# Classical and quantum mechanics of a particle on a rotating loop

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## Abstract

The toy model of a particle on a vertical rotating circle in the presence of uniform gravitational/ magnetic fields is explored in detail. After an analysis of the classical mechanics of the problem we then discuss the quantum mechanics from both exact and semi–classical standpoints. Exact solutions of the Schrödinger equation are obtained in some cases by diverse methods. Instantons, bounces are constructed and semi-classical, leading order tunneling amplitudes/decay rates are written down. We also investigate qualitatively the nature of small oscillations about the kink/bounce solutions. Finally, the connections of these toy examples with field theoretic and statistical mechanical models of relevance are pointed out.

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## I. INTRODUCTION

Toy models in physics play an important role in understanding the basic features of more involved theories and phenomena. In particular, models in one dimensional quantum mechanics, illustrating analogous situations in a field theoretic context have been quite useful for advanced researchers as well as beginning graduate students. Among many such models, those which illustrate non–perturbative aspects of field theory through an analysis of instanton solutions and bounces have been looked at in diverse contexts. The $x^2 - x^4$ potential and the quantum pendulum have been discussed and analyzed in great detail in the past. Apart from aspects which emerge out of solutions and their analysis, it is also important to relate the toy model with different realistic models by mapping rules/dimensional extensions etc. We shall, in this article, try to analyze a novel toy model which has applications in different situations and is also exactly solvable to some extent. In particular, we discuss the problem of *Particle on a Rotating Circle* (henceforth referred to as PORC) in the presence of gravitational/ magnetic field (a charged particle in the case of magnetic field).

Before we indulge into analyzing the salient features of this model, let us first explore the existing literature on it. For a undergraduate/graduate student, a first encounter with this model is likely to occur while doing a course in classical mechanics. The model appears as a problem in the second chapter of Goldstein's book on classical mechanics [1]. It is also elaborately discussed in Arnold [1]. There exists a host of other articles on it, in this journal too [2], the most recent one being published a couple of years ago [3]. The major directions along which this model has been viewed are (i) as an example of spontaneous symmetry breaking (ii) as a mechanical model for second order phase transitions (iii) as an example in quantum mechanics where instanton solutions are obtainable [4](iv) as a lower dimensional analog of certain higher dimensional field theories.

In this article, we shall see that this model (PORC) has the following intriguing features :

- The classical mechanics of the model is exactly solvable in a way similar to that of the pendulum problem

- When extended to full real line (instead of circle $S^1$) the model has intimate connections with several field theoretic and statistical mechanical models (for details see Section III).

- The model in the presence of a uniform gravitational or magnetic field has non-perturbative solutions such as instantons/bounces and the corresponding tunneling/decay rates can be written down and analyzed (for a discussion on the theory of instantons, bounces and their contribution to tunneling/decay rates can be found in [5], [6], [7] ).

- A formal analysis of the Schrödinger equation can be performed and exact solutions obtained. However the full expressions for the eigenvalues and eigenfunctions are encoded in certain continued fractions which we do not discuss in further detail.

- In the presence of a uniform magnetic field the quantum problem is quasi-exactly solvable–we can get at least a few energy eigenstates and eigenvalues analytically [8] [9] by extending the method due to Razavy [10].

## II. THE MODEL

Let us first introduce the model in some detail. We have a point particle of mass $m$ (and charge $q$ if we are looking at the problem in the presence of a magnetic field) constrained to move on a circle ($S^1$). The circle is located vertically and is rotating at an angular frequency $\omega$ about the vertical axis of symmetry. Two special cases will be analyzed – (i) the system in the presence of a uniform gravitational field (ii) the system in the presence of a uniform magnetic field. Specifications about the directions and magnitudes of each of these uniform fields are given below. Note that this problem is a generalisation ($\omega \neq 0$) of the usual pendulum problem ($\omega = 0$) whose quantum mechanics is discussed in [11]

A generic Lagrangian for the such a system can be given as :

$$L_{generic} = \frac{1}{2}mr^2\dot{\theta}^2 - A\cos\theta - B\cos 2\theta\,, \qquad (1)$$

where A and B are constants depending on the various parameters in the model.

We can in principle have four cases depending upon the signs of A and B. These are :

(i) $A > 0, B > 0$ (ii) $A > 0, B < 0$ (iii) $A < 0, B > 0$ (iv) $A < 0, B < 0$

Note that with a proper redefinition of $\theta$ (more precisely $\theta \to \pi - \theta$) we can relate the models (i) and (iii) as well as (ii) and (iv). Therefore, in essence we have only two models to discuss. The two special cases given below exemplify these two cases.

**Case 1 : System in a uniform gravitational field**

$$L_{grav} = \frac{1}{2}mr^2\dot{\theta}^2 - \left(mgr\cos\theta - \frac{1}{2}mr^2\omega^2\sin^2\theta\right), \qquad (2)$$

where $\theta$ is the generalized coordinate required to describe the system.

Thus the effective one–dimensional potential is :

$$V(\theta) = mgr\cos\theta + \frac{1}{4}mr^2\omega^2\cos 2\theta - \frac{1}{4}mr^2\omega^2\,. \qquad (3)$$

Note that this system falls in the $A > 0, B > 0$ class mentioned above. The location of the extrema of the effective potential are as follows :

**Minima :**

(i) $\omega > \omega_0\,:\,\theta = \cos^{-1}(-a)\,, 2\pi - \cos^{-1}(-a)\,.$

$V(\theta_{min}) = V(2\pi - \theta_{min}) = -mr^2\omega_0^2(a^2+1)/a\,.$

(ii) $\omega < \omega_0\,:\,\theta = \pi \quad ; \quad V(\theta_{min}) = -mr^2\omega_0^2\,.$

**Maxima :**

(i) $\omega > \omega_0\,:\,\theta = 0(2\pi),\,\theta = \pi \quad ; \quad V(0) = V(2\pi) = mr^2\omega_0^2\,;\; V(\pi) = -mr^2\omega_0^2\,.$

(ii) $\omega < \omega_0\,:\,\theta = 0(2\pi)\,.$

Thus for $\omega > \omega_0$ one has degenerate minima and maxima as well as a local maxima while for $\omega < \omega_0$ there is no local maxima. The potentials are shown in Fig. 1(a) and 1(b) for the $\omega > \omega_0$ and $\omega < \omega_0$ cases respectively. Here $\omega_0 = \sqrt{g/r}$ while $a = \omega_0^2/\omega^2$.

**Case 2 : System in a uniform magnetic field**

$$L_{mag} = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\omega^2\sin^2\theta + q\mathbf{A}.\mathbf{v}\,, \tag{4}$$

where $q$ denotes the charge of the particle.

Since $\mathbf{B}$ is constant we have $\mathbf{A} = -\frac{1}{2}\mathbf{r}\times\mathbf{B}$. Assuming that the components of $\mathbf{B}$ are $(B_r, B_\theta, 0)$ we find that the vector potential $\mathbf{A}$ has the components $(0,0,A_\phi)$ where $A_\phi = -\frac{1}{2}rB_\theta$. Since $\mathbf{v}$ has components $(0, r\dot{\theta}, \omega r\sin\theta)$ we find that the Lagrangian takes the form :

$$L_{mag} = \frac{1}{2}mr^2\dot{\theta}^2 - \left(\frac{qBr^2\omega}{2}\sin\theta - \frac{1}{2}mr^2\omega^2\sin^2\theta\right)\,. \tag{5}$$

Thus the effective one dimensional potential is

$$V(\theta) = \frac{1}{2}mr^2\omega^2\left(\frac{\omega_c}{\omega}\sin\theta - \sin^2\theta\right)\,, \tag{6}$$

where $\omega_c = \frac{qB}{m}$. With a straightforward redefinition of $\theta$ ($\theta \to \theta - \frac{\pi}{2}$) we see that this model belongs to the other class ($A > 0, B < 0$). The extrema of the effective potential are as follows :

**Minima :**

(i) $\omega > \omega_c/2 : \theta = \frac{\pi}{2}\,, \frac{3\pi}{2}\,,$

$V(\frac{\pi}{2}) = \frac{1}{2}mr^2\omega^2\left(\frac{\omega_c}{\omega} - 1\right) \quad ; \quad V(\frac{3\pi}{2}) = -\frac{1}{2}mr^2\omega^2\left(\frac{\omega_c}{\omega} + 1\right)$

(ii) $\omega < \omega_c/2 : \theta = \frac{3\pi}{2} \quad ; \quad V(3\pi/2) = -\frac{1}{2}mr^2\omega^2(\frac{\omega_c}{\omega} + 1)$

**Maxima :**

(i) $\omega > \omega_c/2 : \theta = \sin^{-1}\frac{\omega_c}{2\omega},\ \pi - \sin^{-1}\frac{\omega_c}{2\omega} \quad ; \quad V(\theta_{max}) = \frac{1}{8}mr^2\omega_c^2$ .

(ii) $\omega < \omega_c/2 : \theta = \frac{\pi}{2} \quad ; \quad V(\pi/2) = \frac{1}{2}mr^2\omega^2(\frac{\omega_c}{\omega} - 1)$ .

Thus for $\omega > \omega_c/2$ the effective potential represents a system which has a false vacuum at $\theta = \frac{\pi}{2}$ and a true vacuum at $\theta = \frac{3\pi}{2}$ while for $\omega < \omega_c/2$ there is no local minima. The two scenarios are plotted in Fig. 2(a) and 2(b) respectively.

¿From the figures for the effective potentials we can conclude the following – (i) in the presence of a uniform magnetic field we may have a true as well as false vacua while (ii) in a uniform gravitational field we can have a degenerate double well potential. Additionally, if we generalize the above over the full (real) line then we have a periodic potential for which the minima are infinite-fold degenerate.

