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# ONE-DIMENSIONAL ADHESION MODEL FOR LARGE SCALE STRUCTURES

KAYYUNNAPARA THOMAS JOSEPH

*Dedicated to the memory of Professor P. L. Sachdev*

Abstract. We discuss initial value problems and initial boundary value problems for some systems of partial differential equations appearing in the modelling for the large scale structure formation in the universe. We restrict the initial data to be bounded measurable and locally bounded variation function and use Volpert product to justify the product which appear in the equation. For more general initial data in the class of generalized functions of Colombeau, we construct the solution in the sense of association.

## 1. Introduction

The Burgers equation in one dimension is a second-order nonlinear parabolic equation balancing quadratic nonlinearity and diffusion and is of the form

$$u_t + uu_x = \epsilon u_{xx}. \tag{1.1}$$

This equation was introduced by Burgers in 1939 as a simplification of the Navier-Stokes equation with the hope of understanding issues such as turbulence. Hopf [10] and Cole [3] showed that Burgers equation can be integrated explicitly. Indeed Burgers equation does not posses the fundamental aspects of turbulence, namely sensitivity to small perturbations in the initial conditions and the spontaneous arise of randomness by chaotic dynamics. Hopf's construction of solution of the inviscid Burgers equation, by passing to the limit as $\epsilon$ goes to zero in the explicit solution of (1.1) with with initial data $u(x,0) = u_0(x)$ in the space of bounded measurable function and the work of Lax [18] for systems laid the mathematical foundation of the theory of Hyperbolic systems of conservation laws.

In the 1980's Burgers equation in one dimensions and in multi-dimensions again started to appear in other fields such as in statistical mechanics and in cosmology and became focus of study again.

In this paper we study Burgers equation which has applications in cosmology where it is closely linked to what is usually referred to as Zeldovich approximation [25]. According to this model the evolution in the last stage of the expansion of the

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universe, matter is described as cold dust moving under gravity alone and the laws are governed by the system

$$\begin{aligned} u_t + (u.\nabla)u = 0, \quad u = \nabla\phi \\ \rho_t + \nabla(\rho u) = 0, \end{aligned} \tag{1.2}$$

where $u$ is velocity and $\rho$ the density of the particles. It is observed that the fastest growing mode in linear theory has decaying vorticity, and this is the reason for the interest is in potential solutions where the velocity $u$ can be represented in terms of a velocity potential $\phi$.

Even with smooth initial data

$$u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x) \tag{1.3}$$

in general, existence of global smooth solutions is not possible as the fastest particles overrun the slowest ones and after some time the density becomes infinite.

Mathematically, the local existence theory given by classical Hamilton Jacobi theory of first order equation is not helpful to construct global solution. The mapping from Lagrangian space$L(y)$ to Eulerian space $E(x)$ given by

$$x = y + tu_0(y) \tag{1.4}$$

is one-one only for short time and

$$u(x,t) = u_0(y), \quad \rho(x,t) = \frac{\rho_0(y)}{\det(\frac{\partial x_i}{\partial y_j})} \tag{1.5}$$

gives only short time existence result.

Gurbatov and Saichev [8], introduced the adhesion model. In adhesion approximation, the motion of particles is described by the motion of sticking particles. The velocity obey the Burgers equation and density by the continuity equation.

$$\begin{aligned} u_t + (u.\nabla)u = \epsilon\Delta u, \quad u = \nabla\phi \\ \rho_t + \nabla(\rho u) = 0 \end{aligned} \tag{1.6}$$

Using a Hopf-Cole transformation Weinberg and Gunn [24], Joseph and Sachdev [16], wrote down the exact formula for the velocity $u^\epsilon$ when the initial data is of the form

$$u(x,0) = \nabla\phi_0(x)$$

and derived a formula for the vanishing viscosity limit, generalizing Hopf's [10] result, namely

$$\lim_{\epsilon\to 0} u^\epsilon(x,t) = u(x,t) = \frac{(x - y(x,t))}{t}$$

where $y(x,t)$ is a minimizer in

$$\min_{y\in R^n} \{\phi_0(y) + \frac{|x-y|^2}{t}\}$$

which exists always and unique for almost every points. Joseph and Sachdev [16] also studied the large time behaviour of the solution for each fixed $\epsilon > 0$. The velocity thus obtained is not smooth and density remains to be determined.

Gurbatov and Saichev [9] proposed a model equation to approximate the adhesion model, to compute density. In the limit $\epsilon$ goes to 0, $y(x,t)$ is interpreted as Lagrangian co-ordinates of particles falling into $x$ at time $t$. The matter density $\rho$

is determined by Euler to Lagrange co-ordinate transformation Jacobian given by (1.5). The idea is formally introduce

$$x = y + tu(x,t) \tag{1.7}$$

where $u$ satisfy the Burgers equation. Interpret this as the mapping from Lagrangian space$L(y)$ to Eulerian space $E(x)$ and the density be defined

$$\rho(x,t) = \rho_0 \quad \det(\frac{\partial y_i}{\partial x_j}).$$

It was shown in [9] that $y = y(x,t)$ defined by (1.7) satisfies the equation

$$y_t + (u.\nabla)y = \epsilon \Delta y,$$

and in one dimension $\rho$ satisfies

$$\rho_t + (u\rho)_x = \epsilon \rho_{xx}, \tag{1.8}$$

where $\rho_0$ is a constant.

For one space dimension, (1.1) together with (1.8) is called the modified adhesion model. Joseph [12] has shown that this modified adhesion model can be linearized using a generalized Hopf-Cole transformation, and showed the formation of $\delta$-waves in the density components in the vanishing viscosity limit when the initial data is of Riemann type. The aim of this paper is to study one dimensional case, and give simpler proofs for known results and give some new results on initial value problems and initial boundary value problems. The focus is on the exact limit, its structure and how to make sense of solution.

