Testing Hall-Post Inequalities
With Exactly Solvable N-Body Problems

Avinash KHARE[1] and Jean-Marc RICHARD[2]

[1] Institute of Physics,
Sachivalaya Marg, Bhubaneswar 751005,India
[2] Institut des Sciences Nucléaires, Université Joseph Fourier–CNRS-IN2P3
53, avenue des Martyrs,F 38026 Grenoble, France.

Abstract

The Hall–Post inequalities provide lower bounds on N-body energies in terms of N'-body energies with $N' < N$. They are rewritten and generalized to be tested with exactly-solvable models of Calogero-Sutherland type in one and higher dimensions. The bound for N spinless fermions in one dimension is better saturated at large coupling than for noninteracting fermions in an oscillator potential.

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It is important to obtain good upper and lower bounds on the binding energy of N-particle systems. Since upper bounds are provided by variational estimates, it is natural that a major emphasis in recent years has been on obtaining good lower bounds.

The Hall–Post inequalities consist of lower bounds to N-body energies in terms of N'-body energies with $N' < N$ and modified constituent masses or coupling constants. [1, 2, 3]. Applications of these inequalities have been proposed for studying the thermodynamic limit of large systems and the stability of matter [3], or the relation of baryons to mesons in hadron spectroscopy [2, 4].

So far, the Hall-Post inequalities have been tested in great detail for few-body systems, for which accurate numerical calculations can be performed, or in the large N limit, within some approximation. It seems appealing to test the inequalities for arbitrary N with energies which are calculated exactly. The purpose of this note is to adapt and apply the Hall–Post inequalities to several exactly-solvable N-body models.

Inspired by the seminal work of Calogero and Sutherland, in recent years, ground state energy and at least a part of the excitation spectrum (if not the full spectrum) has been obtained for several new N-body problems. Some of these Hamiltonians have already found applications in a variety of areas.

We broadly consider four types of N-body problems.

1. N-body problems in one dimension with two-body (and also may be one-body) interaction in the presence of either two-body or one-body oscillator potential. Some examples of this class are A_{N-1} [5], BC_N, D_N [6] models. The case of D dimensions is considered in [7].

2. N-body problems in one dimension but with periodic boundary conditions and with two-body (and also possible one-body interaction). Again, some examples of this class are the A_{N-1} [8] and BC_N, D_N [6] models.

3. N-body problems in one dimension with two-body inverse square interaction in the presence of a hyper-Coulomb potential [9].

4. N-body problems of Calogero-type in two and higher dimensions with two and three-body interactions in the presence of one or two-body oscillator potential [10, 11].

Our conclusions can be summarized as follows. While comparing the first and second type of models with the Hall–Post inequalities, we find that contrary to our initial belief, the bound for spinless fermions in one dimension is better saturated in the large coupling limit rather than for N-noninteracting fermions in an oscillator potential. For the hyper-Coulomb case, using convexity argument, we derive a slightly better lower bound than was known before. Finally, we show that Hall–Post bounds also work reasonably well in models with both two and three-body interactions.

In this paper we shall consider models with N identical particles of mass m (which we shall put equal to 1 without any loss of generality) and whose interaction does not depend on their spins. This corresponds to the Hamiltonian

$$H_N(m, g_1, g_2, g_3) = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + g_1 V(r_i) + g_2 \sum_{i<j} V(r_{ij}) + g_3 \sum_{i<j<k} V(r_{ijk}) \,. \qquad (1)$$

For this case, it has been shown that the corresponding N-particle bound-state energy E_N satisfies the bound [3, 4]

$$E_N(m, g_1, g_2, g_3) \geq \frac{N}{N-1} E_{N-1}\left(m, g_1, \frac{N-1}{N-2}g_2, \frac{N-2}{N-3}g_3\right) \,. \qquad (2)$$

In case the Hamiltonian (1) is translationally invariant, then this bound can be improved yielding the new inequality [1, 2]

$$E_N(m, g_2, g_3) \geq \frac{N-1}{N-2} E_{N-1} \left(m, \frac{N}{N-1} g_2, \frac{N(N-2)}{(N-1)(N-3)} g_3 \right) . \qquad (3)$$