A more general problem is that of a charged particle in a uniform gravitational plus magnetic (and even electric) field. The most general Lagrangian is given by

$$L_{general} = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\omega^2\sin^2\theta - mgr\cos\theta - \frac{qBr^2\omega}{2}\sin\theta + q\phi_{elec}\,, \tag{7}$$

where $\phi_{elec}$ is the electric potential. In general, this problem cannot be exactly solved in either classical or quantum mechanics even though a general analysis of the motion in classical mechanics is possible [3]. However, by suitably choosing the ratio of the electric and magnetic fields we can reduce the problem to that of a particle moving on a rotating circle in uniform gravitational field alone. Similarly, by suitably choosing the ratio of the gravitational and electric fields, one can reduce it to that of a particle moving on a rotating circle in a uniform magnetic field alone. Other combinations and their consequences can also be tried out. We leave it to the reader to figure out the details of these scenarios.

## III. INTER–CONNECTIONS WITH FIELD THEORETIC AND OTHER MODELS

The toy models discussed above have remarkable connections with a wide–ranging variety of field theoretic models. We shall now briefly summarize some of these interconnections.

To begin with, note that there exists a correspondence between the toy models discussed above and $1+1$ dimensional double– sine–Gordon (DSG) field theory defined by the Lagrangian density :

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\nu\phi + Acos\phi + Bcos2\phi\,, \tag{8}$$

in that the potential is formally the same in both the cases. Here A and B could be positive or negative. The field equation that follows from a first variation is

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = A\sin\phi + 2B\sin 2\phi\,. \tag{9}$$

Hence the equation for static (i.e. $\phi$ independent of time) solutions is

$$\frac{d^2\phi}{dx^2} = -\left(Asin\theta + 2Bsin2\theta\right) \ . \tag{10}$$

It may be noted that this equation also follows from our toy model Lagrangian as given by eq. (1) but for an overall sign. In fact if we go to the Euclidean time i.e. let $t = i\tau$ , then the classical equation of motion in the toy model takes the form

$$\frac{d^2\theta}{d\tau^2} = -(Asin\theta + 2Bsin\theta)\,, \tag{11}$$

which is identical to the static field equation as given by eq. (10). Now it is well known [6] that the finite Euclidean action solutions of the Euclidean equations of motion (11) for any system are nothing but the instantons, whose mere existence is due to the presence of degenerate minima in the potential appearing in the theory. Further, once one has obtained these instanton solutions and the corresponding Euclidean actions, the tunneling amplitude for the transition between the degenerate vacua can easily be computed in the dilute (non-interacting) instanton gas approximation. We thus notice that the static finite energy solutions of the DSG equation as given by (10) are the same as the finite (Euclidean) action instanton solutions of our Toy model (with the obvious replacement of $\phi$ with $\theta$ and $x$ with $\tau$).

Since the DSG equation has been extensively studied in the literature [12–16], we can immediately write down the corresponding instanton solutions of our toy model.

It may be added here that the DSG field equation occurs quite naturally in several physically important situations. For example, it appears quite naturally in the study of spin waves in the B-phase of super-fluid $^3He$ [17,18], in the problem of self-induced transparency [19] and in nonlinear excitations in a compressible chain of XY dipoles under conditions of piezoelectric coupling [20]. Further, a model based on the DSG theory has been proposed [21] to describe the poling process in the B-phase of the polymer polyvinylidene fluoride ($PVF_2$). Additionally, the DSG theory has been investigated as a model of a nonlinear system which can support more than one kind of soliton.

The sine-Gordon equation, in turn, is related to the massive Thirring Model [22] while a specific form of DSG has been shown to be equivalent to a generalized massive Thirring model [23]. Thus, any new results obtained in any of these models will immediately have relevance in various other related models.

Finally, we may add that the DSG equation is also related to the anisotropic Heisenberg chain in an applied magnetic field. In particular, by treating spins classically, Leung [12] as well as Pandit et al. [13] have shown that the classical spin dynamics is approximately governed by the generalized double sine Gordon (DSG) model as given by eq. (7).

## IV. CLASSICAL SOLUTIONS

We now move on towards analyzing the real time solutions i.e. the classical mechanics of the above problem. Defining $E = T + V - V_{min}$ so that $E \geq 0$, we integrate the corresponding quadrature and then analyze the various cases separately.

### 1. System in a gravitational field

For the gravitational case, for the two domains of $\omega$ we have the following results..

$\underline{\omega < \omega_0}$

Here $V_{min} = -mr^2\omega_0^2$. Using $\theta = 2\phi$ and then $\tan\phi = y$ we obtain the following generic form of integral which we need to analyze :

$$\int \frac{dy}{\sqrt{y^4 + 2y^2\left(1 - \alpha - \beta\right) + \left(1 - 2\alpha\right)}} = \sqrt{\frac{E}{2mr^2}}\left(t - t_0\right) \tag{12}$$

where

$$\alpha = \frac{m\omega_0^2 r^2}{E} \quad ; \quad \beta = \frac{m\omega^2 r^2}{E}\,. \tag{13}$$

Recall that $V_{max} = 2mgr = 2mr^2\omega_0^2$. We can therefore have two possibilities $E < V_{max}$ and $E > V_{max}$ which we discuss one by one.

(a) $E < V_{max}$

In this energy range we have the following two alternatives depending on the sign of the quadratic term in the denominator of the elliptic integral. These are :

(i) $0 < E < mr^2\left(\omega_0^2 - \omega^2\right)$

In this case the solution is [24]

$$\tan\frac{\theta}{2} = b\frac{dn(X,k)}{\kappa' sn(X,k)} \tag{14}$$

where $X = \frac{b}{\kappa'}\sqrt{\frac{E}{2mr^2}}(t-t_0)$, $a^2b^2 = 2\alpha - 1$, $b^2 - a^2 = 2(\alpha-\beta-1)$ . Here cn(X,k), sn(X,k) and dn(X,k) are Jacobi elliptic functions of real elliptic modulus parameter $k$.

(ii) $mr^2\left(\omega_0^2 - \omega^2\right) < E < V_{max}$

In this case the solution is

$$\tan\frac{\theta}{2} = \frac{adn(X,k)}{\kappa sn(X,k)} \tag{15}$$

where $X = \frac{a}{\kappa}\sqrt{\frac{E}{2mr^2}}(t-t_0)$, $\frac{a}{a^2+b^2} = \kappa$, $\kappa' = \sqrt{1-\kappa^2}$, $a^2b^2 = 2\alpha - 1$, $a^2 - b^2 = 2(1-\alpha+\beta)$.

(b) $E > V_{max}$

The exact solution here is :

$$\tan\frac{\theta}{2} = acn(X,k)/sn(X,k) \tag{16}$$

where $X = a\sqrt{\frac{E}{2mr^2}}(t-t_0)$, $\frac{\sqrt{a^2-b^2}}{a} = \kappa$, $a^2b^2 = 1 - 2\alpha$, $a^2 + b^2 = 2(1-\alpha+\beta)$. Note also that as $b \longrightarrow 0$, $a^2 = 1 + \frac{\omega^2}{\omega_0^2}$ , which implies $E = 2mr^2\omega_0^2$.

$\underline{\omega > \omega_0}$

In this domain of $\omega$ we note that $V_{min} = -\frac{mr^2}{2\omega_0^2}\left(\omega_0^4 + \omega^4\right)$. As before, using $\theta = 2\phi$, $\tan\phi = y$ we get :

$$\int \frac{dy}{\sqrt{y^4\left(E - V_{max} + 2mr^2\omega_0^2\right) + 2y^2\left(E + mr^2\omega_0^2 + mr^2\omega^2 - V_{max}\right) + \left(E - V_{max}\right)}} = \sqrt{\frac{1}{2mr^2}}\left(t - t_0\right) , \tag{17}$$

where $V_{max} = \frac{mr^2}{2\omega^2}\left(\omega_0^2 + \omega^2\right)^2$. The exact solutions in the various sub-cases are analyzed below.

(a) $E < V_{max}$

In a way similar to the $\omega < \omega_0$ case we once again have two possibilities depending on the signs of the coefficients of the various terms appearing in the elliptic integral.

(i) $0 < E < \frac{mr^2}{2\omega^2}\left(\omega^2 - \omega_0^2\right)^2$

The solution for this case is :

$$\tan\frac{\theta}{2} = \frac{1}{adn(X,k)} \tag{18}$$

where $X = a\sqrt{\frac{V_{max}-E}{2mr^2}}(t-t_0)$, $a^2b^2 = \frac{V_{max}-E-2mr^2\omega_0^2}{V_{max}-E}$, $a^2+b^2 = 2\frac{E-V_{max}+mr^2(\omega^2+\omega_0^2)}{V_{max}-E}$.

(ii) $\frac{mr^2}{2\omega^2}\left(\omega^2 - \omega_0^2\right)^2 < E < V_{max}$

Within these bounds of energy, the solution turns out to be

$$\tan\frac{\theta}{2} = \frac{a}{\kappa}\frac{dn(X,k)}{sn(X,k)} \tag{19}$$

with $X = \frac{a}{\kappa}\sqrt{\frac{E+2mr^2\omega_0^2-V_{max}}{2mr^2}}(t-t_0)$, $a^2b^2 = \frac{V_{max}-E}{E+2mr^2\omega_0^2-V_{max}}$, $a^2-b^2 = \frac{E+mr^2(\omega^2+\omega_0^2)-V_{max}}{E+2mr^2\omega_0^2-V_{max}}$.