## 2. Initial value problem

In one space dimension the Zeldovich approximation takes the form

$$\begin{gathered} u_t + (u^2/2)_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{gathered} \tag{2.1}$$

With initial data

$$u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x) \tag{2.2}$$

Equation (2.1) has a local smooth solution if the initial data is smooth. The equation for $u$ says that $u$ is constant along the characteristics given by (1.4). The construction given by (1.4)-(1.5) is not valid when the characteristics cross, the solution is multi-valued and shocks develop.

In this section we take the initial data (2.2) and use the model (1.1) and (1.8) to construct explicit formula for the solution of (2.1) and (2.2) as $\epsilon$ goes to 0. For the case the initial data in $L^\infty \cup BV_{\rm loc}$, we show that this explicit formula for the inviscid problem is same as the one obtained in LeFloch [19] and in Tan et al. [22] by different methods. The Riemann problem was treated earlier by Joseph [12]. The solution shows that the particles move with constant velocities until they collide and at collision, the colliding particle form a new massive particle; $u$ remains bounded pointwise and $\rho$ is not bounded pointwise but are measures. Also there appear vacuum. We use the Volpert product [23] to justify the product $\rho u$ appearing in the equation. Note that the use of Volpert product was first advocated for hyperbolic problems by LeFloch [20] and this provides only one particular approach among a family of definitions described in Dal Maso, LeFloch and Murat [7].

Next we analyze the modified adhension model in the light of Colombeau theory [4, 5, 6]. We consider initial datas in more general class and construct solution in the algebra of generalized functions of Colombeau [4].

2.1. **Explicit formula for the modified adhesion model.** The modified adhesion model see Gurbatov and Saichev [9] in one space dimension is

$$\begin{aligned} u_t + (u^2/2)_x &= \frac{\epsilon}{2} u_{xx}, \\ \rho_t + (\rho u)_x &= \frac{\epsilon}{2} \rho_{xx}. \end{aligned} \tag{2.3}$$

We solve (2.3) with initial data

$$u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x) \tag{2.4}$$

using a generalized Hopf-Cole transformation see Joseph [12], namely

$$u = -\epsilon \frac{a_x}{a}, \;\; \rho = (\frac{b}{a})_x \tag{2.5}$$

the problem (2.3) and (2.4) is reduced to the linear problem

$$\begin{gathered} a_t = \frac{\epsilon}{2} a_{xx}, \quad a(x,0) = e^{-\frac{U_0(x)}{\epsilon}} \\ b_t = \frac{\epsilon}{2} b_{xx}, \quad b(x,0) = R_0(x) e^{-\frac{U_0(x)}{\epsilon}} \\ R_0(x) = \int_0^x \rho(y)dy, \quad U_0(x) = \int_0^x u_0(y)dy \end{gathered} \tag{2.6}$$

Solving (2.6) and substituting in (2.5), we have the following theorem.

**Theorem 2.1.** *Assume $u_0$ and $v_0$ are bounded measurable or integrable. Then*

$$\begin{gathered} u^\epsilon(x,t) = \int_{R^1} \frac{(x-y)}{t} d\mu^\epsilon_{(x,t)}(y) \\ \rho^\epsilon(x,t) = \partial_x R^\epsilon(x,t) \\ R^\epsilon(x,t) = \int_{R^1} R_0(y) d\mu^\epsilon_{(x,t)}(y). \end{gathered} \tag{2.7}$$

*where for each $(x,t)$, and $\epsilon > 0$, the probability measure $d\mu^\epsilon_{(x,t)}(y)$ defined by*

$$\begin{gathered} d\mu^\epsilon_{(x,t)}(y) = \frac{e^{-\frac{\theta(x,y,t)}{\epsilon}} dy}{\int_{R^1} e^{-\frac{\theta(x,y,t)}{\epsilon}} dy} \\ \theta(x,y,t) = U_0(y) + \frac{(x-y)^2}{2t}. \end{gathered} \tag{2.8}$$

*is a solution to* (2.3) *and* (2.4).

2.2. **Vanishing viscosity limit.** In this section, we find an explicit formula for global solution of (2.1) with initial data (2.2) in the space of bounded measurable functions which are locally BV functions. Here we use the modified adhesion model (2.3) and follow the analysis of Hopf [10] and Lax [18] and the properties of the minimizers of

$$\min_{-\infty < y < \infty} \theta(x,y,t) \tag{2.9}$$

proved in these papers. We state the results.

**Theorem 2.2.** *Assume $u_0$ and $\rho_0$ are bounded measurable and in $BV_{\rm loc}$. For each fixed $t>0$, except for a countable $x$, there exits a unique minimizer $y(x,t)$ for* (2.9) *and at these points*

$$u(x,t)=\lim_{\epsilon\to 0}u^{\epsilon}(x,t)=\frac{(x-y(x,t))}{t}, \tag{2.10}$$

$$R(x,t)=\lim_{\epsilon\to 0}R^{\epsilon}(x,t)=\int_0^{y(x,t)}\rho_0(z)dz. \tag{2.11}$$

*The functions $u(x,t)$ and $R(x,t)$ are well defined a.e. and are functions of bounded variation.*

*Further for each $t>0$, and $x\in R^1$, $u(x+,t)$ and $u(x-,t)$ $R(x-,t)$ and $R(x+,t)$ exits.*

*Also $u(x,t)$ satisfies the entropy condition $u(x-,t)\geq u(x+,t)$.*

*Finally*

$$\rho(x,t)=\lim_{\epsilon\to 0}\rho^{\epsilon}(x,t)=\partial_x(\int_0^{y(x,t)}\rho_0(z)dz) \tag{2.12}$$

*in the sense of distributions. Further, $(u,\rho)$ satisfies* (2.1) *in the sense of distribution and satisfies the initial conditions* (2.2).

