UICHEP-TH/99-3, IP/BBSR/99-10, February 9, 2008

New Solvable and Quasi Exactly Solvable Periodic Potentials

Avinash Khare[a,1] **and Uday Sukhatme**[b,2]

a) Institute of Physics, Sachivalaya Marg, Bhubaneswar 751005, Orissa, India

b) Department of Physics, University of Illinois at Chicago, Chicago, IL 60607-7059, U.S.A.

Abstract

Using the formalism of supersymmetric quantum mechanics, we obtain a large number of new analytically solvable one-dimensional periodic potentials and study their properties. More specifically, the supersymmetric partners of the Lamé potentials $ma(a+1)\ \mathrm{sn}^2(x,m)$ are computed for integer values $a = 1, 2, 3, ...$ For all cases (except $a = 1$), we show that the partner potential is distinctly different from the original Lamé potential, even though they both have the same energy band structure. We also derive and discuss the energy band edges of the associated Lamé potentials $pm\ \mathrm{sn}^2(x,m) + qm\ \mathrm{cn}^2(x,m)/\mathrm{dn}^2(x,m)$, which constitute a much richer class of periodic problems. Computation of their supersymmetric partners yields many additional new solvable and quasi exactly solvable periodic potentials.

[1]khare@iopb.res.in

[2]sukhatme@uic.edu

1. Introduction:

The energy spectrum of electrons on a lattice is of central importance in condensed matter physics. In particular, knowledge of the existence and locations of band edges and band gaps determines many physical properties. Unfortunately, even in one dimension, there are very few analytically solvable periodic potential problems in quantum mechanics. The aim of this paper is to extend the small currently known set of analytically solvable periodic potentials.

For a potential with period L, one is seeking solutions of the Schrödinger equation subject to the Bloch condition

$$\psi(x) = e^{ikL}\ \psi(x+L)\,, \tag{1}$$

where k denotes the crystal momentum. The spectrum shows energy bands whose edges correspond to $kL = 0, \pi$, that is the wave functions at the band edges satisfy $\psi(x) = \pm\psi(x+L)$. For periodic potentials, the band edge energies and wave functions are often called eigenvalues and eigenfunctions, and we will also use this terminology. The classic text book example which is used to demonstrate band structure is the Kronig-Penney model

$$V(x) = \sum_{n=-\infty}^{\infty} V_0\delta(x-nL)\ \ .$$

It should be noted that the band edges for the Kronig-Penney model can only be computed by solving a transcendental equation. Another well studied class of periodic potentials is

$$V(x) = pm\ \mathrm{sn}^2(x,m)\ ,\ p \equiv a(a+1)\ \ . \tag{2}$$

Here $\mathrm{sn}(x,m)$ is a Jacobi elliptic function of real elliptic modulus parameter m $(0 \le m \le 1)$ with period $4K(m)$. For simplicity, from now onward, we will not explicitly display the modulus parameter m as an argument of Jacobi elliptic functions [1]. The elliptic function potentials of eq. (2) have a period $L = 2K(m)$, and will be referred to as Lamé potentials, since the corresponding Schrödinger equation is called Lamé's equation [2, 3]. It is well known that for any integer value $a = 1, 2, 3, \ldots$, the corresponding Lamé potential (2) has a bound bands followed by a continuum band [2, 3]. All band edge energies and wave functions are analytically known.

At this point it is worth recalling that supersymmetric quantum mechanics (SUSYQM) has proved useful in discovering many, new, analytically solvable potentials on both the full as well as the half line [4]. It is then natural to enquire if one can also use similar techniques to discover new solvable periodic potentials. In this paper, we demonstrate that this is indeed possible.

Our work is inspired by several recent papers [5, 6, 7, 8] which discuss various general aspects of SUSYQM for periodic potentials. In particular, Dunne and Feinberg [5] defined and developed the concept of "self-isospectral" periodic potentials in detail. A one dimensional potential $V_-(x)$ of period L is said to be self-isospectral if its supersymmetric partner potential $V_+(x)$ is just the original potential upto a discrete transformation - a translation by any constant amount, a reflection, or both. A common example is translation by half a period, in which case the condition for self-isospectrality is

$$V_+(x) = V_-(x - L/2) \quad . \tag{3}$$

It is easily checked that if the superpotential W satisfies

$$W(x) = -W(x - L/2)\,, \tag{4}$$

then condition (3) immediately follows. In this sense, any self-isospectral potential is rather uninteresting, since application of the SUSYQM formalism [4] to it just yields a discrete transformation and basically nothing new. We have recently pointed out [9] that the Lamé potentials given in eq. (2) are not self-isospectral for $a \geq 2$, and hence SUSYQM generates new exactly solvable periodic problems. This point is further developed in detail in this paper.

We expand our discussion to the band edges and wave functions of a much richer class of periodic potentials given by

$$V(x) = pm\,\mathrm{sn}^2(x) + qm\,\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)}\,, \quad p \equiv a(a+1)\,,\ q \equiv b(b+1) \quad , \tag{5}$$

where, like $\mathrm{sn}(x)$, the Jacobi elliptic functions $\mathrm{cn}(x)$ and $\mathrm{dn}(x)$ also have a modulus parameter m which, for notational convenience, is not explicitly displayed. The potentials of eq. (5) are called associated Lamé potentials, since the corresponding Schrödinger equation is called the associated Lamé equation [3]. More precisely, we often refer to the associated Lamé potential of eq. (5) as the (p, q) potential and note that $(p, 0)$ potentials are just the ordinary Lamé potentials. Although some results for (p, q) potentials are available in scattered form in the mathematical literature, many of our results are new. In particular, we obtain band edge energies and wave functions for the special case $p = q = a(a + 1)$ for $a = 1, 2, 3, \ldots$. We study many (p, q) potentials and check whether they are self-isospectral by constructing and examining the supersymmetric partner potentials. In most cases, $V_-(x)$ is not self-isospectral, and consequently $V_+(x)$ is a new, exactly or quasi-exactly solvable periodic potential.