(b) $E > V_{max}$

The solution here is given by:

$$\tan\frac{\theta}{2} = a\frac{cn(X,k)}{sn(X,k)} \tag{20}$$

where $X = a\sqrt{\frac{E-V_{max}+2mr^2\omega_0^2}{2mr^2}}(t-t_0)$.

**2. System in a magnetic field :**

In the magnetic case, we follow the same procedure as above. The only major difference is that, unlike the gravitational case, $V_{min} = -\frac{1}{2}mr^2\omega^2\left(1+\frac{\omega_c}{\omega}\right)$ is the same irrespective of whether $\omega > \frac{\omega_c}{2}$ or $\omega < \frac{\omega_c}{2}$. Using $\theta = \phi - \frac{\pi}{2}$, $\frac{\phi}{2} = \eta$ and $\tan\eta = y$ we find that we have to handle the following integral :

$$\int \frac{dy}{\sqrt{\left[(1-\alpha)\,y^4 + (2-\alpha-\beta)\,y^2 + 1\right]}} = (t-t_0)\sqrt{\frac{E}{2mr^2}}\,, \tag{21}$$

where $\alpha = \frac{mr^2\omega\omega_c}{E}, \beta = \frac{2mr^2\omega^2}{E}$.

It is convenient to make another transformation $y = \frac{1}{u}$ and then analyse the resulting integral, which is, generically, of the form

$$-\int \frac{du}{\left[u^4 + u^2\,(2-\alpha-\beta) + (1-\alpha)\right]^{\frac{1}{2}}} = (t-t_0)\sqrt{\frac{E}{2mr^2}}\,. \tag{22}$$

The various cases are now analyzed below.

$\underline{\omega < \frac{\omega_c}{2}}$

(a) $E < V_{max}$

Here, depending on the signs of the various coefficients we will have two cases :

(i) $\frac{mr^2\omega\omega_c}{2}\left(1+\frac{2\omega}{\omega_c}\right) < E < V_{max}$

The solution here is :

$$\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \frac{k}{a}\frac{sn(X,k)}{dn(X,k)} \tag{23}$$

where $\kappa = \frac{a}{\sqrt{a^2+b^2}}$, $a^2b^2 = \alpha - 1$, $a^2 - b^2 = 2 - \alpha - \beta$ and $X = \frac{a}{\kappa}\sqrt{\frac{E}{2mr^2}}(t - t_0)$.

(ii) $0 < E < \frac{mr^2\omega\omega_c}{2}\left(1+\frac{2\omega}{\omega_c}\right)$

The solution here is the same as before except that now $a^2 < b^2$ and therefore one may obtain it from the previous solution by just interchanging $a$ and $b$.

(b) $E > V_{max}$

$$\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \frac{sn(X,k)}{acn(X,k)} \tag{24}$$

where $X = a\sqrt{\frac{E}{2mr^2}}(t - t_0)$, $a^2b^2 = 1 - \alpha$, $a^2 + b^2 = 2 - \alpha - \beta$ .

$\underline{\omega > \frac{\omega_c}{2}}$

(a) $E < V_{max}$

The two cases depending on the signs of the various coefficients are :

(i) $0 < E < mr^2\omega\omega_c$ Here the solution is :

$$\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \frac{k'^2}{b}\frac{sn(X,k)}{dn(X,k)}\,, \tag{25}$$

where $a^2b^2 = \alpha - 1$, $b^2 - a^2 = \alpha + \beta - 2$, $\kappa = \frac{a}{\sqrt{a^2+b^2}}$ and of course $k'^2 + k^2 = 1$.

(ii) $mr^2\omega\omega_c < E < V_{max}$

The solution is

$$\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \frac{1}{a}sn[a\sqrt{\frac{E}{2mr^2}}(t - t_0)]\,. \tag{26}$$

(b) $E > V_{max}$

The two cases are given below :

(i) $V_{max} < E < mr^2\left(\omega^2 + \frac{\omega\omega_c}{2}\right)$

The solution is

$$\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \frac{1}{a} sn(X, k)\,. \tag{27}$$

(ii) $E > mr^2\left(\omega^2 + \frac{\omega\omega_c}{2}\right)$

The solution is

$$\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \frac{1}{a}\frac{sn(X, k)}{cn(X, k)}\,, \tag{28}$$

where in both the above solutions $X = a\sqrt{\frac{E}{2mr^2}}(t - t_0)$.

Before concluding this section, we briefly discuss a couple of special cases in which the solution can be written in terms of the well known trigonometric and hyperbolic functions.

Special solutions in the gravitational case

For $\omega < \omega_0$ if we choose $E = 2mr^2\omega_0^2$ then we obtain the following special solution

$$\tan\frac{\theta}{2} = \pm\frac{1}{a} cosech\left(\sqrt{\omega_0^2 + \omega^2}(t - t_0)\right)\,, \tag{29}$$

where $a^2 = \frac{\omega_0^2}{\omega^2 + \omega_0^2}$. This is known as the 'sticking' solution–i.e. as $t \to \pm\infty$, $\theta \to 0$ (maxima), while for $t \to t_0$, $\theta \to \pi$ (minima).

If $\omega > \omega_0$, the same solution holds (with the same form of $a$) but at $E = V_{max} = \frac{mr^2}{2\omega^2}(\omega^2 + \omega_0^2)^2$. It is also a sticking solution but it goes from local to absolute maxima as $t$ goes from $t_0$ to $\pm\infty$. It is the *value* of $\omega$ which makes the two solutions functionally different.

Special solutions in the magnetic case

Here, for $\omega < \frac{\omega_c}{2}$ and $E = V_{max} = mr^2\omega\omega_c$ we find the following solution :

$$\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \sqrt{\frac{1}{1 - \frac{2\omega}{\omega_c}}}\sinh\left(\sqrt{\frac{\omega\omega_c}{2}\left(1 - \frac{2\omega}{\omega_c}\right)}(t - t_0)\right) \tag{30}$$

This is once again the so–called 'sticking' solution–the particle reaches the maxima ($\pi/2$) as $t \to \pm\infty$ while it is at the minimum ($3\pi/2$) as $t \to t_0$.

Another solution for $\omega < \frac{\omega_c}{2}$ is obtained at $E = \frac{mr^2}{8}\left(\omega_c + 2\omega\right)^2 > V_{max}$. This is given by

$$\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)=\sqrt{\frac{\omega_c+2\omega}{\omega_c-2\omega}}\tan\left(\sqrt{1-\frac{4\omega^2}{\omega_c^2}}\frac{\omega_c}{4}(t-t_0)\right). \qquad (31)$$

This solution oscillates around the minima at $\theta=\frac{3\pi}{2}$.

For $\omega>\frac{\omega_c}{2}$ and $E=mr^2\omega\omega_c<V_{max}$ we have a solution which oscillates around the minimum at $\theta=\frac{3\pi}{2}$. This is given by

$$y=\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)=\sqrt{\frac{\omega_c}{2\omega-\omega_c}}\sin\left(\sqrt{1-\frac{\omega_c}{2\omega}}\omega(t-t_0)\right). \qquad (32)$$

Finally, for $\omega>\frac{\omega_c}{2}$ and $E=\frac{mr^2}{8}(\omega_c+2\omega)^2=V_{max}$ we have the solution

$$\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)=\sqrt{\frac{2\omega+\omega_c}{2\omega-\omega_c}}\tanh\left(\sqrt{1-\frac{\omega_c^2}{4\omega^2}}\omega(t-t_0)\right). \qquad (33)$$

This is again a sticking solution which goes from the absolute minimum to the maximum as $t$ goes from $t_0$ to $\pm\infty$.

## V. QUANTUM MECHANICS OF A PORC IN A CONSTANT GRAVITATIONAL FIELD–INSTANTONS

In the gravitational case, the potential has a pair of degenerate minima. Therefore, one has the possibility of constructing instanton solutions. Since the two minima are separated by two different barriers there are two types of instantons. In Appendix we also look at the possibility of constructing exact solutions of the Schrödinger equation in this case.

### A. Instanton solutions

As mentioned earlier, the effective potential has degenerate minima and therefore we will obviously have instanton solutions. Since the minima are separated by two kinds of barriers we will have two different instantons –one across the barrier at $0(2\pi)$ and another across the barrier at $\pi$.

It is worth reminding that instantons are the finite Euclidean action solutions of the Euclidean equations of motion of any theory. Since we are dealing with particle mechanics we look at solutions to the Euclideanised Newton's second law. These solutions have the interpretation in terms of quantum tunneling. For a particle in a double well potential, classically, it is not possible to cross the barrier connecting the two vacua. However, when we Euclideanise the Newton's second law, we are essentially looking at the motion in Euclidean time in an inverted potential. These Euclidean solutions (called instantons) begin at one vacuum at $\tau \to -\infty$ and end at another at $\tau \to +\infty$. In the saddle–point approximation, the tunneling amplitude goes as $\exp(\frac{-S_E}{\hbar})$.

The instanton solutions and aspects of quantum tunneling in this model have been worked out in an earlier paper by one of us [4]. We therefore only summarize these results here and refer the reader to that article for the relevant details. This section is included here entirely for the sake of completeness.

Below we write down the instanton solutions and the corresponding Euclidean actions. To arrive at the instantons we need to solve the Euclidean equation of motion which is given by

$$\theta'' = -\omega^2(\cos\theta + a)\sin\theta \quad ; \quad a = \frac{\omega_0^2}{\omega^2}\,, \tag{34}$$

where the prime denotes differentiation w.r.t. $\tau = -it$ (Euclidean time). It is easy to convert the above equation into one for $\theta'$ alone and then integrate the resulting first order equation.