*Proof.* For initial data $u_0$ integrable or bounded measurable, one get that for each fixed $(x,t)$, $\theta(x,y,t)$ has a global minimum as a function of $y$ and minimum is achieved at some point. Let $y(x,t)$ be a point where global minimum is achieved in (2.8). This minimizing point may not be unique. Hopf [10] and Lax [18] analysis shows that he largest and the smallest of these minimizers $y^{+}(x,t)$ and $y^{-}(x,t)$ are increasing functions of $x$ and hence the point of discontinuities are at most countable and except these points they are equal $y(x,t)=y^{-}(x,t)=y^{+}(x,t)$. At these points where $y(x,t)$ is unique the measure $d\mu_(x,t)^{\epsilon}(y)\to\delta_{y(x,t)}$ as $\epsilon$ goes to 0, in measure in the sense that for any continuous function $g(y)$ on $R^1$

$$\int g(y)d\mu^{\epsilon}_{(x,t)}(y)\to\langle\delta_{y(x,t)},g(y)\rangle.$$

We immediately get the limit (2.10) and (2.11).

To show $(u,\rho)$ satisfies (2.1), we should interpret the product $u\rho$ suitably as we only know that $u$ and $R$ are functions of bounded variation and hence $\rho=R_x$ is a Radon measure. We follow Volpert [23]. With respect to $u$, a function of bounded variation, we have the decomposition of the domain $R^1\times[0,\infty)$

$$[0,\infty)\times[0,\infty)=S_c\cup S_j\cup S_0$$

where $S_c$ and $S_j$ are points of approximate continuity of $u$ and points of approximate jump of $u$ and $S_0$ is a set of one dimensional Hausdorff-measure zero. At any point $(x,t)\in S_j$, $u(x-0,t)$ and $u(x+0,t)$ denote the left and right values of $u(x,t)$. For any continuous function $g:R^1\to R^1$, the Volpert product $g(u)\rho=g(u)R_x$ is defined as a Borel measure in the following manner. Consider the averaged superposition of $g(u)$ (see Volpert [23])

$$\overline{g(u)}(x,t)=\begin{cases}g(u(x,t), & \text{if } (x,t)\in S_c,\\ \int_0^1 g((1-\alpha)(u(x-,t)+\alpha u(x+,t))d\alpha, & \text{if } (x,t)\in S_j.\end{cases} \tag{2.13}$$

Volpert [23] proves that $\overline{g(u)}$ is measurable and locally integrable with respect to the Borel measure $R_x$, so that the nonconservative product $\overline{g(u)}R_x$ has a meaning as a locally finite Borel measure. In deed

$$[g(u)R_x](A) = \int_A \overline{g(u)}(x,t)R_x$$

where $A$ is a Borel measurable subset of $S_c$ and

$$[g(u)R_x](\{(x,t)\}) = \overline{g(u)}(x,t)(R(x+0,t) - R(x-0,t))$$

provided $(x,t) \in S_j$.

To show $(u,\rho)$ is a solution, we need to show

$$(u,\phi_t) + \Big(\frac{(u^2)}{2}, \phi_x\Big) = 0, \quad (\rho,\phi_t) + (\overline{u}\rho, \phi_x) = 0 \tag{2.14}$$

for all test functions $\phi$. The first is standard and follows in the limit as $\epsilon$ goes to zero, by an application of dominated convergence theorem after multiplying the first equation in (2.3) by a test function and integrating by parts.

To show that $\rho$ satisfies the second equation we show that

$$\mu = R_t + \overline{u}R_x = 0 \tag{2.15}$$

in the sense of measures. This is in LeFloch [19] and for completeness we give the details of his arguments for our special case. Let $(x,t) \in S_c$ and $u = \frac{x-y(x,t)}{t}$, since $u$ satisfies (2.1), we have

$$-\frac{(x-y(x,t))}{t^2} - \frac{\partial_t y(x,t)}{t} + u(a,t)\frac{(1-\partial_x y(x,t))}{t} = 0.$$

It follows that

$$\partial_t y(x,t) + u\partial_x y(x,t) = 0.$$

Now

$$\partial_t R(x,t) + u\partial_x R(x,t) = (\frac{dv_0}{dx})(y(x,t)\{\partial_t y(x,t) + u\partial_x y(x,t)\}$$

and we get

$$\partial_t R(x,t) + u\partial_x R(x,t) = 0.$$

Now we consider a point $(s(t),t) \in S_j$, then

$$\frac{ds(t)}{dt} = \frac{u(s(t)+,t)) - u(s(t)-,t)}{2}$$

is the speed of propagation of the discontinuity at this point.

$$\begin{aligned}
&\mu\{(s(t),t)\}\\
&= -\frac{ds(t)}{dt}(R(s(t)+,t) - R(s(t)-,t))\\
&\quad + \int_0^1 (u(s(t)-,t) + \alpha(u(s(t)+,t) - u(s(t)-,t))d\alpha(R(s(t)+,t) - R(s(t)-,t))\\
&= [-\frac{ds(t)}{dt} + \frac{(u(s(t)+,t)) + (u(s(t)-,t))}{2}](R(s(t)+,t) - R(s(t)-,t))\\
&= 0.
\end{aligned}$$

This proves (2.15). Since $R_{tx} = R_{xt}$, in the sense of distributions, and $\rho = R_x$, differentiating (2.15) with respect to $x$ gives

$$\rho_t + (\overline{u}\rho)_x = 0$$

in the sense of distributions and (2.14) follows.