The associated Lamé potentials given by eq. (5) can also be re-written in the alternative form

$$V(x) = pm\ \mathrm{sn}^2(x) + qm\ \mathrm{sn}^2(x + K(m)) \quad , \tag{6}$$

since [1]

$$\mathrm{sn}(x+K) = \mathrm{cn}(x)/\mathrm{dn}(x)\ ,\ \mathrm{cn}(x+K) = -\sqrt{1-m}\ \mathrm{sn}(x)/\mathrm{dn}(x)\ ,\ \mathrm{dn}(x+K) = \sqrt{1-m}/\mathrm{dn}(x)\ .$$

It is clear from (6) that potentials (p, q) and (q, p) have the same energy spectra with wave functions shifted by $K(m)$. Therefore, it is sufficient to restrict our attention to $p \geq q$.

Before actually solving the Schrödinger equation for the associated Lamé potential (5), let us make a few general comments. Throughout this paper, we have chosen units with $\hbar = 1$, and taken the particle mass in the Schrödinger equation to be 1/2. Note that in the limit when the elliptic modulus parameter $m = 0$, the potential vanishes and one has a rigid rotator problem of period $2K(0) = \pi$, whose energy eigenvalues are at $E = 0, 1, 4, 9, ...$ with all the nonzero values being two-fold degenerate. On the other hand, the limit $m \to 1$ is much trickier since $K(m)$ tends to infinity and the periodic nature of the potential is obscured. The Schrödinger equation for finding the eigenstates for an arbitrary periodic potential is called Hill's equation in the mathematics literature [3]. A general property of Hill's equation is the oscillation theorem which states that for a potential with period L, the band edge wave functions arranged in order of increasing energy $E_0 \leq E_1 \leq E_2 \leq E_3 \leq E_4 \leq E_5 \leq E_6 \leq ...$ are of period $L, 2L, 2L, L, L, 2L, 2L, ...$ The corresponding number of wave function nodes in the interval L are $0, 1, 1, 2, 2, 3, 3, ...$ and the energy band gaps are given by $\Delta_1 \equiv E_2 - E_1,\ \Delta_2 \equiv E_4 - E_3,\ \Delta_3 \equiv E_6 - E_5,\ ....$ We shall see that the expected $m = 0$ limit and the oscillation theorem are very useful in identifying if all band edge eigenstates have been properly determined or if some have been missed.

The plan of the paper is as follows. In Sec. 2, we briefly review the basic ideas of SUSYQM. A detailed discussion of Lamé potentials and their supersymmetric partners is given in Sec. 3. Solutions of the Schrödinger equation for the associated Lamé potentials are presented in Sec. 4. Many key new results are summarized in Table 3. It is shown that the locus of quasi exactly solvable problems [10, 11] in the (p, q) plane are parabolas about the line $p = q$. Our solutions are valid for any real choice of the parameters a, b (recall $p = a(a + 1), q = b(b + 1)$). Integer and half-integer values of a, b , including the very interesting special case $a = b$ =integer, are treated in detail in Sec. 5. In most cases, the application of SUSYQM gives new solvable periodic potentials, many of which are illustrated in the figures. Finally, Sec. 6 contains some concluding remarks.

2. Supersymmetric Quantum Mechanics Formalism:

The supersymmetric partner potentials $V_\pm(x)$ are defined in terms of the superpotential $W(x)$ by

$$V_\pm(x) = W^2(x) \pm W'(x)\,. \tag{7}$$

The corresponding Hamiltonians $H_\pm$ can be factorized as

$$H_- = A^+A,\ H_+ = AA^+\,, \tag{8}$$

where

$$A = \frac{d}{dx} + W(x)\,,\ A^+ = -\frac{d}{dx} + W(x)\,, \tag{9}$$

so that the spectra of $H_\pm$ are nonnegative. It is also clear that on the full line, both $H_\pm$ cannot have zero energy modes since both $\psi_0^{(\pm)}$ given by

$$\psi_0^{(\pm)}(x) = \exp(\pm \int^x W(y)dy) \quad , \tag{10}$$

cannot be simultaneously normalized.

On the other hand, when the superpotential $W(x)$ is periodic $(W(x+L) = W(x))$ then the potentials $V_-(x)$ and $V_+(x)$ are isospectral - their spectra match completely, including the zero modes, and SUSY is unbroken provided

$$\int_0^L W(y)dy = 0\,. \tag{11}$$

It is worth noting that in this case both $\psi_0^{(\pm)}$ belong to the Hilbert space. Thus in this case even though SUSY is unbroken, the Witten index is zero [5]. The condition (11) is trivially satisfied in case $W(x)$ is an odd function of x and throughout this paper we shall only consider superpotentials W which are odd function of x. Further, using the known eigenfunctions $\psi_n^{(-)}(x)$ of $V_-(x)$ one can immediately write down the corresponding un-normalized eigenfunctions $\psi_n^{(+)}(x)$ of $V_+(x)$. In particular, from eq. (10) it follows that the ground state of $V_+(x)$ is given by

$$\psi_0^{(+)}(x) = \frac{1}{\psi_0^{(-)}(x)} \quad , \tag{12}$$

while the excited states $\psi_n^{(+)}(x)$ are obtained from $\psi_n^{(-)}(x)$ by using the relation

$$\psi_n^{(+)}(x) = \left[\frac{d}{dx} + W(x)\right]\psi_n^{(-)}(x)\ ,\ (n \geq 1)\ . \tag{13}$$

Thus by starting from an exactly solvable periodic potential $V_-(x)$, one gets another isospectral periodic potential $V_+(x)$. As emphasized previously, if $V_-(x)$ is not self-isospectral, then $V_+(x)$ is a new solvable periodic potential!