We first deal with the case of $\omega > \omega_0$. The instanton/anti–instanton and it's Euclidean action across the barrier at $\theta = 2\pi(0)$ are given by

$$\theta_1(\tau) = \pm 2\tan^{-1}\left(\sqrt{\frac{1+a}{1-a}}\tanh\left[\sqrt{\frac{1}{4}(1-a^2)}\omega\tau\right]\right)\,. \tag{35}$$

$$S_{E1} = 4mg^{1/2}r^{3/2}\left[\sqrt{\frac{1-a^2}{a}} + 2\sqrt{a}\tan^{-1}\left(\sqrt{\frac{1+a}{1-a}}\right)\right]\,. \tag{36}$$

The instanton/anti–instanton and it's Euclidean action across the barrier at $\theta = \pi$ are given by :

$$\theta_2(\tau) = \pm 2\tan^{-1}\left(\sqrt{\frac{1+a}{1-a}}\coth\left[\sqrt{\frac{1}{4}(1-a^2)}\omega\tau\right]\right)\,. \tag{37}$$

$$S_{E2} = 4mg^{1/2}r^{3/2}\left[\sqrt{\frac{1-a^2}{a}} - 2\sqrt{a}\tan^{-1}\left(\sqrt{\frac{1+a}{1-a}}\right)\right] . \tag{38}$$

Notice that for $a \sim 1$ (i.e. $\omega \sim \omega^0$) the second term in each Euclidean action has the larger value. This implies that the Euclidean action is proportional to $\pm\frac{4\pi mgr}{\hbar\omega}$ and the tunneling amplitude will be dominated by the instanton across the $\theta = \pi$ barrier. On the other hand for $a \sim 0$ (i.e. $\omega \sim \infty$ or $r \to \infty$–very high frequencies or very large radius ) the Euclidean action is proportional to $\frac{4m\omega^2 r}{\hbar\omega}$ and the contribution to the tunneling probablity from both the instantons are the same. In the latter case, the effect of gravity is washed out while in the former gravity predominates.

For the $\omega < \omega_0$ case we have a single well in $S^1$ but infinite number of degenerate minima on the full line and the instanton/ anti–instanton as well as it's Euclidean action are given by

$$\theta_3(\tau) = \pm 2\tan^{-1}\left(\frac{p\sinh(\omega_0\tau}{p)}\right) . \tag{39}$$

$$S_{E3} = 4mg^{1/2}r^{3/2}\left[\sqrt{a}\tan^{-1}\left(\frac{1}{\sqrt{a-1}}\right) + \sqrt{\frac{a-1}{a}}\right] \tag{40}$$

with $p = \sqrt{\frac{a}{a-1}}$.

For domains of $a \sim 1$ (when the first term in $S_{E3}$ dominates ) and $a >> 1$ (when the second term in $S_{E3}$ gives the largest contribution) one can see the effects similar to the ones stated earlier.

## VI. QUANTUM MECHANICS OF A PORC IN A CONSTANT MAGNETIC FIELD –BOUNCES, INSTANTONS AND EXACT SOLUTIONS

### A. Exact Solutions

We now write down the Schrödinger equation for this system and look for exact solutions [8] following the method introduced by Razavy for bistable potentials [10].

For this case, the Schrödinger equation turns out to be

$$\frac{d^2\Psi}{dx^2} + \left[\frac{8mr^2E}{\hbar^2} + \frac{2m^2r^4\omega^2}{\hbar^2} - \frac{4m^2r^4\omega\omega_c}{\hbar^2}\cos 2x + \frac{2m^2r^4\omega^2}{\hbar^2}\cos 4x\right]\Psi = 0\,, \tag{41}$$

where we have introduced the following redefinitions $\theta --> \frac{\pi}{2} - \theta$ and then $\theta --> 2x$. We further rewrite the Schrödinger equation in the following form :

$$\frac{d^2\Psi}{dx^2} + \left[\bar{\epsilon} - (n+1)\xi\cos 2x + \frac{1}{8}\xi^2\cos 4x\right]\Psi = 0 \tag{42}$$

where $\epsilon = \frac{8mr^2E}{\hbar^2}$ ; $\bar{\epsilon} = \epsilon + \frac{1}{8}\xi^2$ ; $(n+1) = \frac{mr^2\omega_c}{\hbar}$ ; $\xi = \frac{4mr^2\omega}{\hbar}$ .

We shall now write down the exact solutions for different values of $n$ which, of course, correspond to different Lagrangians/Hamiltonians with the same functional form. In fact, as is clear from the definition given above, $n$ is related to the ratio of the *critical* angular momentum $mr^2\omega_c$ and Planck's constant $\hbar$.

Following Razavy, we first write down a solution for $n = 0$ (and also $\bar{\epsilon} = 0$) and then define the wave functions for $n > 0$ as the product of the $n = 0$ solution and an unknown function $\Phi(x)$. We thus start with the ansatz

$$\Psi(x) = \exp\left(-\frac{\xi}{4}\cos 2x\right)\Phi(x)\,. \tag{43}$$

The resulting differential equation for $\Phi(x)$ turns out to be

$$\Phi'' + \xi\sin 2x\Phi' + \left(\bar{\epsilon} + \frac{\xi^2}{8} - n\xi\cos 2x\right)\Phi = 0 \tag{44}$$

with the primes denoting differentiation w.r.t. $x$. It is easily shown that this is a quasi-exactly solvable (QES) system [9]. In particular, one can easily show that the above equation is equivalent to a three term difference equation which is exactly solvable in case $n$ is a non-negative integer. In particular, for $n = 0, 1, 2, ...$, first $(n+1)$ solutions of period $\pi$ $(2\pi)$ are obtained in case $n$ is even (odd). For example, the solutions for $n = 0, 1, 2, 3$ are

$\mathbf{n = 0}$ :

$$\Phi_0 = 1 \quad ; \quad E_0 = -\frac{1}{2}\hbar\omega\frac{\omega}{\omega_c}$$

.

$\mathbf{n=1}$ :

$$\Phi = \sin x \quad ; \quad E = \tfrac{1}{16}\hbar\omega_c - \tfrac{1}{2}\hbar\omega \left(\tfrac{1+2\omega}{\omega_c}\right)$$

.

$$\Phi = \cos x \quad ; \quad E = \tfrac{1}{16}\hbar\omega_c + \tfrac{1}{2}\hbar\omega \left(1 - 2\tfrac{\omega}{\omega_c}\right)$$

.

$\mathbf{n=2}$ :

$$\Phi_0 = 2\xi + a_- \cos 2x \quad ; \quad E_0 = \tfrac{\hbar\omega_c}{12} - \tfrac{3}{2}\hbar\omega\tfrac{\omega}{\omega_c} - \tfrac{\hbar\omega_c}{12}\sqrt{1 + 144\tfrac{\omega^2}{\omega_c^2}}$$

.

$$\Phi = \sin 2x \quad ; \quad E = \tfrac{1}{6}\hbar\omega_c - \tfrac{3}{2}\hbar\omega\tfrac{\omega}{\omega_c}$$

.

$$\Phi = 2\xi + a_+ \cos 2x \quad ; \quad E = \tfrac{\hbar\omega_c}{12} - \tfrac{3}{2}\hbar\omega\tfrac{\omega}{\omega_c} + \tfrac{\hbar\omega_c}{12}\sqrt{1 + 144\tfrac{\omega^2}{\omega_c^2}}$$

.

$\mathbf{n=3}$ :

$$\Phi = 3\xi\cos x + b_0 \cos 3x \quad ; \quad E = \tfrac{\hbar\omega_c}{32}\left(-\tfrac{1}{4}\xi^2 + 5 + \xi - 2\sqrt{\xi^2 + 4 - 2\xi}\right)$$

.

$$\Phi = 3\xi\sin x + b_1 \sin 3x \quad ; \quad E = \tfrac{\hbar\omega_c}{32}\left(-\tfrac{1}{4}\xi^2 + 5 - \xi - 2\sqrt{\xi^2 + 4 + 2\xi}\right)$$

.

$$\Phi = 3\xi\cos x + b_2 \cos 3x \quad ; \quad E = \tfrac{\hbar\omega_c}{32}\left(-\tfrac{1}{4}\xi^2 + 5 + \xi + 2\sqrt{\xi^2 + 4 - 2\xi}\right)$$

.

$$\Phi = 3\xi\sin x + b_3 \sin 3x \quad ; \quad E = \tfrac{\hbar\omega_c}{32}\left(-\tfrac{1}{4}\xi^2 + 5 - \xi + 2\sqrt{\xi^2 + 4 + 2\xi}\right)$$

.

where

$a_\pm = 2 \pm 2\sqrt{1+\xi^2} \quad ; \quad b_{0,2} = 4 - \xi \pm 2\sqrt{\xi^2 - 2\xi + 4}\,; \; b_{1,3} = 4 + \xi \pm 2\sqrt{\xi^2 + 2\xi + 4}$ .

A remark is in order at this stage. Since we are treating the problem of a point particle on circle $S^1$, and since the potential in the Schrödinger eq. (41) satisfies the boundary condition

$$V(x+\pi) = V(x)\,, \tag{45}$$

hence physical considerations demand that the corresponding wave functions must also satisfy the boundary condition

$$\psi(x+\pi) = \psi(x)\,. \tag{46}$$

In that case, the solutions obtained for $n = 1, 3$ are unacceptable as they do not satisfy this boundary condition but rather they change sign under $x \to x + \pi$. However, if we are considering it as a periodic problem on the full line then of course these are acceptable solutions.

We may add that the ground state energies obtained here (for $n = 0, 2$) are useful in another context. In particular, it has been shown using the transfer integral method [25] that the classical free energy of the soliton bearing field theories at low temperatures is given by the ground state energy of the corresponding Schrödinger like equation. Thus the ground state energies obtained here for $n = 0, 2$ are of direct relevance in the context of the classical free energy of the corresponding double sine-Gordon field theory.