As a representative of the first type of models, we consider the original Calogero problem [5] for which the N-body Hamiltonian is given by ($\hbar = m = 1$)

$$H = -\frac{1}{2} \sum_{i=1}^{N} \frac{d^2}{dx_i^2} + \sum_{i<j=1}^{N} \left[\frac{\omega^2}{4} (x_i - x_j)^2 + \frac{g}{(x_i - x_j)^2} \right] . \qquad (4)$$

As shown by Calogero, the ground-state energy for this N-particle system is given by

$$E_N = \sqrt{\frac{N}{8}} \left[N^2 - 1 + (\beta - 1)N(N-1) \right] \omega , \qquad (5)$$

where

$$g = \beta(\beta - 1), \quad \text{i.e.,} \quad \beta = \frac{1}{2} \pm \frac{\sqrt{1+4g}}{2} , \qquad (6)$$

while the corresponding ground-state eigenfunction is given by

$$\psi = \prod_{i=1}^{N} (x_i - x_j)^\beta \times \exp \left[-\frac{\omega}{\sqrt{2N}} \sum_{i<j=1}^{N} (x_i - x_j)^2 \right] . \qquad (7)$$

Note that we have to choose the positive square root in eq. (6) so that $g = 0$ corresponds to $\beta = 1$ and hence to fermions.

For translationally invariant Hamiltonians with two-body interaction, as the one given in eq. (4), the bound (3) takes the form

$$E_N(m, \omega, g) \geq \frac{N-1}{N-2} E_{N-1} \left(m, \omega \sqrt{\frac{N}{N-1}}, \frac{N}{N-1} g \right) . \qquad (8)$$

Using the bound-state energy expression (5) as derived for the Hamiltonian (4), it is easily seen that

$$E_{N-1} \left(m, \omega \sqrt{\frac{N}{N-1}}, \frac{N}{N-1} g \right) = \sqrt{\frac{N}{8}} \left[N(N-2) + (\beta' - 1)(N-1)(N-2) \right] \omega , \qquad (9)$$

where

$$\beta' = \frac{1}{2} + \frac{\sqrt{N(2\beta-1)^2 - 1}}{2\sqrt{N-1}} . \qquad (10)$$

Using eqs. (5), (8) and (9) we then get

$$R_N \equiv \frac{E_N(m, \omega, g)}{\frac{N-1}{N-2} E_{N-1}(m, \omega \sqrt{\frac{N}{N-1}}, \frac{N}{N-1} g)} \geq \frac{N+1+(\beta-1)N}{N+(\beta'-1)(N-1)} . \qquad (11)$$

In the non-interacting fermion limit (i.e., $g = 0$ or $\beta = \beta' = 1$), the right-hand side is $\frac{N+1}{N}$ while in the strong-coupling limit (i.e., $g \to \infty$) the r.h.s. is $\frac{\sqrt{N}}{\sqrt{N-1}}$. On the other hand, as $N \to \infty$ in
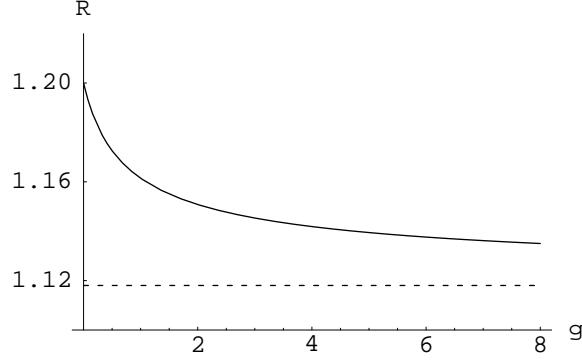
Figure 1: Hall–Post ratio R_N, as defined in eq. (11) for the Calogero model (4), in the case of $N = 5$ particles, as a function of the coupling constant g. The dotted line is the $g \to \infty$ limit.

one dimension, then the ratio goes to 1, for all values of g. Thus we find that contrary to our naive expectation, the bound is better satisfied by strongly interacting rather than non-interacting fermions in an oscillator potential.