To show that the solution satisfies the initial conditions, first we observe by Lax's [18] argument, $\lim_{t\to 0} u(x,t) = u_0(x)$, a.e. $x$. Now since $y(x,t) - x = -tu(x,t)$, it follows that $y(x,t) \to x$ as $t \to 0$ a.e. $x$. So we get $\int_0^{y(x,t)} v_0(z)dz \to \int_0^x v_0(z)dz$ as $t \to 0$ for a.e $x$. $u$ satisfies the entropy condition $u(x-0,y) \geq u(x+0,t)$ follows from the increasing nature of $y^+(x,t)$ and $y^-(x,t)$ as a function of $x$, for each $t > 0$ and the formula (2.10) for $u$. The proof of the theorem is complete. $\square$

It is instructive to deduce the formula for $(u,\rho)$ obtained in [12] for special initial data namely the Riemann initial data from (2.10), (2.11) and (2.12). LeFloch [19] noted that there are infinite number of solutions for the Riemann problem, here we select the one given by the vanishing viscosity solution.

### 2.3. **Formula for some special initial data.**

First we take Riemann type initial data, namely

$$u_0(x) = \begin{cases} u_l, & \text{if } x < 0, \\ u_r, & \text{if } x > 0, \end{cases} \qquad \rho_0(x) = \begin{cases} \rho_l, & \text{if } x < 0, \\ \rho_r, & \text{if } x > 0 \end{cases} \tag{2.16}$$

We have the following formula for the vanishing viscosity limit.

**Theorem 2.3.** *Let $u^\epsilon$ and $\rho^\epsilon$ are solutions given by (2.7) with initial data of Riemann type (2.14) and $u(x,t) = \lim_{\epsilon\to 0} u^\epsilon(x,t)$ and $\rho(x,t) = \lim_{\epsilon\to 0} u^\epsilon(x,t)$, the $(u,\rho)$ have the following form.*

***Case 1*** $u_l = u_r = u_0$**:**

$$u(x,t) = u_0$$

$$\rho(x,t) = \begin{cases} \rho_l, & \textit{if } x < u_0 t, \\ \rho_r, & \textit{if } x > u_0 t \end{cases}$$

***Case 2*** $u_l < ur$**:**

$$u(x,t) = \begin{cases} u_l, & \textit{if } x < u_l t, \\ x/t, & \textit{if } u_l t < x < u_r t \\ u_r, & \textit{if } x > u_r t \end{cases}$$

*and*

$$\rho(x,t) = \begin{cases} \rho_l, & \textit{if } x < u_l t, \\ 0, & \textit{if } u_l t < x < u_r t \\ \rho_r, & \textit{if } x > u_r t \end{cases}$$

***Case 3*** $u_r < u_l$**:**

$$u(x,t) = \begin{cases} u_l, & \textit{if} x < st, \\ (1/2)(u_l + u_r), \textit{if } x = st \\ u_r, \textit{if } x > st \end{cases}$$

$$d\rho(x,t) = \begin{cases} \rho_l dx, & \textit{if } x < st, \\ (1/2)(u_l - u_r)(\rho_l + \rho_r)t\delta_{x=st}, & \textit{if } x = st \\ \rho_r dx, & \textit{if } x > st \end{cases}$$

*where* $s = \frac{u_l + u_r}{2}$

*Proof.* By the previous theorem it is sufficient to compute $y(x,t)$ for the minimizer in (2.9). An easy computation shows that $y(x,t)$ takes the following form for each of the cases: For case 1, $u_l = u_r = u_0$, $y(x,t) = x - tu_0$. For case 2, $u_l < u_r$,

$$y(x,t) = \begin{cases} x - tu_l, & \text{if } x < u_l t, \\ 0, & \text{if } u_l t < x < u_r t \\ x - tu_r, & \text{if } x > u_r t \end{cases}$$

and for case $u_r < u_l$,

$$y(x,t) = \begin{cases} x - tu_l, & \text{if } x < st, \\ x - tu_r, & \text{if } x > st \end{cases}$$

Substituting these values of $y(x,t)$, in (2.10) - (2.12), the formula for $(u,\rho)$ follows.

An independent proof that $(u,\rho)$ satisfies the equation is instructive and is taken from Joseph [12]. The first equation is standard, we only deal with the second equation of (2.1). An easy calculation shows that that

$$\begin{aligned}
(\rho,\phi_t) + (u\rho,\phi_x) &= \int_0^\infty (u_l - u_r)(\frac{\rho_l+\rho_r}{2} t\phi_t(st,t)dt \\
&\quad + \int_0^\infty (\frac{u_l+u_r}{2})(u_l - u_r)(\frac{\rho_l+\rho_r}{2})t\phi_x(st,t)dt \\
&\quad + \int_{x<st} (\rho_r\phi_t + \rho_r u_r \phi_x)dxdt + \int_{x>st} (\rho_r\phi_t + \rho_r u_r\phi_x)dxdt \\
&= \int_0^\infty \{R(u_l,u_r,\rho_l,\rho_r)\}\phi(st,t)dt
\end{aligned}$$

where

$$R(u_l,u_r,\rho_l,\rho_r) = (u_l - u_r)(\frac{\rho_l+\rho_r}{2}) - \frac{(\rho_r-\rho_l)(u_l+u_r) - 2\rho_r u_r + 2\rho_l u_l}{2}$$

Here we used

$$\begin{gathered}
\int_{x<st} (\rho_l\phi_t + \rho_l u_l \phi_x)dxdt = -\int_0^\infty (s\rho_l - \rho_l u_l)\phi(st,t)dt, \\
\int_{x>st} (\rho_r\phi_t + \rho_r u_r\phi_x)dxdt = \int_0^\infty (s\rho_r - \rho_r u_r)\phi(st,t)dt, \\
\frac{\phi(st,t)}{dt} = s\phi_x(st,t) + \phi_t(st,t), \\
\int_0^\infty t\frac{d\phi(st,t)}{dt}dt = -\int_0^\infty \phi(st,t)dt\,.
\end{gathered}$$