3. Lamé Potentials (p,0) and Their Supersymmetric Partners:

The supersymmetric quantum mechanics formalism of the previous section will now be applied to the Lamé potentials $ma(a+1)\,\mathrm{sn}^2(x,m)$. Analytic solutions are known for integer values of a [2], and the supersymmetric partner potentials can be readily computed. We first discuss the results for small integer values of a, and then present some eigenstate results for arbitrary integer values of a.

A. Lamé potentials with a = 1,2,3 :

a=1: The $a=1$ Lamé potential $V_- = 2m\,\mathrm{sn}^2(x) - m$ is known to be self-isospectral [5] since its SUSY partner satisfies $V_+(x) = V_-(x - K(m))$. Both $V_+(x)$ and $V_-(x)$ have one energy band ranging from energy 0 to energy $1-m$, with a continuum starting at energy 1 [2]. Note that at $m=0$ one has energy eigenvalues at $0, 1$ as expected for a rigid rotator and as $m \to 1$, one gets $V_-(x) \to 1 - 2\mathrm{sech}^2 x$, the band width $1-m$ vanishes as expected, and one has an energy level at $E=0$.

a=2: For the $a=2$ case, the Lamé potential (2) has 2 bound bands and a continuum band. The energies and wave functions of the five band edges are well known [2, 3]. The lowest energy band ranges from $2+2m-2\delta$ to $1+m$, the second energy band ranges from $1+4m$ to $4+m$ and the continuum starts at energy $2+2m+2\delta$, where $\delta = \sqrt{1-m+m^2}$. The wave functions of all the band edges are given in Table 1. Note that in the interval $2K(m)$ corresponding to the period of the Lamé potential, the number of nodes increases with energy. In order to use the SUSYQM formalism, we must shift the Lamé potential by a constant to ensure that the ground state i.e. (the lower edge of the lowest band) has energy $E=0$. As a result, the potential

$$V_-(x) = -2 - 2m + 2\delta + 6m\mathrm{sn}^2(x) \tag{14}$$

has its ground state energy at zero with a corresponding un-normalized wave function [2]

$$\psi_0^{(-)}(x) = 1 + m + \delta - 3m\mathrm{sn}^2(x) \quad . \tag{15}$$

The corresponding superpotential is

$$W = -\frac{d}{dx}\log\psi_0^{(-)}(x) = \frac{6m\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x)}{\psi_0^{(-)}(x)}\,, \tag{16}$$

and hence the partner potential $V_+(x)$ for the potential $V_-(x)$ given in eq. (14) is

$$V_+(x) = -V_-(x) + \frac{72m^2\text{sn}^2(x)\text{cn}^2(x)\text{dn}^2(x)}{[1+m+\delta-3m\text{sn}^2(x)]^2} \quad . \tag{17}$$

Although the SUSYQM formalism guarantees that the potentials $V_\pm$ are isospectral, they are not self-isospectral, since they do not satisfy eq. (3) [9]. Therefore, $V_+(x)$ as given by eq. (17) is a new periodic potential which is strictly isospectral to the potential (14) and hence it also has 2 bound bands and a continuum band. In Fig. 1 we have plotted the potentials $V_\pm(x)$ corresponding to $a = 2$ for three different values of the parameter m. The values are $m = 0.5, 0.8, 0.998$. The difference in shape between $V_-(x)$ and $V_+(x)$ is manifest from the figures, especially for large m. Using eqs. (12) and (13) and the known eigenstates of $V_-(x)$, we can immediately compute all the band-edge Bloch wave functions for $V_+(x)$. In Table 1 we have given the energy eigenvalues and wave functions for the isospectral partner potentials $V_\pm(x)$. At $m = 0$ one has energy eigenvalues $0, 1, 4$ as expected for a rigid rotator. As $m \to 1$, one gets $V_-(x) \to 4 - 6\text{sech}^2 x$, the band widths vanish as expected, and one has two energy levels at $E = 0, 3$, with a continuum above $E = 4$.

a=3: For the $a = 3$ Lamé potential, the ground state wave function is

$$\psi_0^{(-)}(x) = \text{dn}(x)[2m+\delta_1+1-5m\,\text{sn}^2(x)] \quad ,$$

the corresponding superpotential is [9]

$$W = \frac{m\,\text{sn}(x)\text{cn}(x)}{\text{dn}(x)} \, \frac{[2m+\delta_1+11-15m\,\text{sn}^2(x)]}{[2m+\delta_1+1-5m\,\text{sn}^2(x)]} \quad , \tag{18}$$

and the partner potentials $V_\pm(x)$ are [9]

$$V_-(x) = -2-5m+2\delta_1+12m\;\text{sn}^2(x) \;,\; \delta_1 \equiv \sqrt{1-m+4m^2} \;, \tag{19}$$

and

$$V_+(x) = -V_-(x) + \frac{2m^2\text{sn}^2(x)\text{cn}^2(x)}{\text{dn}^2(x)} \frac{[2m+\delta_1+11-15m\;\text{sn}^2(x)]^2}{[2m+\delta_1+1-5m\;\text{sn}^2(x)]^2} \quad . \tag{20}$$