### B. Euclidean time solutions–Bounces and Instantons

Since the effective potential also has a false vacuum, we expect that 'bounce' solutions exist to the classical equations of motion in Euclidean time. The classical equation of motion in Euclidean time which we solve is given by

$$\theta'^2 = \omega^2 \left[(1-\sin\theta)(1+\sin\theta - a)\right] , \tag{47}$$

where we have added a constant to the original effective potential in order to write it in the above form (note that this does not effect the classical equations of motion which we intend to solve).

Here $a = \frac{\omega_c}{\omega} < 2$. However, this choice of the effective potential forces us to confine ourselves to $a < 2$. For $a = 2$ the equation is meaningless.

A straightforward integration of the above equation yields the following solutions :

$$\theta_1(\tau) = \frac{\pi}{2} + 2\tan^{-1}\left[\frac{1}{\alpha} sech\sqrt{\omega(\omega - \frac{\omega_c}{2})}\tau\right] . \tag{48}$$

$$\theta_2(\tau) = \frac{5\pi}{2} - 2\tan^{-1}\left[\frac{1}{\alpha} sech\sqrt{\omega(\omega - \frac{\omega_c}{2})}\tau\right] . \tag{49}$$

The first of these is the bounce across the barrier at $\sin^{-1}\frac{\omega_c}{2\omega}$ while the second one is across the barrier at $\pi - \sin^{-1}\frac{\omega_c}{2\omega}$. Note that the Euclidean actions for both these solutions are the same–this is because the barrier heights which separate the false and the true vacua are the same. The expression for the Euclidean action turns out to be

$$S_E = 2mr^2\omega_c\sqrt{1-\frac{a}{2}}\left[\frac{2+a}{a} - \sqrt{\frac{2-a}{2}}\sinh^{-1}\sqrt{\frac{2-a}{2}}\right] \tag{50}$$

where $\alpha^2 = \frac{a}{2-a}$. Using the Euclidean action one can now evaluate the decay rate of the false vacuum by the formula $\Gamma \sim \exp\left(-\frac{S_E}{\hbar}\right)$.

For a periodic potential generalization of the magnetic case problem one has degenerate absolute minima at $\frac{(2n+1)\pi}{2}$ and one can obtain an 'instanton' which starts out at, say, $\theta = -\frac{\pi}{2}$ (as $\tau \to -\infty$) and crosses the local minimum at $\theta = \frac{\pi}{2}$ (as $\tau \to 0$) to end up at $\theta = \frac{3\pi}{2}$ (as $\tau \to \infty$). To that end, let us first note that by adding a suitable constant one can also write the classical equation of motion in Euclidean time in an alternative form as

$$\theta'^2 = \omega^2\left[(1+\sin\theta)(1-\sin\theta + a)\right] . \tag{51}$$

It is now easily shown that irrespective of whether $\omega >$ or $< \omega_c/2$, the instanton solution is given by

$$\theta_3 = \frac{\pi}{2} + 2\tan^{-1}\left[\frac{1}{\sqrt{1+\frac{2\omega}{\omega_c}}}\sinh\left(\sqrt{(\omega_c+2\omega)2\omega}\frac{\tau}{2}\right)\right] , \tag{52}$$

with the corresponding action being

$$S_E = -\frac{1}{2a^2} + \frac{1}{4a^3}\frac{(4a^2-1)}{\sqrt{a^2-1}}\ln\left[\frac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}\right] . \tag{53}$$

where $a = \sqrt{1+\frac{2\omega}{\omega_c}}$.

## VII. SMALL OSCILLATIONS ABOUT INSTANTONS AND BOUNCES–QUALITATIVE ANALYSIS

In this section we present a qualitative analysis of the problem of small oscillations about an instanton/bounce solution. We do *not* explicitly solve the corresponding Schrödinger-like equations but discuss qualitatively the nature of the effective potentials, the possible existence of negative eigenvalues and bound states. It may however be added that in all the cases we have been able to reduce the stability equation to Heun's equation. Additionally, the analysis for the small oscillations about the DSG kink has been discussed in [26].

The equation governing small oscillations is given by:

$$\left(-\partial_\tau^2 + V''(\theta(\tau))\right)\chi_n = \lambda_n \chi_n\,. \tag{54}$$

This is a 'time–independent' Schrödinger equation with the potential $V''(\theta(\tau))$. We therefore need to solve the corresponding eigenvalue problem and look for the existence of a negative eigenvalue which will indicate an instability. Below we write down the 'potential' in the two cases of the gravitational and magnetic fields.

**1. : System in a gravitational field**

Since for $\omega > \omega_0$ we have two types of instantons we need to evaluate $V''(\theta)$ for each of these cases separately.

(a) Using eq. (reft1), $V''(\theta)$ for the instanton across the $2\pi$ barrier turns out to be

$$V''(\theta_1(\tau)) = -mr^2\omega^2(1-a^2)\frac{\left[(1-a)+(1+a)\tanh^4 b\omega\tau - 6\tanh^2 b\omega\tau\right]}{\left[(1-a)+(1+a)\tanh^2 b\omega\tau\right]^2}\,. \tag{55}$$

This is shown in Fig 5.

(b) Using eq. (37), $V''(\theta(\tau))$ for the instanton across the $\pi$ barrier turns out to be

$$V''(\theta_2(\tau)) = -mr^2\omega^2(1-a^2)\frac{\left[(1-a)+(1+a)\coth^4 b\omega\tau - 6\coth^2 b\omega\tau\right]}{\left[(1-a)+(1+a)\coth^2 b\omega\tau\right]^2}\,. \tag{56}$$

This is plotted in Fig 6.

A condition for the stability of a given instanton is the existence of a node-less zero–mode solution (solution with a zero eigenvalue) to the Schrödinger equation governing small oscillations. The zero mode solution can be written down very easily– $\chi_0$ is just equal to $\frac{d\theta(\tau)}{d\tau}$ (details are there is [5], [6] and [7]).

For the instanton across $\theta = 2\pi$, using eq. (35), one finds that

$$\chi_0 = \frac{d\theta_1(\tau)}{d\tau} = (1 - a^2)\omega \frac{1}{[2\cosh^2\beta\tau - (1+a)`]} \tag{57}$$

where $\beta = \sqrt{\frac{1-a^2}{2}}\omega$. Note that this solution is node-less which implies that the corresponding instanton is stable.

On the other hand, it can be readily shown that the instanton across the $\theta = \pi$ barrier is also stable. In particular, using eq. (37), the zero–mode solution is given by

$$\chi_0 = \frac{d\theta_2(\tau)}{d\tau} = -(1 - a^2)\omega \frac{1}{[(1+a) + 2\sinh^2\beta\tau]} \,. \tag{58}$$

which is clearly nodeless as well.

In the case of $\omega < \omega_0$ there is only one instanton and using eq. (39), $V''(\theta)$ turns out to be

$$V''(\theta_3(\tau)) = mr^2\omega^2 \frac{\left[(a-1)p^4\sinh^4 y + 6p^2\sinh^2 y - (a+1)\right]}{(1 + p^2\sinh^2 y)} \,, \tag{59}$$

and the corresponding zero-mode solution is given by

$$\chi_0 = \frac{d\theta_3(\tau)}{d\tau} = \frac{2\omega_0 \cosh y}{1 + p^2\sinh^2 y} \,, \tag{60}$$

which is clearly node-less. Here $y = \frac{\omega_0\tau}{p}$ with $a = \frac{\omega_0^2}{\omega^2}$ and $p = \sqrt{\frac{a}{a-1}}$.

**Case 2: System in a magnetic field**

For the bounce solution in the PORC in a magnetic field, using eq. (48) (or (49)), the potential $V''(\theta)$ in the Schrödinger-like equation turns out to be

$$V''(\theta_{1,2}(\tau)) = \left(\frac{mr^2\omega^2}{2-a}\right) \frac{\left[-a^2\cosh^4\beta\tau + 8a\cosh^2\beta\tau - (4 - a^2)\right]}{(1 + \alpha^2\cosh^2\beta\tau)^2} \,. \tag{61}$$

This is plotted in Fig.7. The corresponding zero-mode solution is given by

$$\chi_0 = \frac{d\theta_{1,2}(\tau)}{d\tau} = \alpha\omega\sqrt{2(2-a)}\left(\frac{\sinh y}{1 + \alpha^2\cosh^2 y}\right) \,, \tag{62}$$

which clearly has a node at $\tau = 0$. Here $y = \frac{(2-a)\omega\tau}{2}$.

On the other hand, corresponding to the instanton solution in a magnetic field, using eq. (52), $V''(\theta)$ turns out to be

$$V''(\theta_3(\tau) = \frac{mr^2\omega^2(a+1)}{a} \frac{\left[a(6+a) - 2(1+3a)\cosh^2 y + \cosh^4 y\right]}{(a+\cosh^2 y)^2} . \tag{63}$$

The corresponding zero-mode solution is given by

$$\chi_0 = \frac{d\theta_3(\tau)}{d\tau} \propto \left(\frac{\cosh y}{a+\cosh^2 y}\right) , \tag{64}$$

which clearly is node-less. Here $a = \frac{2\omega}{\omega_c}$, $y = \sqrt{a(a+1)}\frac{\omega_c\tau}{2}$.

