2(\alpha(x - vt)) \ , \ \alpha = \sqrt{\frac{1 - v^2}{8}} \ , \tag{102}$$

Now consider the linear superposition

$$u(x,t) = -2\alpha^2 \sum_{i=1}^{p} d_i^2(\alpha(x - vt)) + \beta\alpha^2 \ , \quad (p = 1, 2, 3...) \ . \tag{103}$$

It is easy to check that this is an exact solution to the Boussinesq eq. (99) provided

$$2[4 - 2m - 3\beta - 6A_1(p,m)]\alpha^2 = v^2 - 1 \ . \tag{104}$$

Here use has been made of the identity (6) with $A_1(p,m)$ given by eqs. (7) and (10). It may be noted that $A_1(p,m) \leq 0$.

In this article we have shown that in view of the remarkable identities satisfied by Jacobi elliptic functions, a kind of linear superposition principle works for several nonlinear equations, some of which are associated with integrable systems, the others not. It would indeed be worthwhile to obtain such solutions for other nonlinear systems where elliptic functions play a role in the space of exact solutions. A question which comes to mind is how the solutions obtained in this paper are related to previously known solutions. At first sight, it would appear that our procedure has given new solutions, but a closer investigation reveals that our solutions are expressible in terms of previous solutions via a non-trivial generalization of Landen's formulas which connect Jacobi elliptic functions with two different modulus parameters [11]. The reader is referred to Ref. [11] for more details.

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