Clearly, the potential $V_-(x)$ is not self-isospectral. In fact, $V_-(x)$ and $V_+(x)$ are distinctly different periodic potentials which have the same seven band edges corresponding to three bound bands and a continuum band [2]. In Fig. 2 we have plotted the potentials $V_\pm(x)$ corresponding to $a = 3$ for several different values of the parameter m. The values of m are 0.5, 0.8, 0.998. It is clear from the figure that the potentials $V_+(x)$ and $V_-(x)$ have different shapes and are far from being self-isospectral. Using eqs. (12) and (13) and the known eigenstates of $V_-(x)$, we can immediately compute all the 7 band edges corresponding to the known 3 bound bands and a continuum band [2, 3]. For example, the ground state $\psi_0^{(+)}$ is given by

$$\psi_0^{(+)}(x) = \frac{1}{\psi_0^{(-)}(x)} = \frac{1}{\mathrm{dn}(x)[1 + 2m + \delta_1 - 5m\ \mathrm{sn}^2(x)]} \ . \tag{21}$$

The wave functions for the remaining six states are similarly written down by using eq. (13). These are shown in Table 2. The band edge energies for the $a = 3$ Lamé potential (12,0) as a function of the elliptic modulus parameter m are plotted in Fig. 3. Note that at $m = 0$ one has energy eigenvalues at $0, 1, 4, 9$ as expected for a rigid rotator and as $m \to 1$, one gets $V_-(x) \to 9 - 12\mathrm{sech}^2 x$, the band widths vanish as expected, and one has three energy levels at $E = 0, 5, 8$ with a continuum above $E = 9$.

B. Results for general integer values of a:

The extension to higher values of a is straightforward. It is possible to make several general comments about the form of the band edge wave functions for the partner potentials $V_+(x)$. This is most conveniently done by separately discussing the cases of even and odd values of a.

a= even integer: For a even, say $a = 2N$, it is known [2] that there are $N + 1$ solutions of the form $F_N(\mathrm{sn}^2 x)$, and N solutions each of the three forms

$$\mathrm{sn}x\ \mathrm{cn}x\ F_{N-1}(\mathrm{sn}^2 x)\ ,\ \mathrm{sn}x\ \mathrm{dn}x\ F_{N-1}(\mathrm{sn}^2 x)\ ,\ \mathrm{cn}x\ \mathrm{dn}x\ F_{N-1}(\mathrm{sn}^2 x)\ .$$

Here F_r denotes a polynomial of degree r in its argument. The ground state $\psi_0^-(x)$ (which is the lower edge of the lowest band) is of the form $F_N(\mathrm{sn}^2 x)$. It is easily checked using eq. (13) that the corresponding partner potential $V_+(x)$ has N solutions each of the four forms

$$\frac{\mathrm{dn}x\ G_N(\mathrm{sn}^2 x)}{\psi_0^-(x)}\ ,\ \frac{\mathrm{sn}x\ G_N(\mathrm{sn}^2 x)}{\psi_0^-(x)}\ ,\ \frac{\mathrm{cn}x\ G_N(\mathrm{sn}^2 x)}{\psi_0^-(x)}\ ,\ \frac{\mathrm{sn}x\ \mathrm{cn}x\ \mathrm{dn}x\, G_{N-1}(\mathrm{sn}^2 x)}{\psi_0^-(x)}\ ,$$

while the ground state is given by $\psi_0^+(x) = 1/\psi_0^-(x)$.

a= odd integer: For a odd, say $a = 2N + 1$, it is known [2] that the Lamé potentials have $N + 1$ solutions each of the three forms

$$\mathrm{sn}x\ F_N(\mathrm{sn}^2 x)\ ,\ \mathrm{cn}x\ F_N(\mathrm{sn}^2 x)\ ,\ \mathrm{dn}x\ F_N(\mathrm{sn}^2 x)$$

and N solutions of the form

$$\mathrm{sn}x\ \mathrm{cn}x\ \mathrm{dn}x\ F_{N-1}(\mathrm{sn}^2 x).$$

The ground state $\psi_0^-(x)$ is of the form dn$x$ $F_N(\text{sn}^2x)$. We can then easily deduce that the corresponding partner potentials $V_+(x)$ will have $N+1$ solutions each of the two forms

$$\frac{\text{sn}x\ G_{N+1}(\text{sn}^2x)}{\psi_0^-(x)}\ ,\ \frac{\text{cn}x\ G_{N+1}(\text{sn}^2x)}{\psi_0^-(x)}\ ,$$

and N solutions each of the two forms

$$\frac{\text{dn}x\ G_{N+1}(\text{sn}^2x)}{\psi_0^-(x)}, \frac{\text{sn}x\ \text{cn}x\ \text{dn}x\ G_N(\text{sn}^2x)}{\psi_0^-(x)},$$

while as usual, the ground state is given by $\psi_0^+(x) = 1/\psi_0^-(x)$.

In summary, for integral a, Lamé potentials with $a \geq 2$ are not self isospectral. They have distinct supersymmetric partner potentials even though both potentials have the same $(2a+1)$ band edge eigenvalues.

4. Associated Lamé Potentials (p,q) and Their Supersymmetric Partners:

In contrast to the Lamé potentials discussed above, there seems to be no systematic treatment of associated Lamé potentials in the literature. Therefore, we will first devote some time to discuss the properties of associated Lamé potentials, show that they are quasi exactly solvable and then proceed to construct and study their isospectral supersymmetric partner potentials.