It is easy to see $R(u_l,u_r,\rho_l,\rho_r) = 0$. □

**An example with finite mass.** Another interesting case is initial data of finite mass and of the form

$$u_0(x) = \begin{cases} 0, & \text{if } -\infty < x < 0 \text{ or } 1 < x < \infty, \\ 1, & \text{if } 0 < x < 1 \end{cases}$$

$$\rho_0(x) = \begin{cases} 0, & \text{if } -\infty < x < 0 \text{ or } 1 < x < \infty, \\ \rho_c, & \text{if } 0 < x < 1 \end{cases}$$

The explicit solution is of the following form, for $0 \le t \le 2$,

$$u(x,t) = \begin{cases} 0, & \text{if } x < 0, \\ x/t, & \text{if } 0 \le x \le t \\ 1, & \text{if } t \le x \le t/2+1 \\ 0, & \text{if } x \ge t/2+1 \end{cases}$$

$$d\rho(x,t) = \begin{cases} 0, & \text{if } x < t, \\ \rho_c, & \text{if } t \le x \le t/2+1 \\ \rho_c/2t\delta_{x=t/2+1}, & \text{if } x = t/2+1 \\ 0, & \text{if } x > t/2+1 \end{cases}$$

and for $t > 2$:

$$u(x,t) = \begin{cases} 0, & \text{if } x < 0, \\ x/t, & \text{if } 0 \le x \le (2t)^{1/2} \\ 0, & \text{if } x \ge (2t)^{1/2} \end{cases}$$

$$d\rho(x,t) = \rho_c\delta_{x=(2t)^{1/2}}$$

Here we remark that the vanishing viscosity limit (2.12) for $\rho$ is an extension of the formula (1.5), or equivalently $\rho(x,t) = \rho_0(y(x,t))\det(\frac{\partial y(x,t)}{\partial x})$. In (1.5) solution exists only short time where as here the solution is global, $y(x,t)$ is defined through a minimization problem (2.9) and the product has to be interpreted suitably as a measure. Both are same up to the time smooth solution exists.

2.4. **Generalized solutions in the sense of Colombeau.** We have seen that when initial data $(u_0, \rho_0)$ in the space of bounded measurable functions which is locally BV $u$ contains a classical shock but density is not a function; it contain a $\delta$ measure concentrated along the shock. As we know $D'$ the space of distributions is not an algebra there is a problem in the product $\rho u$ with appear in the second equation. We defined the product as a Radon measure proposed by Volpert [23] and used some ideas of LeFloch [19] to show it is the solution of the problem (2.1) and (2.2) when the initial data is in the space of bounded measurable functions which is locally a BV function.

In this section we consider a larger class of function as initial data and adopt the approach of Colombeau [2, 4, 5, 6], and construct solution for of (2.1) and (2.2) with equality replaced by association in the sense of Colombeau. This approach takes into account not only the final limit but the microscopic structure of the shock due to the viscous effects in the solutions.

First we describe the algebra of generalized functions of Colombeau in $\Omega = \{(x,t), x \in R^1, t > 0\}$, denoted by $\mathcal{G}(\Omega)$. Let $C^\infty(\Omega)$, the class of infinitely differentiable functions in $\Omega$ and consider the infinite product $\mathcal{E}(\Omega) = [C^\infty(\Omega)]^{(0,1)}$. Thus any element $v$ of $\mathcal{E}(\Omega)$ is a map from $(0,1)$ to $C^\infty(\Omega)$ and is denoted by $v = (v^\epsilon)_{0<\epsilon<1}$. An element $v = (v^\epsilon)_{0<\epsilon<1}$ is called moderate if given a compact subset K of $\Omega$ and $j$ and $\ell$ non negative integers,there exists $N > 0$ such that

$$\|\partial_t^j \partial_x^\ell v^\epsilon\|_{L^\infty(K)} = \mathcal{O}(\epsilon^{-N}), \tag{2.17}$$

as $\epsilon$ tends to 0. An element $v = (v^\epsilon)_{0<\epsilon<1}$ is called null if for all compact subsets K of $\Omega$ and for all nonnegative integers $j$ and $\ell$ and for all $M > 0$

$$\|\partial_t^j \partial_x^\ell v^\epsilon\|_{L^\infty(K)} = \mathcal{O}(\epsilon^{M}), \tag{2.18}$$

as $\epsilon$ approaches 0. The set of all moderate elements is denoted by $\mathcal{E}_M(\Omega)$ and the set of null elements is denoted by $\mathcal{N}(\Omega)$. It is easy to see that $\mathcal{E}_M(\Omega)$ is an algebra with partial derivatives, the operations being defined point wise on representatives and $\mathcal{N}(\Omega)$ is an ideal which is closed under differentiation. The quotient space denoted by

$$\mathcal{G}(\Omega) = \frac{\mathcal{E}_M(\Omega)}{\mathcal{N}(\Omega)} \tag{2.19}$$

is an algebra with partial derivatives, the operations being defined on representatives. The algebra $\mathcal{G}(\Omega)$ is called the algebra of generalized functions of Colombeau.

Two elements $u$ and $v$ in $\mathcal{G}(\Omega)$ are said to be associated, if for some (and hence all) representatives $(u^\epsilon)_{0<\epsilon<1}$ and $(v^\epsilon)_{0<\epsilon<1}$, of $u$ and $v$ , $u_\epsilon - v_\epsilon$ goes to 0 as $\epsilon$ tends to 0, in the sense of distribution and is denoted by "$u \approx v$". Here we remark that this notion is different from the notion of equality in $\mathcal{G}(\Omega)$, which means that $u - v \in \mathcal{N}(\Omega)$, or in other words,

$$\|\partial_t^j \partial_x^\ell (u^\epsilon - v^\epsilon)\|_{L^\infty(K)} = \mathcal{O}(\epsilon^M)$$

for all M, for all compact subsets K of $\Omega$ for all $j$, and $\ell$ nonnegative integers.
We refer to the works [2, 4, 5, 6, 15] and the references therein that use the Colombeau algebra to find global solutions of initial value problems when nonconservative product appears in the equation.