## VIII. SUMMARY AND OUTLOOK

We have discussed a variety of aspects of the toy model of a particle on a rotating loop. After defining a generic class of models (via a choice of a class of trigonometric potentials ) we identified two specific cases. These included the effects of a (i) uniform gravitational field and (ii) a uniform magnetic field. The classical mechanics for both these cases was worked out. Subsequently we dealt with the quantum mechanical problem in either case from the (i) exact as well as the (ii) semi-classical standpoints. For the gravitational case we found exact solutions by identifying the time–independent Schrödinger equation with the Whittaker–Hill equation. These solutions were *formal* in the sense that the energy eigenvalues remain hidden in the continued fractions. We have not attempted to arrive at the expressions for the energy eigenvalues by solving the continued fractions. Hence this discussion appears in the Appendix to the paper. Apart from the exact solutions we have also looked at the instantons in this model. We constructed the instantons and the corresponding Euclidean actions which gave us a feeling for the quantum tunneling phenomena in this example. In the magnetic case the potential also has a false vacuum. Further, for special values of the parameters we were able to obtain some of the energy eigenvalues and eigenfunctions. Thereafter, we constructed the corresponding bounce (instanton) solutions and their Euclidean actions to arrive at the decay rate of the false vacuum (feel for quantum tunneling phenomena). We then looked into the problem of small oscillations about the instantons and bounces by investigating the nature of the potential in the Schrödinger–type equation which govern these perturbations. This section is largely qualitative. Even though we did not solve the relevant Schrödinger-like equation, we have shown that in all the cases the Schrödinger equation reduces to Heuns equation which is known to have four regular singular points. Further, as expected we have seen that the zero mode is stable (unstable) depending on whether one is considering an instanton (bounce) solution. Finally, the connection of these toy examples with several models in field theory and statistical mechanics was pointed out.

The aim of the paper has been to point out the diverse aspects of the problem of a particle on a rotating loop. It is quite illuminating that a lot of important concepts in field theory can be illustrated through the study of this simplistic example. We conclude by pointing out some other aspects which may be dealt with in future.

- Aspects of chaos in PORC with a $\delta$ function kick–the generalization of the kicked pendulum/kicked rotor
- Investigating what happens at finite temperature by utilizing techniques of finite temperature quantum mechanics/quantum field theory
- Exact classical and quantum solutions for the model with a gravitational, an electric and a magnetic field all put in together

## ACKNOWLEDGEMENTS

SK thanks IUCAA, Pune for financial support during the period when this work began.

## APPENDIX : Exact solutions for PORC in a gravitational field

The time-independent Schrödinger equation for this system is given by

$$\frac{d^2\psi}{d\theta^2} + \frac{2m}{\hbar^2}\left[E + \frac{1}{4}mr^2\omega^2 - mgr\cos\theta - \frac{1}{4}mr^2\omega^2\cos 2\theta\right]\psi = 0\,. \tag{65}$$

Introducing a new variable $\alpha = \theta/2$ we arrive at the following equation

$$\frac{d^2\psi}{d\alpha^2} + \left[\left(\frac{8mE}{\hbar^2} + \frac{2m^2r^2\omega^2}{\hbar^2}\right) - \frac{8m^2gr}{\hbar^2}\cos 2\alpha - \frac{2m^2r^2\omega^2}{\hbar^2}\cos 4\alpha\right]\psi = 0\,. \tag{66}$$

This equation is the Whittaker-Hill equation.We now digress briefly to a discussion of the solutions of this equation.

In the theory of periodic differential equations, the Whittaker-Hill equation is a special case of the general Hill equation given by

$$\frac{d^2\psi}{d\theta^2} + \left(\sum_{n=0}^{\infty} A_n\cos 2n\theta\right)\psi = 0\,. \tag{67}$$

Whittaker-Hill equation is a special case of this general Hill equation and is also known as the three term Hill equation, i.e. we have only the $n = 0, 1, 2$ terms remaining in the infinite series given above. Thus we have

$$\frac{d^2\psi}{d\theta^2} + (A_0 + A_1\cos 2\theta + A_2\cos 4\theta)\,\psi = 0\,. \tag{68}$$

Such equations arise when the Helmholtz equation $(\nabla^2 + k^2)\,\psi = 0$ is separated in a general paraboloidal coordinate system. We shall be concerned here with the form of equation for which $A_1$ and $A_2$ are both negative.Following [27] we define

$$|A_2| = \frac{1}{8}\omega^2, A_0 = \eta + \frac{1}{8}\beta^2, A_1 = -\rho\beta\,. \tag{69}$$

Hence we have

$$\frac{d^2\psi}{d\theta^2} + \left(\eta + \frac{1}{8}\beta^2 - \rho\beta\cos 2\theta - \frac{1}{8}\beta^2\cos 4\theta\right)\psi = 0\,. \tag{70}$$

The transformation $\psi = \phi\exp(-\frac{1}{4}i\beta\cos 2\theta)$ reduces (68) to Ince's equation

$$\frac{d^2\phi}{d\theta^2} + i\beta \sin 2\theta \frac{d\phi}{d\theta} + [\eta + \beta(\rho + i) \cos 2\theta]\, \phi = 0\,. \tag{71}$$

Solutions to the above equation falls into four classes depending on whether the function $\phi$ is even or odd and has period $\pi$ or $2\pi$. These solutions are in terms of infinite Fourier series with the relation between the coefficients of the series given in terms of three term recurrence relations. The recurrence relations can be reduced to an infinite continued fraction which is known as the corresponding characteristic equation for the relevant function. We shall primarily be concerned with solutions of period $\pi$. This is because the equation governing our physical problem is written in terms of a new variable $\alpha = \theta/2$. Thus $\pi$ periodic solution in $\alpha$ will turn out to be $2\pi$ periodic solutions in $\theta$. We now list the even and odd $\pi$ periodic solutions of the Whittaker-Hill equation and the relevant continued fractions. Details about how these are obtained and a more thorough treatment of various issues related to periodic differential equations can be found in [27].

### A. Even Solutions of Period $\pi$

$$\phi(\theta) = \sum_{r=0}^{\infty} A_{2r} \cos 2r\theta \equiv gc_{2n}(\theta, \omega, \rho)\,;\ A_{2r} = X_{2r} B_{2r}\,, \tag{72}$$

where

$$X_{2r} = (-\rho + i)(-\rho + 3i) \cdots [-\rho + (2r-1)i]\ ; r \geq 1\,,\ X_0 = 1\,, \tag{73}$$

and

$$\frac{B_{2r}}{B_{2r+2}} = \frac{\frac{1}{2}\beta}{4r^2 - \eta -}\, \frac{\frac{1}{4}\beta^2\{\rho^2 + (2r+1)^2\}}{4(r+1)^2 - \eta -}\, \frac{\frac{1}{4}\beta^2\{\rho^2 + (2r+3)^2\}}{4(r+2)^2 - \eta -} \cdots\,. \tag{74}$$

The Continued fraction for characteristic values is given by

$$-\frac{2\eta}{\beta(\rho^2 + 1)} = \frac{\beta}{4 - \eta -}\, \frac{\frac{1}{4}\beta^2(\rho^2 + 9)}{16 - \eta -}\, \frac{\frac{1}{4}\beta^2(\rho^2 + 25)}{36 - \eta -} \cdots\,. \tag{75}$$

### B. Odd Solutions of Period $\pi$

$$\phi(\theta) = \sum_{r=1}^{\infty} C_{2r} \sin 2r\theta \equiv gs_{2n+2}(\theta, \omega, \rho)\,;\; C_{2r} = Y_{2r} D_{2r}\,, \tag{76}$$

where

$$Y_{2r} = (-\rho + 3i)(-\rho + 5i)\cdots[-\rho + (2r-1)i]\;,\; r \geq 2\,,\; Y_2 = 1\,, \tag{77}$$

and

$$\frac{D_{2r}}{D_{2r-2}} = \frac{\frac{1}{2}\beta}{4r^2 - \eta -}\,\frac{\frac{1}{2}\beta^2\{\rho^2 + (2r+1)^2\}}{4(r+1)^2 - \eta -}\,\frac{\frac{1}{2}\beta^2\{\rho^2 + (2r+3)^2\}}{4(r+2)^2 - \eta -}\cdots\,. \tag{78}$$

The Continued fraction for characteristic values is given by

$$\frac{4 - \eta}{\frac{1}{2}\beta(\rho^2 + 9)} = \frac{\frac{1}{2}\beta}{16 - \eta -}\,\frac{\frac{1}{4}\beta^2(\rho^2 + 25)}{36 - \eta -}\,\frac{\frac{1}{4}\beta^2(\rho^2 + 49)}{64 - \eta -}\cdots\,. \tag{79}$$

Equations (75) and (79) are the equations which will give us the energy levels. Comparing eqs. (66) and (70) we find that

$$\eta = \frac{8mEr^2}{\hbar^2}\,,\; \frac{1}{8}\beta^2 = \frac{2m^2 r^4 \omega^2}{\hbar^2}\,,\; \rho = \frac{2mgr}{\hbar\omega}\,. \tag{80}$$

Fixing $m, r$ and $\omega$ we can evaluate the energy levels of the system by solving the respective characteristic equations (for even and odd functions) iteratively.