A. Description of associated Lamé potentials:

As mentioned before, we will refer to the associated Lamé potentials given by eq. (5) or the equivalent form eq. (6) as the (p,q) potential. The special cases $p=0$, as well as $q=0$, correspond to ordinary Lamé potentials.

In general, for any value of p and q, the associated Lamé potentials have a period $2K(m)$ since

$$\text{sn}(x+2K) = -\text{sn}(x)\ ,\ \text{cn}(x+2K) = -\text{cn}(x)\ ,\ \text{dn}(x+2K) = \text{dn}(x)\ .$$

However, for the special case $p=q$, eq. (6) shows that the period is $K(m)$. From a physical viewpoint, if one thinks of a Lamé potential $(p,0)$ as due to a one-dimensional regular array of atoms with spacing $2K(m)$, and "strength" p, then the associated Lamé potential (p,q) results from two alternating types of atoms spaced by $K(m)$ with "strengths" p and q respectively. If the two types of atoms are identical [which makes $p=q$], one expects a potential of period $K(m)$.

Extrema (defined for this discussion as either local or global maxima and minima) of associated Lamé potentials are easily found by setting $dV(x)/dx = 0$. This gives

$$\text{sn}(x)\ \text{cn}(x)\ [p\ \text{dn}^4(x) - q(1-m)] = 0\ \ .$$

Extrema occur when (i) $\text{sn}(x) = 0$, that is $x = 0, \ \pm 2K(m), \ \pm 4K(m), \ldots$; (ii) $\text{cn}(x) = 0$, that is $x = \ \pm K(m), \ \pm 3K(m), \ldots$; (iii) $\text{dn}^4(x) = (1-m)q/p$. At the points specified by (i) and (ii), one always has extrema and $V(x)$ has values pm and qm. In addition, since $\text{dn}^4(x)$ has a minimum value $(1-m)^2$ and a maximum value unity [1], condition (iii) yields additional extrema provided

$$(1-m)^2 \leq (1-m)q/p \leq 1 \ .$$

For given fixed values of q and m, this condition has a solution provided p lies in the critical range

$$q(1-m) \leq p \leq q/(1-m) \ .$$

Alternatively, for given fixed values of p and q with $p \geq q$, condition (iii) has a solution provided m is greater than the critical value $1 - q/p$.

The associated Lamé potentials for $q = 2, m = 0.5$ and several values of p are plotted in Fig. 4(a). In the critical range of p values $1 \leq p \leq 4$, one expects additional extrema, and these are clearly seen in Fig. 4(a). In general the period is $2K(0.5) = 3.708$, but for $p = q = 2$, the period $K(0.5)$ is evident. Note that as p increases, any given extremum changes character. For example, at $x = 0$, as p increases, one goes from a maximum to a local minimum to an absolute minimum. In Fig. 4(b) we have plotted associated Lamé potentials for $p = 4, q = 2$ and several values of m. As expected from the above discussion, one always sees extrema at the points specified by conditions (i) and (ii), and additional extrema coming from condition (iii) are evident for $m \geq 1/2$.

B. Solutions of the associated Lamé equation - parabolas of solvability:

The associated Lamé equation is just the Schrödinger equation for the potential in eq. (5).

$$-\frac{d^2\psi}{dx^2} + [pm \ \text{sn}^2(x) + qm \ \frac{\text{cn}^2(x)}{\text{dn}^2(x)} - E]\psi = 0 \ . \tag{22}$$

On substituting

$$\psi(x) = [\text{dn}(x)]^{-b} \ y(x) \, , \tag{23}$$

it is easily shown that $y(x)$, satisfies the Hermite elliptic equation [3]

$$y''(x) + 2bm\frac{\text{sn}(x)\text{cn}(x)}{\text{dn}(x)}y'(x) + [\lambda - (a+1-b)(a+b)m\text{sn}^2(x)]y(x) = 0 \, , \tag{24}$$

where

$$p = a(a+1) \ , \ q = b(b+1) \ , \ E = \lambda + mb^2 \, . \tag{25}$$

On further substituting

$$\mathrm{sn}(x) = \sin t\,, \quad y(x) \equiv z(t)\,, \tag{26}$$

one obtains Ince's equation

$$(1 - m\sin^2 t)z''(t) + (2b-1)m\sin t\cos t\ z'(t) + [\lambda - (a+1-b)(a+b)m\sin^2 t]z(t) = 0\,, \tag{27}$$

which is a well known QES equation [3]. In particular, on substituting

$$\cos t = u\,, \quad z(t) \equiv w(u)w(u) = \sum_{n=0}^{\infty} \frac{u^n R_n}{n!}\,, \tag{28}$$

it is easily shown that R_n satisfies a three-term recursion relation. In particular if $a+b+1 = n$ ($n = 1,2,3,...$) then one obtains n QES solutions. Actually n QES solutions are also obtained in case $b - a = -n(n = 1,2,3,...)$ but since q is unchanged under $b \to -b-1$, no really new solutions are obtained in this case. The QES solutions for $n = 1,2,3,4,5$ are given in Table 3. In particular, for any given choice of $p = a(a+1)$, Table 3 lists the eigenstates of the associated Lamé equation for various values of q.

For $q = a(a-1)$, there is just one eigenstate with energy ma^2 and wave function $\psi = dn^a(x)$. Since the wave function has period $2K(m)$ and is nodeless, this is clearly the ground state wave function of the $(a(a+1), a(a-1))$ potential for any real choice of the parameter a. The equations $p = a(a+1)$ and $q = a(a-1)$ are the parametric forms of the equation of the parabola $(p-q)^2 = 2(p+q)$, which is plotted in Fig. 5 and labeled $P1$. For any point on the parabola, one knows the ground state wave function and energy $E_0 = ma^2$. The parabola $P1$ includes the points (2,0) and (6,2).