Fig. 1 displays the ratio R_N as a function of the coupling constant g, in case $N = 5$. One starts at $g = 0$ from the value $R_5 = 6/5$ corresponding to a pure oscillator. Then the ratio evolves regularly towards its large g limit $\sqrt{5/4}$.

We have also examined how Hall-Post inequalities behave in similar models like BC_N, D_N, and even periodic models like those of Sutherland. In all these cases we found a behaviour similar to that given in Fig. 1. We might add here that many of these models are not translationally invariant and instead of the lower bound (8), we have to use the simpler bound (2).

As a second example, we now consider a model of the third type [9], i.e.,

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{d^2}{dx_i^2} + \sum_{i<j=1}^{N}\frac{g}{(x_i - x_j)^2} - \frac{\alpha^2}{\sqrt{\sum_{i<j=1}^{N}(x_i - x_j)^2}} \ , \tag{12}$$

which contains a hypercentral Coulomb interaction. The N-particle ground-state binding energy has been shown to be

$$E_N = -\frac{\alpha^2}{N[N - 2 + N(N-1)\beta]^2} \ , \tag{13}$$

with β and g being again related by eq. (6).

We now show that using a convexity argument, one can adapt the bound (3) to this case. In particular, we use the fact that if a function f is a convex function, then

$$f\left(\frac{\alpha a + \beta b + \cdots}{\alpha + \beta + \cdots}\right) \geq \frac{\alpha}{\alpha + \beta + \cdots}f(a) + \frac{\beta}{\alpha + \beta + \cdots}f(b) + \cdots \ , \tag{14}$$

for positive weight factors α, β, $\dots$. On using $f(x) = -1/\sqrt{x}$, it is easily shown that in this case the Hall-Post inequality takes the form

$$E_N(m, g, \alpha) \geq \frac{N-1}{N-2}E_{N-1}\left(m, \frac{N}{N-1}g, \frac{(N-1)\sqrt{(N-2)}}{N^{3/2}}\alpha\right) \ . \tag{15}$$
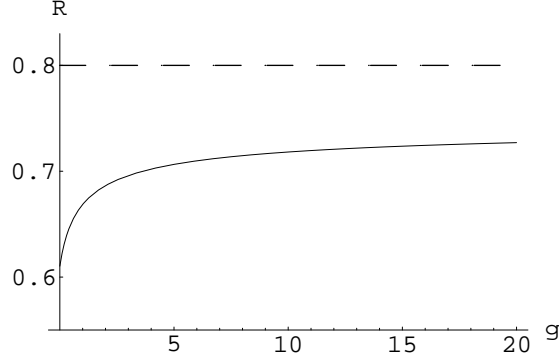
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Figure 2: Ratio R_N, as given in eq. (13), for the model of eq. (12), in the case of $N = 5$ particles, as a function of the coupling constant g. The dotted line is the $g \to \infty$ limit.

How good is this inequality? We can test that by using the energy eigenvalue (13) and computing E_{N-1} for the appropriate couplings. On noting the fact that the binding energy is negative, the Hall-Post inequality (15) in our case takes the form

$$R_N \equiv \frac{E_N(m, g, \alpha)}{\frac{N-1}{N-2} E_{N-1}(m, \frac{N}{N-1}g, \frac{(N-1)\sqrt{(N-2)}}{N^{3/2}}\alpha)}$$
$$\leq \frac{(N)^2[N-3+(N-1)(N-2)\beta']^2}{(N-1)^2[N-2+N(N-1)\beta]^2} \,, \tag{16}$$

where

$$\beta' = \frac{1}{2} + \frac{\sqrt{(N-1)(2\beta-1)^2 - 1}}{2\sqrt{N-2}} \,. \tag{17}$$

We find that but for $N \to \infty$, g arbitrary, this bound is not as good as (11) in the Calogero case. In Fig. 2, we have plotted the ratio R_N as a function of g in case $N = 5$ which gives an indication of how good this inequality is for finite N.