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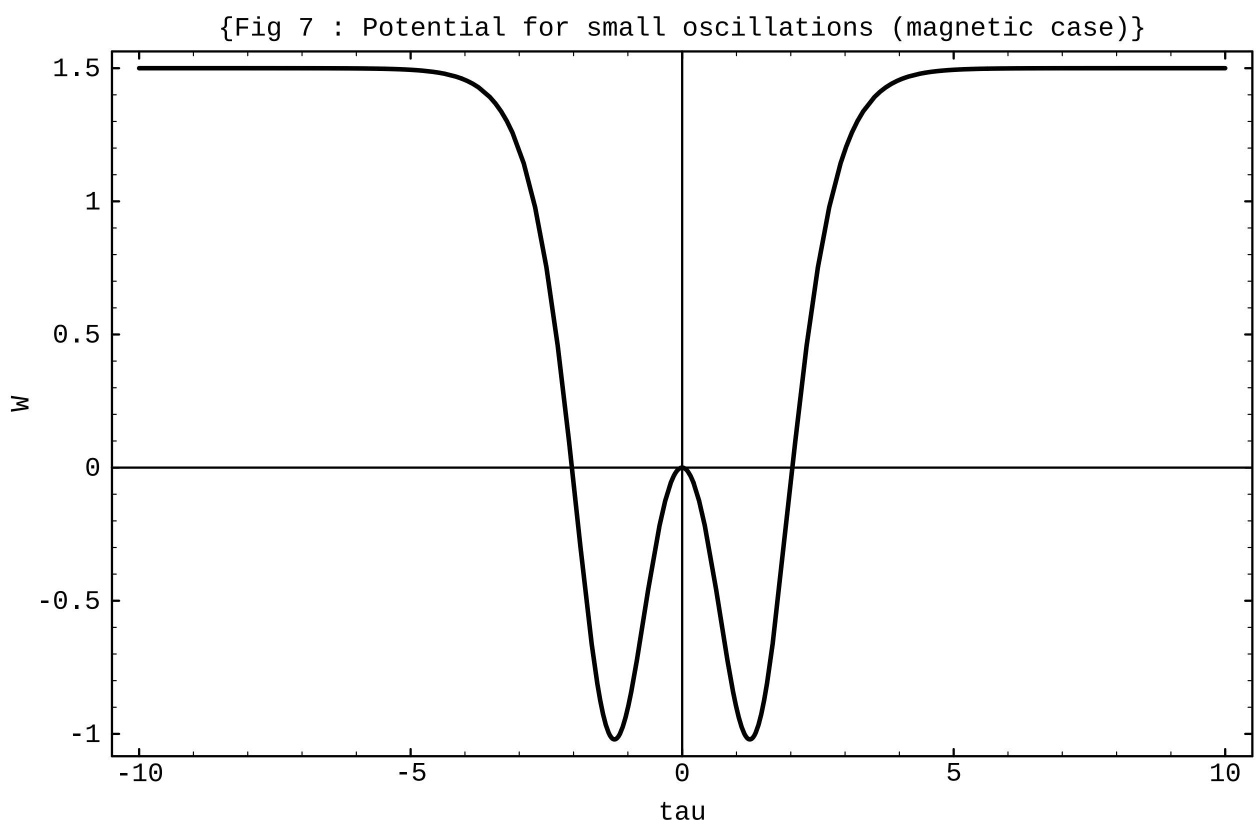
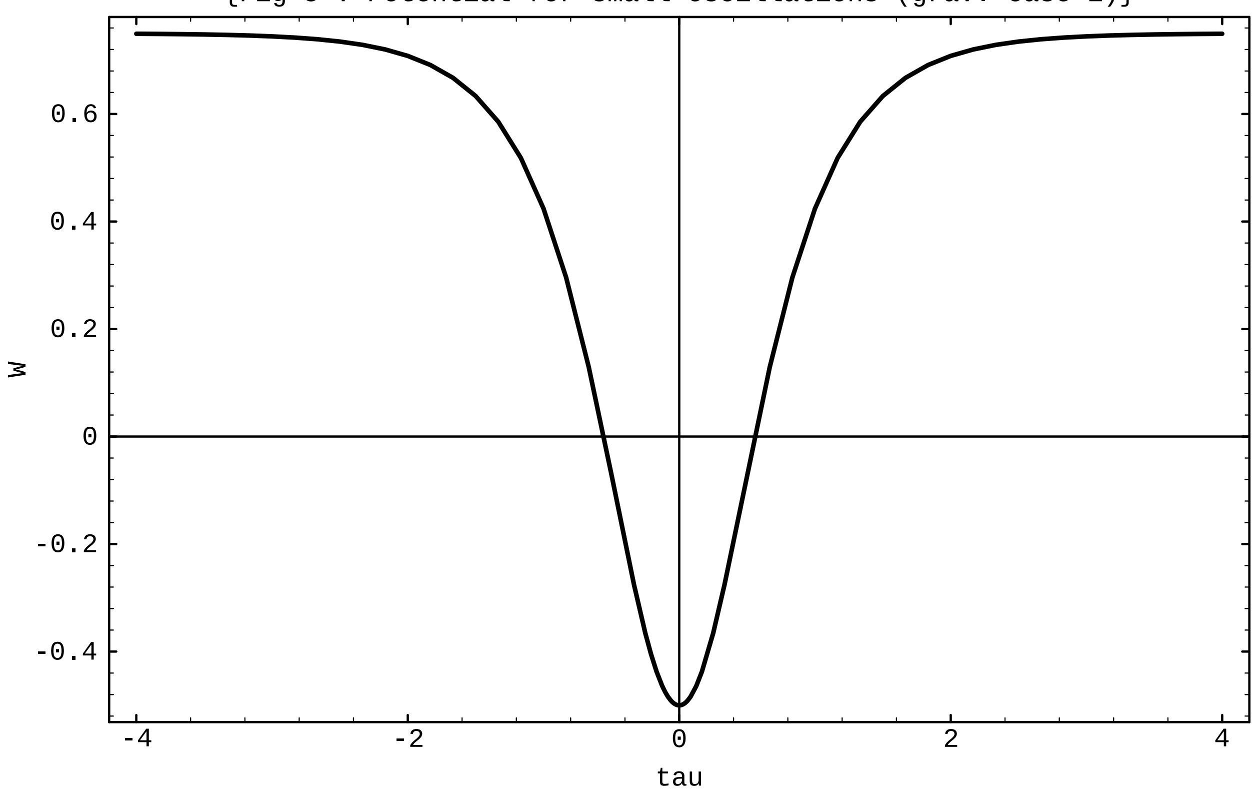
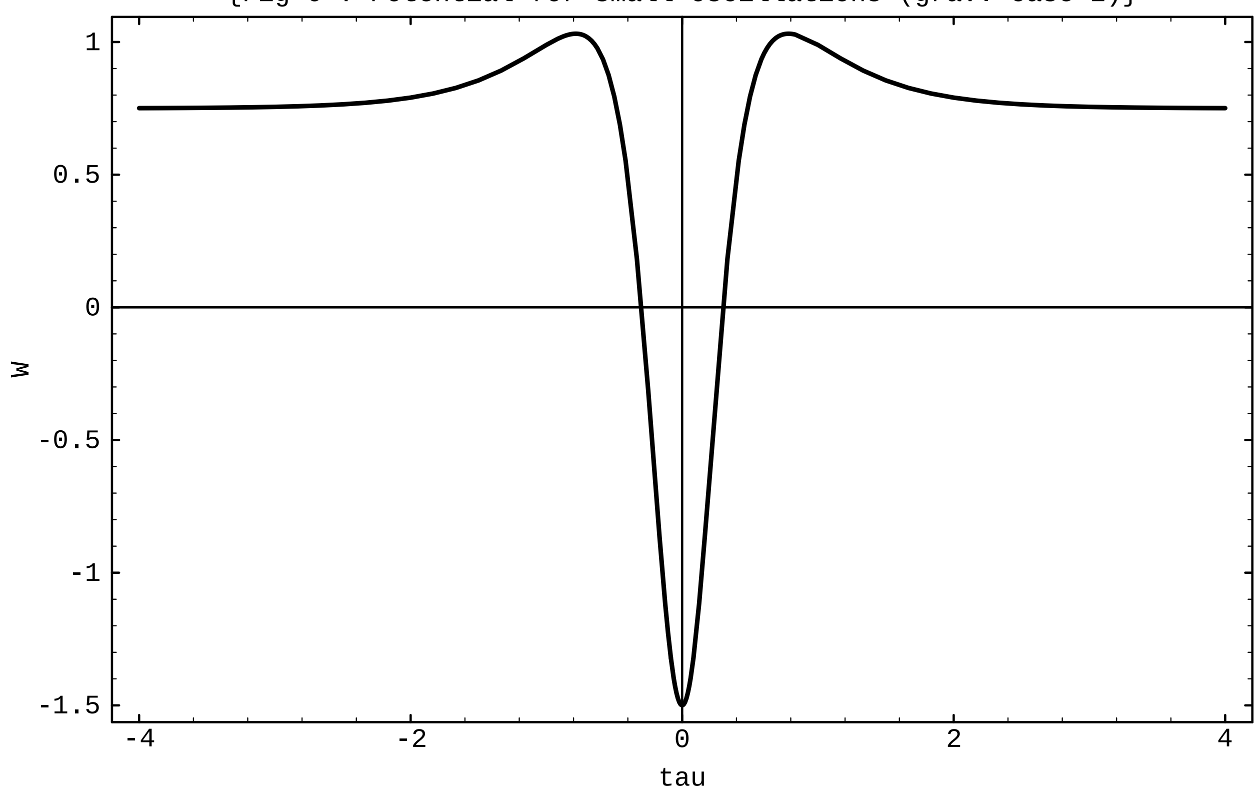
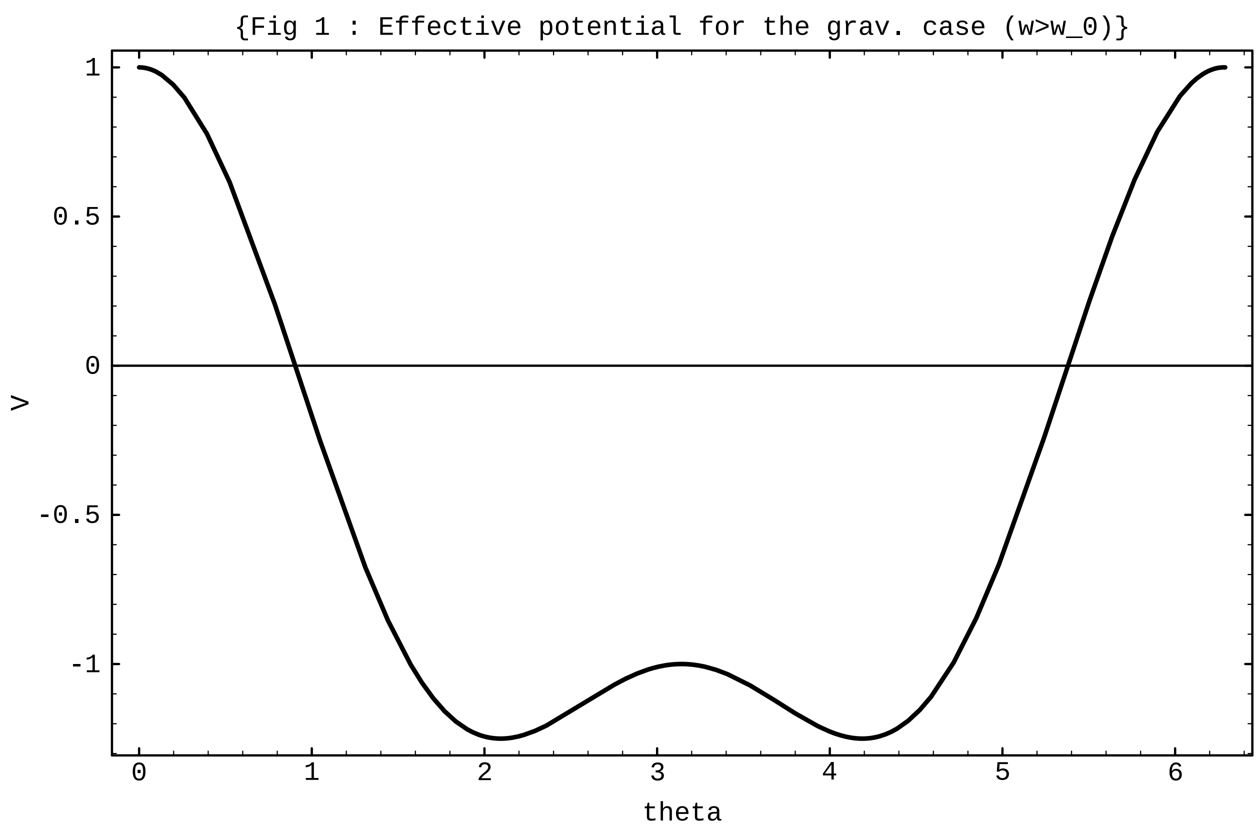