Now, roughly speaking, we show that $(u, \rho) = (u^\epsilon(x,t))_{0<\epsilon<1}, \rho = (\rho^\epsilon(x,t))_{0<\epsilon<1}$, with $u^\epsilon$ and $\rho^\epsilon$ given by (2.7) satisfies the equation (2.1), in the quarter plane $\{(x,t) : x \in R^1, t > 0\}$ in the sense of association:

$$\begin{gathered} u_t + (\frac{u^2}{2})_x \approx 0 \\ v_t + (u\rho)_x \approx 0. \end{gathered} \tag{2.20}$$

More precisely, consider $(u, v)$ where $u = (u^\epsilon(x,t))_{0<\epsilon<1}$, and $\rho = (\rho^\epsilon(x,t))_{0<\epsilon<1}$, with $u^\epsilon$ and $\rho^\epsilon$ are solutions of equation (2.3)

$$\begin{gathered} u_t^\epsilon + (\frac{u^{\epsilon 2}}{2})_x = \frac{\epsilon}{2} u_{xx}^\epsilon, \\ \rho_t^\epsilon + (u^\epsilon \rho^\epsilon)_x = \frac{\epsilon}{2} \rho_{xx}^\epsilon, \end{gathered} \tag{2.21}$$

in $\{(x,t) : x \in R^1, t > 0\}$, supplemented with an initial condition at $t = 0$

$$u^\epsilon(x,0) = u_0^\epsilon(x), \quad \rho^\epsilon(x,0) = \rho_0^\epsilon(x), \tag{2.22}$$

where $u_0 = (u_0^\epsilon(x))_{0<\epsilon<1}$, $\rho_0 = (v_0^\epsilon(x))_{0<\epsilon<1}$, are in $\mathcal{G}(R^1)$, the algebra of generalized functions of Colombeau. We assume that $u_0^\epsilon$, and $v_0^\epsilon$ are bounded $C^\infty$ functions of $x$ with the following estimates, for $j = 0, 1, 2, \dots$,

$$\|\partial_x{}^j u_0^\epsilon\|_{L^\infty([0,\infty))} = O(\epsilon^{-j}) \|\partial_x{}^j \rho_0^\epsilon\|_{L^\infty([0,\infty))} = O(\epsilon^{-j}) \tag{2.23}$$

and $u_0^\epsilon(x) \to u_0(x)$, $\rho_0^\epsilon(x) \to \rho_0(x)$ point wise a.e. These conditions are satisfied for example if we take bounded measurable functions on $R^1$, and then take convolution with the Friedrichs mollifiers with scale $\epsilon$. We shall prove the following result.

**Theorem 2.4.** *Assume that* $u_0 = (u_0^\epsilon(x)_{0<\epsilon<1}$, $\rho_0 = (\rho_0^\epsilon(x)_{0<\epsilon<1}$, *are in* $\mathcal{G}(R^1)$, *with the estimates* (2.23) *and as described before. Let* $(u^\epsilon, \rho^\epsilon)$ *be given by the formula* (2.7) *with* $(u_0(x), \rho_0(x))$ *replaced by* $(u_0^\epsilon(x), \rho_0^\epsilon(x))$, *for* $\epsilon > 0$, *then* $u = (u^\epsilon)_{0<\epsilon<1}$ *and* $\rho = (\rho^\epsilon)_{0<\epsilon<1}$ *are in* $\mathcal{G}(\Omega)$ *and* $(u, v)$ *is a solution to* (2.20) *with initial condition* $(u_0, \rho_0) = (u_0^\epsilon(x), \rho_0^\epsilon(x)_{0<\epsilon<1}$

*Proof.* To show that $u$ and $\rho$ are in $\mathcal{G}(\Omega)$ we need to prove that $u^\epsilon$ and $\rho^\epsilon$ satisfies the estimate (2.17). From the formulas (2.7) and and the estimate (2.23), it is easy to see the estimates

$$\|u^\epsilon\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(R^N)}, \quad \|\rho^\epsilon\|_{L^\infty(\Omega)} \le \|\rho_0\|_{L^\infty(R^N)}. \tag{2.24}$$

An application of the Leibinitz's rule and the estimate (2.23), gives us

$$\|\partial_x^k u^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-2k}), \quad \|\partial_x^\alpha \rho^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-2k}). \tag{2.25}$$

Since $(u^\epsilon, \rho^\epsilon)$ is a solution of (2.21), using the estimate (2.25), we get

$$\|\partial_t u^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-2}), \quad \|\partial_t \rho^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-2}). \tag{2.26}$$

Now applying the differential operator ${\partial_t}^j {\partial_x}^k$ on both sides of (2.21), first $k = 1, j = 0, 1, 2, \dots$ and then $k = 2$, $j = 0, 1, 2, \dots$ successively and using (2.25) and (2.26) we obtain

$$\begin{aligned} \|\partial_t^j \partial_x^k u^\epsilon\|_{L^\infty(\Omega)} &= \mathcal{O}(\epsilon^{-2(j+k)}) \\ \|\partial_t^j \partial_x^k \rho^\epsilon\|_{L^\infty(\Omega)} &= \mathcal{O}(\epsilon^{-2(j+k)}) \end{aligned}$$

These estimates show that $u$ and $v$ are in $\mathcal{G}(\Omega)$.