For $q = (a-1)(a-2)$, we see from Table 3 that two eigenstates with energies $1 + m(a-1)^2$ and $1+ma^2$ are known. Since they have period $4K(m)$ and just one node in the interval $L = 2K(m)$, they must correspond to the first and second band edge energies E_1 and E_2 of the $(a(a+1), (a-1)(a-2))$ potential. Eliminating a from the equations $p = a(a+1)$ and $q = (a-1)(a-2)$ gives the "parabola of solvability" $(p-q)^2 = 8(p+q) - 12$, which is plotted in Fig. 5 and labeled $P2$. This parabola includes the points (2,0) and (6,0) which correspond to Lamé potentials. Similarly, the parabolas of solvability Pn ($n = 0,1,2,...$) corresponding to $q = (a-n+1)(a-n)$ in Table 3 are plotted. n eigenstates are known for any point on the parabola of solvability Pn.

C. Supersymmetric partner potentials:

It is easily checked from Table 3 that the solution corresponding to $q = a(a-1)$ as well as one of the $q = (a-2)(a-3)$ solutions are nodeless and correspond to the ground state. Hence, for these cases, one can obtain the superpotential and hence the partner potential V_+ and enquire if V_- is self-isospectral. For example, consider the case of $p = a(a+1), q = a(a-1)$ in which case W is given by

$$W \equiv -\frac{\psi_0'(x)}{\psi_0(x)} = am\frac{\mathrm{sn}(x)\mathrm{cn}(x)}{\mathrm{dn}(x)} \,, \tag{29}$$

so that the corresponding partner potentials are

$$\begin{aligned} V_- &= (a-1)am\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} + ma(a+1)\mathrm{sn}^2(x) - ma^2 \quad , \\ V_+ &= a(a+1)m\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} + m(a-1)a\mathrm{sn}^2(x) - ma^2 \,. \end{aligned} \tag{30}$$

It is easily seen that these partner potentials satisfy eq. (3), are consequently self-isospectral and SUSY gives nothing new in this case. It is amusing to note that the superpotential W obtained here was in fact discussed in ref. [5] (see their eq. (32)).

Let us now consider the SUSY partner potential computed from the ground state for the $p = a(a+1), q = (a-2)(a-3)$ case. It is given by (see Table 3)

$$\psi_0(x) = [m(a-1) - 1 - \delta_1 + m(2a-1)\mathrm{sn}^2(x)](\mathrm{dn}(x))^{a-2} \quad , \tag{31}$$

where $\delta_1 = \sqrt{1 - m + m^2(a-1)^2}$. The corresponding superpotential W turns out to be

$$W = \frac{m(a-2)\mathrm{sn}(x)\mathrm{cn}(x)}{\mathrm{dn}(x)} - \frac{2m(2a-1)\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x)}{[m(1-a) - 1 - \delta_1 + m(2a-1)\mathrm{sn}^2(x)]} \,. \tag{32}$$

Hence the corresponding partner potentials are

$$V_-(x) = ma(a+1)\mathrm{sn}^2(x) + m(a-3)(a-2)\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} - 2 - m(a^2 - 2a + 2) + 2\delta_1 \,, \tag{33}$$

$$V_+(x) = -V_-(x) + 2W^2(x) \,. \tag{34}$$

It is easily checked that these potentials are not self-isospectral since they do not satisfy the condition (3). Thus one has discovered a whole class of new elliptic periodic potentials $V_+(x)$ as given by eq. (34) for which three states are analytically known no matter what a is. In particular, the energy eigenfunctions for V_+ of these three states are easily obtained by using the corresponding energy eigenstates of V_- as given in Table 3 and using eqs. (12) and (13).

5. Associated Lamé Potentials with Special Values of p and q:

We shall now discuss associated Lamé potentials $(a(a+1), b(b+1))$, where a and b are either both positive integers or half-integers. In most cases, we show that although several band edge energies are exactly known from Table 3, one usually does not know all the band edge energies, that is one has a quasi exactly solvable problem. However, in the special case of $p = q (a = b = \text{integer})$, we show that all the band edge eigenstates can be obtained and one has an exactly solvable periodic problem.

A. a,b = integer, a $\neq$ b:

First, let us note that the Lamé potentials $(a(a+1), 0)$ are in this category when a = integer and $b = 0$. For example, when $a = 3$, one has the (12,0) potential. We see from Fig. 5 that two parabolas of solvability pass through the point (12,0). From Table 3 it follows that 3 band edges of period $2K(m)$ are obtained from $q = (a-2)(a-3)$ and 4 band edges of period $4K(m)$ are obtained from $q = (a-3)(a-4)$. Altogether, arranging in order of increasing nodes, one has 7 band edges with periods $2K, 4K, 4K, 2K, 2K, 4K, 4K$ with $0, 1, 1, 2, 2, 3, 3$ nodes respectively. There are no missing states, and as discussed in Sec. 3A, this gives three bound bands and a continuum band.