{Fig 7 : Potential for small oscillations (magnetic case)}
W
tau
1.5
1
0.5
0
-0.5
-1
-10
-5
0
5
10

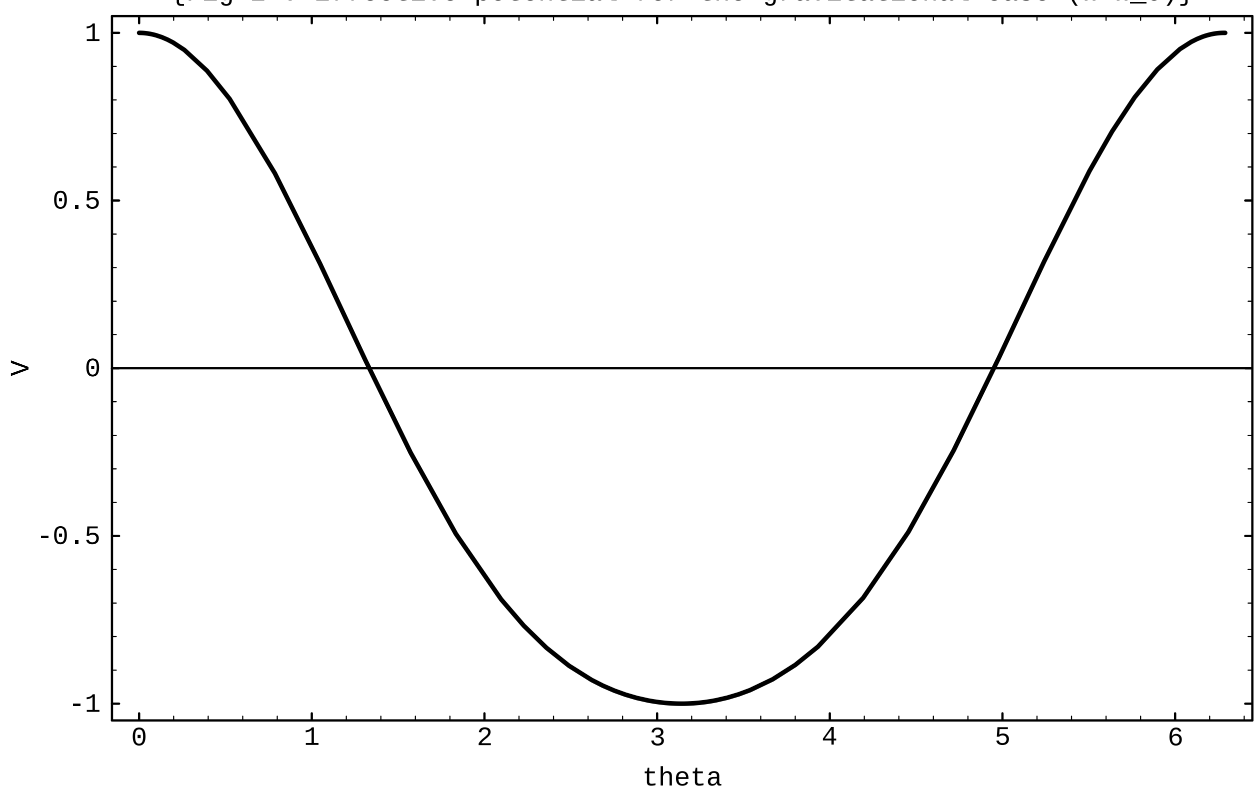
{Fig 5 : Potential for small oscillations (grav. case 1)}



{Fig 6 : Potential for small oscillations (grav. case 2)}

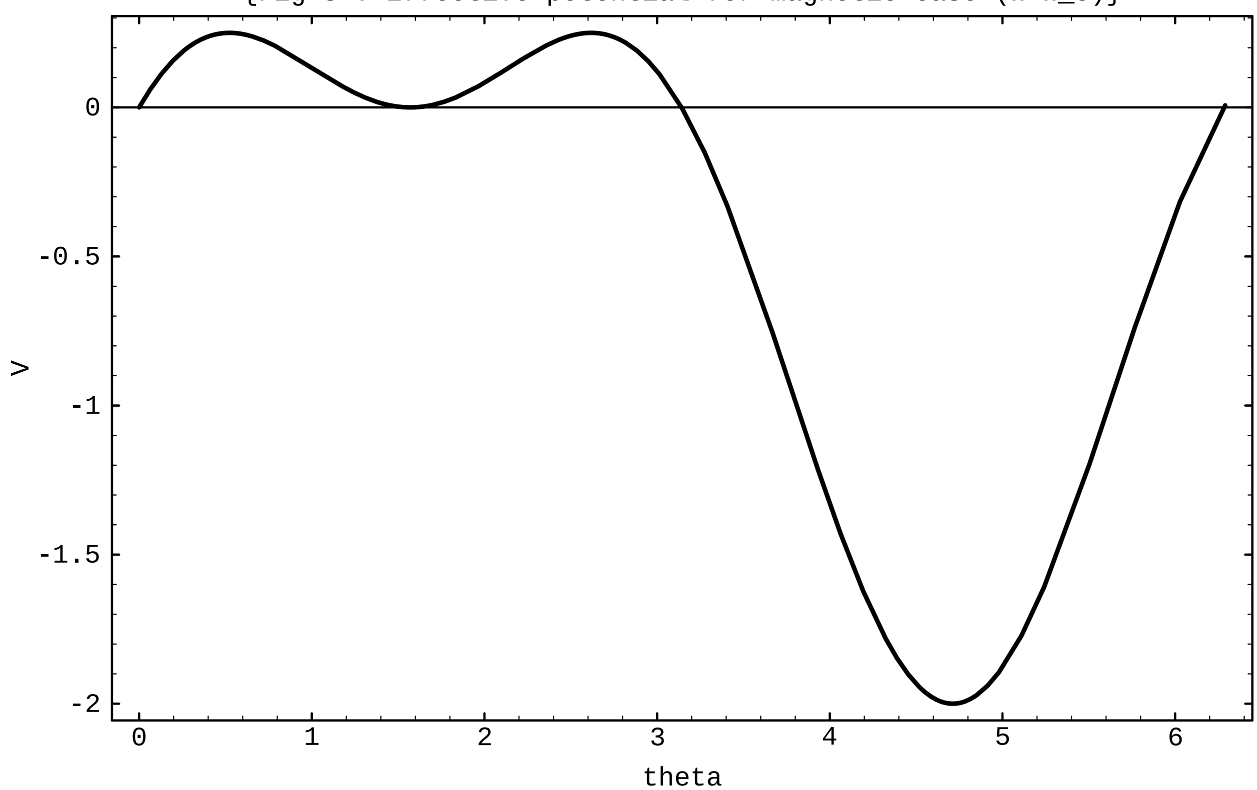



{Fig 1 : Effective potential for the grav. case (w>w_0)}
1
0.5
0
-0.5
-1
V
0
1
2
3
4
5
6
theta




{Fig 2 : Effective potential for the gravitational case (w<w_0)}

{Fig 3 : Effective potential for magnetic case (w>w_c)}



{Fig 4 : Effective potential for the magnetic case (w<w_c)}