Now to show that $u$ and $v$ satisfy (2.20) in the sense of association. We multiply (2.21) by a test function $\phi \in C_0^\infty(\Omega)$ and integrate we get

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty (u_t^\epsilon + (1/2)(u^{\epsilon 2})_x \phi \, dx \, dt &= \frac{\epsilon}{2} \int_0^\infty \int_{-\infty}^\infty u^\epsilon \phi_{xx} \, dx \, dt, \\ \int_0^\infty \int_{-\infty}^\infty (\rho_t^\epsilon + (u^\epsilon \rho^\epsilon)_x \phi \, dx \, dt &= -\frac{\epsilon}{2} \int_0^\infty \int_{-\infty}^\infty \frac{b^\epsilon}{a^\epsilon} \phi_{xxx} \, dx \, dt. \end{aligned}$$

Now we have to show that the right hand side approaches zero as $\epsilon$ approaches zero. This easily follows by an application of dominated convergence theorem as $u^\epsilon(x,t)$ and $\epsilon b^\epsilon / a^\epsilon$ are bounded and converge point wise almost every where. This completes the proof of the theorem. □

## 3. Boundary value problem

In this section we consider the system

$$\begin{aligned} u_t + (u^2/2)_x &= 0, \\ \rho_t + (\rho u)_x &= 0 \end{aligned} \tag{3.1}$$

in the quarter plane $\{(x,t) : x > 0, t > 0\}$, supplemented with an initial condition at $t = 0$

$$\begin{pmatrix} u(x,0) \\ \rho(x,0) \end{pmatrix} = \begin{pmatrix} u_0(x) \\ \rho_0(x) \end{pmatrix} \tag{3.2}$$

and a weak form of the Dirichlet boundary condition,

$$\begin{pmatrix} u(0,t) \\ \int_0^\infty \rho(y,t) dy \end{pmatrix} = \begin{pmatrix} u_b(t) \\ R_b(t) \end{pmatrix} \tag{3.3}$$

where $u_0(x)$ and $\rho_0(x)$ are integrable functions of $x$ and $u_b(t)$ and $R_b(t)$ are Lipschitz continuous functions of $t$. Indeed with strong form of Dirichlet boundary conditions (3.3), there is neither existence nor uniqueness as the speed of propagation $\lambda = u$ and does not have a definite sign at the boundary $x = 0$. We note that the speed is completely determined by the first equation. We use the Bardos Leroux and

Nedelec [1] formulation of the boundary condition for the $u$ component which for our case is equivalent to the following condition :

$$u(0+,t) \in E(u_B(t)) \tag{3.4}$$

where the admissible set $E(u_B(t))$ is defined by

$$E(u_B(t)) = \begin{cases} (-\infty, 0], & \text{if } u_B(t) \leq 0, \\ (-\infty, -u_B(t)) \cup \{u_B(t)\}, & \text{if } u_B(t) > 0 \end{cases}$$

Here $u_b^+(t) = \max u_b(t), 0$. There are explicit representations of the entropy weak solution of of the first component $u$ of (1.1) with initial condition $u(x,0) = u_0(x)$ and the boundary condition (3.4) by Joseph [11] and by Joseph and Gowda [13]. We use the formula in [13] for $u$ which involve a minimization of functionals on certain class of paths and generalized characteristics. Once $u$ is obtained, the equation for $v$ is linear equation with a discontinuous coefficient $u(x,t)$. Now $\int_0^\infty \rho(0+,t)dt = R_b(t)$ is prescribed only if the characteristics at $(0,t)$ has positive speed, i.e., $u(0+,t) > 0$. So the weak form of boundary conditions for $\rho$ component is

$$\text{if } u(0+,t) > 0, \text{ then } \int_0^\infty \rho(y,t)dy = R_b(t). \tag{3.5}$$

To state the results we introduce some notation.

For each fixed $(x,y,t)$, $x > 0$, $y \geq 0$, $t > 0$, $C(x,y,t)$ denotes the following class of paths $\beta$ in the quarter plane $D = \{(z,s) : z \geq 0, s \geq 0\}$. Each path is connected from the initial point $(y,0)$ to $(x,t)$ and is of the form $z = \beta(s)$, where $\beta$ is a piecewise linear function of maximum three lines and always linear in the interior of $D$. Thus for $x > 0$ and $y > 0$, the curves are either a straight line or have exactly three straight lines with one lying on the boundary $x = 0$. For $y = 0$ the curves are made up of one straight line or two straight lines with one piece lying on the boundary $x = 0$. Associated with the data $u_b(t)$, we define the functional $J(b)$ on $C(x,y,t)$

$$J(\beta) = -\int_{\{s:\beta(s)=0\}} \frac{(u_B(s)^+)^2}{2} ds + \int_{\{s:\beta(s)\neq 0\}} \frac{(\frac{d\beta(s)}{ds})^2}{2} ds. \tag{3.6}$$

We call $\beta_0$ is straight line path connecting $(y,0)$ and $(x,t)$ which does not touch the boundary $x = 0$, $\{(0,t), t > 0\}$, then let

$$A(x,y,t) = J(\beta_0) = \frac{(x-y)^2}{2t}. \tag{3.7}$$

For any $\beta \in C^*(x,y,t) = C(x,y,t) - \beta_0$, that is made up of three straight lines connecting $(y,0)$ to $(0,t_1)$ in the interior and $(0,t_1)$ to $(0,t_2)$ on the boundary and $(0,t_2)$ to $(x,t)$ in the interior, it can be easily seen from (3.6) that

$$J(\beta) = J(x,y,t,t_1,t_2) = -\int_{t_1}^{t_2} \frac{(u_B(s)^+)^2}{2} ds + \frac{y^2}{2t_1} + \frac{x^2}{2(t-t_2)^2}. \tag{3.8}$$

For the curves made up two straight lines with one piece lying on the boundary $x = 0$ which connects $(0,0)$ and $(0,t_2)$ and the other connecting $(0,t_2)$ to $(x,t)$.