As a second example with $q \neq 0$, consider the (6,2) associated Lamé potential, that is $p = 6, q = 2$. In this case, taking $a = 2$, one can get five band edges from Table 3 - one solution of period $2K$ is obtained from $q = a(a-1)$, while the remaining four solutions of period $4K$ are obtained from $q = (a-3)(a-4)$. The eigenvalues and eigenfunctions are given in Table 4 along with the number of nodes in one period $2K$. It is clear that there are two solutions of period $2K$ with 2 nodes in the interval $2K$ which have to be present but have not been obtained. This is also clear from the $m = 0$ limit, since the energies from Table 5 are 0,1,1,9,9 and the states at 4,4 are missing. Thus, this is a QES problem. Fig. 6 illustrates the (6,2) associated Lamé potential and its supersymmetric partner for three choices of m. The self-isospectral nature of the (6,2) potential is evident from Fig. 6 - it also follows from eqs. (30) with $a = 2$. The band edge energies for the (6,2) associated Lamé potential as a function of the elliptic modulus parameter m is shown in Fig. 7. The two unobtained band edges of period $2K$ will have energies $E = 4$ at $m = 0$ and $E = 3$ at $m = 1$.

Let us now discuss the general associated Lamé potential $(a(a+1), b(b+1))$. Without any loss of generality let us assume that $a > b$. Using Table 3, we obtain $(a-b)$ states of period $2K(4K)$ for $q = [a-(a-b)][a-(a-b-1)]$ for $(a-b)$ odd (even), and $(a+b+1)$ states of period $4K(2K)$for $q = [a-(a+b+1)][a-(a+b)]$ for $(a-b)$ odd (even). It can be established that some states are missing by looking at the node structure as well as the $m = 0$ limit. Hence we again have a QES problem.

B. a = b = integer:

Let us now discuss the special case of $p = q = a(a+1),\ a = 1, 2, ...$. In this case the associated Lamé potential (5) has period K, rather than $2K$. It then follows from the oscillation theorem that with increasing energy, the band edges must have periods $K, 2K, 2K, K, K, ...$ and in the $m = 0$ limit the eigenvalues must go to $E = 0, 4, 16, 36, ...$ with all nonzero eigenvalues being doubly degenerate. It is easy to check from Table 5 that one case for which we already have exact results is when $p = q = 2$. In particular, consider the special case $a = 1$, for which $V_-(x)$ of eq. (33) takes the form

$$V_-(x) = 2m\mathrm{sn}^2(x) + 2m\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} - 2 - m + 2\sqrt{1-m}\,. \tag{35}$$

Using Table 5, we can calculate three energy eigenvalues and eigenfunctions of V_- taking $a = 1$ in $q = (a-2)(a-3)$. These are given in Table 5. Whereas the ground state is of period K, the next two states in Table 5 indeed have period $2K$. Using $a = 1$ in eqs. (31) to (34), we find that the corresponding SUSY partner potential is

$$V_+(x) = 2 - m - 2\sqrt{1-m} - \frac{8\sqrt{1-m}m^2\mathrm{sn}^2(x)\mathrm{cn}^2(x)}{[\mathrm{dn}^2(x) + \sqrt{1-m}]^2}\,. \tag{36}$$

Are the potentials $V_\pm(x)$ self-isospectral? Using the relations

$$\mathrm{sn}(x + K(m)/2) = (1 + \sqrt{1-m})^{1/2}\left[\frac{\sqrt{1-m}\ \mathrm{sn}(x) + \mathrm{cn}(x)\mathrm{dn}(x)}{\mathrm{dn}^2(x) + \sqrt{1-m}}\right], \tag{37}$$

$$\mathrm{cn}(x + K(m)/2) = (1 + \sqrt{1-m})^{1/2}(1-m)^{1/4}\left[\frac{(1 + \sqrt{1-m})^{1/2}\mathrm{cn}(x) - \mathrm{sn}(x)\mathrm{dn}(x)}{\mathrm{dn}^2(x) + \sqrt{1-m}}\right], \tag{38}$$

$$\mathrm{dn}(x + K(m)/2) = (1-m)^{1/4}\left[\frac{(1 + \sqrt{1-m})\mathrm{dn}(x) - m\mathrm{sn}(x)\mathrm{cn}(x)}{\mathrm{dn}^2(x) + \sqrt{1-m}}\right], \tag{39}$$

a little algebra reveals that indeed $V_\pm$ are self-isospectral and satisfy eq. (3).

Are the higher members of the $p = q$ family (i.e. $p = q = 6, 12, 20, ...$) also self-isospectral? If our experience with the Lamé case is any guide then we would doubt it. Indeed, we will now show that the (6,6) associated Lamé potential is not self-isospectral. We get five band edges analytically from Table 3. In particular, take $a = 2$ and consider the case of $q = (a-4)(a-5)$, for which we know two eigenstates as given in Table 3. In fact, in this case three more eigenstates can be analytically obtained but the corresponding eigenvalues and eigenfunctions have not been given in Table 3 since the energy eigenvalues are solutions of a cubic equation whose exact solution for arbitrary a can not be written in a compact form. However, for $a = 2$, we are able to solve the cubic equation and obtain the three eigenvalues in a closed simple form. In particular consider an ansatz of the form

$$y = A + B\mathrm{sn}^2 x + D\mathrm{sn}^4 x\,. \tag{40}$$

On substituting this ansatz in eq. (24) it is easy to show that the energy eigenvalue $\lambda(= E - m(a-4)^2)$ must obey the cubic equation

$$\lambda^3 + [28m - 20 - 12am]\lambda^2 + [64 - 304m + 160ma + 32m^2(a-2)(a-3)]\lambda - 64m(2a-3)(2-2m+ma) = 0\,. \tag{41}$$