As the third and the last example, we consider a model of fourth type [10, 11] which is a Calogero-type model but in D-dimensions given by

$$H = \frac{1}{2}\sum_{i=1}^{N} \mathbf{p}_i^2 + \frac{\omega^2}{4}\sum_{i<j=1}^{N} \mathbf{r}_{ij}^2 + g\sum_{i<j=1}^{N}\frac{1}{\mathbf{r}_{ij}^2} + G\sum_{\substack{i<j,k \\ k\neq i,j}}\frac{\mathbf{r}_{ki}.\mathbf{r}_{kj}}{\mathbf{r}_{ki}^2\mathbf{r}_{kj}^2} \,, \tag{18}$$

where one also requires a three-body potential in addition to the two-body potential. The N-boson ground-state energy in

$$E_N(m, g, G(g)) = \sqrt{\frac{N}{8}}[D(N-1) + N(N-1)\beta]\omega \,, \tag{19}$$

provided the two-body coupling g and the three-body coupling G are related by

$$g = G + (D-2)\sqrt{G} \,, \quad G = \beta^2 \,, \tag{20}$$

this defining $G(g)$. Note that unlike in one-dimension case, here $\beta = 0$ corresponds to $g = G = 0$ and hence to bosons in a pure oscillator potential.

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Let us now examine how the Hall-Post inequality (3) fares in this case. It is not possible to test this inequality directly since g and G have a specific relation $G(g)$ between them which in general will not be satisfied if g and G are changed to $\frac{N}{N-1}g$ and $\frac{N(N-2)}{(N-1)(N-3)}G$, respectively. However, we make use of the fact that both the two and three body terms in eq. (18) are positive in any dimension $D(\geq 2)$ and hence we can write an appropriate inequality. For example we obviously have

$$
\begin{aligned}
E_N\left(m,\omega,g,G(g)\right) &\geq \frac{N-1}{N-2}E_{N-1}\left(m,\omega\sqrt{\frac{N}{N-1}},\frac{N}{N-1}g,\frac{N(N-2)}{(N-1)(N-3)}G(g)\right) \\
&\geq \frac{N-1}{N-2}E_{N-1}\left(m,\omega\sqrt{\frac{N}{N-1}},\frac{N}{N-1}g,G(\frac{N}{N-1}g)\right) ,
\end{aligned}
\tag{21}
$$

since

$$
\frac{N(N-2)}{(N-1)(N-3)}G(g) \geq G(\frac{N}{N-1}g).
\tag{22}
$$

How good is this inequality? Using the exact N-particle binding energy (19) it is easily seen that

$$
\frac{E_N(m,\omega,g,G(g))}{\frac{N-1}{N-2}E_{N-1}(m,\omega\sqrt{\frac{N}{N-1}},\frac{N}{N-1}g,G(\frac{N}{N-1}g))} \geq \frac{D+N\beta}{D+(N-1)\beta'} ,
\tag{23}
$$

where

$$
\beta' = -\frac{D-2}{2} + \frac{\sqrt{N(2\beta+D-2)^2-(D-2)^2}}{2\sqrt{N-1}} .
\tag{24}
$$

It is interesting that this bound is saturated for noninteracting bosons ($\beta = 0$ i.e., $g = G = 0$) in an oscillator potential as well as for large number of particles, no matter what the coupling is. For large coupling but finite N, the ratio is $\sqrt{N/(N-1)}$. The bound is also saturated at large D, as seen explicitly in eq. (23). This property of the large D limit was noted by Gonzalez-Garcia [12].

Final remarks are in order:

1. It would be interesting to extend the investigation to the case of unequal masses. Some of the exactly-solvable models considered here can be generalized as to accommodate different constituent masses [13]. The bound for unequal masses was discussed in the case of $N = 3$ or $N = 4$ particles interacting through simple pairwise potentials [14].

2. Exactly solvable models are also available for particles with residual interaction with nearest and next-to-nearest neighbour only [15]. It would be interesting to extend the Hall-Post inequalities to this situation.

3. One of the striking feature of some of the models considered here in one or two dimensions is that the statistics evolves continuously as one changes the coupling constant. Still, if one looks at the E_N to E_{N-1} ratio measuring how far the inequalities are from saturation, it evolves with a rather smooth and monotonic behaviour. Fermions can approach saturation provided the coupling and the number of particles are large enough.

We hope to address some of the open issues in the near future.

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