$$J(\beta) = J(x,y,t,t_1 = 0,t_2) = -\int_0^{t_2} \frac{(u_B(s)^+)^2}{2} ds + \frac{x^2}{2(t-t_2)^2}.$$

It was proved in [11, 13], that there exists a $\beta^* \in C^*(x,y,t)$ or correspondingly $t_1(x,y,t)$, $t_2(x,y,t)$ so that

$$\begin{aligned} B(x,y,t) &= J(\beta^*) \\ &= \min\{J(\beta) : \beta \in C^*(x,y,t)\} \\ &= \min\{J(x,y,t,t_1,t_2) :\ 0 \le t_1 < t_2 < t\} \\ &= J(x,y,t,t_1(x,y,t),t_2(x,y,t)) \end{aligned} \tag{3.9}$$

is a Lipschitz continuous so that

$$Q(x,y,t) = \min\{J(\beta) : \beta \in C(x,y,t)\} = \min\{A(x,y,t), B(x,y,t)\}, \tag{3.10}$$

and

$$U(x,t) = \min\{Q(x,y,t) + U_0(z),\ 0 \le y < \infty\} \tag{3.11}$$

are Lipschitz continuous functions in their variables, where $U_0(y) = \int_0^y u_0(z)dz$. Further minimum in (3.11) is attained at some value 0f $y \ge 0$ which depends on $(x,t)$, we call it $y(x,t)$. If $A(x,y(x,t),t) \le B(x,y(x,t),t)$

$$U(x,t) = \frac{(x-y(x,t))^2}{2t}) + U_0(y), \tag{3.12}$$

and if $A(x,y(x,t),t) > B(x,y(x,t),t)$,

$$U(x,t) = J(x,y(x,t),t,t_1(x,y(x,t),t),t_2(x,y(x,t),t)) + U_0(y). \tag{3.13}$$

Here and hence forth $y(x,t)$ is a minimizer in (3.11) and in the case of (3.13), $t_2(x,t) = t_2(x,y(x,t),t)$ and $t_1(x,t) = t_1(x,y(x,t),t)$.

**Theorem 3.1.** *For every $(x,t)$ minimum in (3.11) is achieved by some $y(x,t)$, and $U(x,t)$ is a Lipschitz continuous and for almost every $(x,t)$ there is only one minimizer $y(x,t)$.*

*For every point $(x,t)$ satisfying $U(x,t) = A(x,y(x,t),t) \le B(x,y(x,t),t)$, define*

$$\begin{aligned} u(x,t) &= \frac{x-y(x,t)}{t} \\ \rho(x,t) &= \partial_x(R_0(y(x,t)). \end{aligned}$$

*and for the points $(x,t)$ where $B(x,y(x,t),t) < A(x,y(x,t),t)$,define*

$$\begin{aligned} u(x,t) &= \frac{x}{t-t_2(x,t)} \\ \rho(x,t) &= \partial_x(R_b(t_2(x,t)). \end{aligned}$$

*Then the function $(u(x,t),\rho(x,t))$ is a weak solution of (3.1), satisfying the initial condition (3.2) and boundary conditions (3.4) and (3.5)*

The proof this theorem follows immediately from that of [17, Theorem 2.1] and is omitted.

## 4. Concluding remarks

The system of partial differential equation for $u_j, j = 1,2,\dots,n$

$$(u_j)_t + (1/2)\sum_{i=1}^{j}(u_i u_{j-i+1})_x = \frac{\epsilon}{2}(u_j)_{xx}, \tag{4.1}$$

is an extension of the adhesion model (2.3). Joseph and Vasudeva Murthy [14] showed that with initial conditions,

$$u_j(x,0) = u_{0j}(x), j = 1, 2, \dots, n, \tag{4.2}$$

Equation (4.1) can be linearized using a generalized Hopf-Cole transformation and the solution can be explicitly written down for $\epsilon > 0$. The explicit solution shows that the order of singularities of solutions of (4.1) increases as $n$ increases in the passage to the limit $\epsilon \to 0$.

When $n = 1$, and $u = u_1$, (4.1) becomes the Burgers equation (1.1) and was studied by Hopf [10] as $\epsilon \to 0$. The limit is bounded $BV_{\text{loc}}$ functions if initial data is bounded and is a solution to the inviscid Burgers equation which is an example of conservation laws with genuinely nonlinear characteristic fields. Here we have classical shocks.

When $n = 2$ and $u = u_1$, $\rho = u_2$, (4.1) is the modified adhesion model (2.3), where $u$ remains bounded but $\rho$ is not bounded functions but a Radon measure, in the limit $\epsilon \to 0$, when the initial data are bounded measurable functions which are locally BV.

When $n = 3$ and $u = u_1$, $\rho = u_2$, $w = u_3$, in (4.1) the equation becomes

$$\begin{aligned} u_t + (u^2/2)_x &= \frac{\epsilon}{2} u_{xx}, \\ \rho_t + (\rho u)_x &= \frac{\epsilon}{2} \rho_{xx} \\ w_t + (\rho^2 + uw)_x &= \frac{\epsilon}{2} w_{xx}. \end{aligned}$$

Joseph [15] observed that the nature of singularity is worse than $\delta$ -measures and constructed solution in the algebra of generalized functions of Colombeau. Shelkovich [21] used the explicit solution of [15] to derive a formula for the solution for the inviscid case, in the limit $\epsilon \to 0$ and showed that the solution contains $\delta$ and its derivative $\delta'$ when the initial data is of Riemann type.

The exact nature of singularities of solutions of (4.1), as $\epsilon$ goes to zero is still not known for the cases $n \geq 4$, even with Riemann type initial data.

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Kayyunnapara Thomas Joseph
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
*E-mail address*: `ktj@math.tifr.res.in`