The solution of this equation is in general quite lengthy but in the special case of $a = 2$ this cubic equation is easily solved yielding three eigenvalues in a compact form. On combining them with the two levels given in Table 3, we obtain the eigenvalues and eigenfunctions of all the five band edges for the case $p = q = 6$. These are given in Table 6. We have also verified that these five eigenstates in ascending order of energy indeed have periods $K, 2K, 2K, K, K$ respectively and taht the energy eigenvalues have expected limits at $m = 0$. In particular the associated Lamé potential $V_-(x)$ is

$$V_-(x) = 6m\mathrm{sn}^2(x) + 6m\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} - 8 - 2m + 2\delta_8\,, \tag{42}$$

whose ground state energy is zero while the corresponding eigenfunction ψ_0^- is

$$\psi_0^-(x) = \frac{\left[1 - (4 - m - \delta_8)\mathrm{sn}^2(x) + (4 - 2m - \delta_8)\mathrm{sn}^4(x)\right]}{\mathrm{dn}^2(x)}\,, \quad \delta_8 = \sqrt{16 - 16m + m^2}\,. \tag{43}$$

Hence the corresponding superpotential is

$$W(x) = \frac{-2m\mathrm{sn}(x)\mathrm{cn}(x)}{\mathrm{dn}(x)} + \frac{2\mathrm{sn}(x)\mathrm{cn}(x)}{\mathrm{dn}(x)\psi_0^-(x)}\left[(4 - m - \delta_8) - 2(4 - 2m - \delta_8)\mathrm{sn}^2(x)\right]\,, \tag{44}$$

and the partner potential $V_+(x)$ which is isospectral to $V_-(x)$ is

$$V_+(x) = -V_-(x) + 2W^2(x)\,. \tag{45}$$

It is not difficult to see that the W as given by eq. (44) does not satisfy the self-isospectral condition (4) and hence unlike the $p = q = 2$ case, the $p = q = 6$ potential is *not* self-isospectral. In Fig. 8, we have plotted the potentials $V_\pm(x)$ corresponding to $p = q = 6$ for several different values of the parameter m. The figures confirm that the potentials are far from being self-isospectral. Thus we have obtained a new exactly solvable periodic potential (45) which has two bound bands and a continuum band, with five band edges and the corresponding eigenfunctions being exactly known using Table 6 and eqs. (12) and (13). In Fig. 9, we plot the band edge energies for the (6,6) potential as a function of the elliptic modulus parameter m.

It is also clear from here that even the higher associated Lamé potentials with $p = q = 12, 20, ...$ which have 7,9,... band edges are also exactly solvable in principle and none of them will be self-isospectral, so that in each case one obtains a new exactly solvable periodic potential. In particular, for $p = q = n(n+1)$ there will be $(2n+1)$ band edges in both $V_{\pm}(x)$ whose energy eigenvalues can be obtained from Table 3 when q has the form $[n-2n][n-(2n+1)]$. Out of the $(2n+1)$ band edges in $V_-(x)$, $(n+1)$ solutions (including the ground state) have the form $\frac{F_n(\mathrm{sn}^2 x)}{\mathrm{dn}^n x}$ while n solutions have the form $F_{n-1}(\mathrm{sn}^2 x)\frac{\mathrm{sn}x\mathrm{cn}x}{dn^n x}$. On the other hand, as far as the $(2n+1)$ solutions of the partner potential V_+ are concerned, there are n states each of the two forms

$$\frac{\mathrm{sn}x\mathrm{cn}x G_n(\mathrm{sn}^2 x)}{\mathrm{dn}^{2n-1}x\psi_0^-(x)}, \frac{G_{n+1}(\mathrm{sn}^2 x)}{\mathrm{dn}^{2n-1}x\psi_0^-(x)},$$

while the ground state (i.e. the lower edge of the lowest band) is given by $\psi_0^+(x) = 1/\psi_0^-(x)$.

C. a,b = half-integer:

Let us now specialize to the case when both a, b are half integral with $a > b$. As an illustration, let us first consider the case of $a = 3/2, b = 1/2$ so that $p = 15/4, q = 3/4$. In this case, the oscillation theorem requires band edges with periods $2K, 4K, 4K, 2K, 2K, ...$ Using Table 3 and Fig. 5, we see one gets three eigenstates when $q = (a-2)(a-3)$ with $a = 3/2$, all with period $2K$. The ground state is at $E_0 = \frac{9m}{4}$ while there are two degenerate levels at $E_3 = E_4 = 4 + \frac{m}{4}$. To understand this degeneracy better, let us go along the parabola of solvability $P2$ given by $q = (a-2)(a-3)$. The band gap is given by $\Delta_2 \equiv |-2+m+2\sqrt{1-m+m^2(a-1)^2}|$ and is plotted in Fig. 10. It vanishes at $a = 3/2$ (15/4, 3/4) potential, and has the correct values $\Delta_2 = 2\sqrt{1-m+m^2} - 2 + m$ for $a = 2$ and $\Delta_2 = 2\sqrt{1-m+4m^2} - 2 + m$ for $a = 3$ which correspond to the (6,0) and (12,0) Lamé potentials. The vanishing of Δ_2 at $a = 3/2$ occurs because the eigenfunctions corresponding to E_3 and E_4 cross over as one goes along the parabola $P2$.

These arguments are easily generalised in case $p = (n+1/2)(n+3/2), q = (k+1/2)(k+3/2)$ with $n > k$. The energy eigenvalues of $(n-k)$ states can be obtained by using Table 3 in case q is of the form $q = [n+1/2-(n-k)][n+1/2-(n-k-1)]$ and the corresponding eigenstates have period $2K(4K)$ depending on whether $(n-k)$ is odd (even). On the other hand, the energy of $(n+k+2)$ states is obtained when q is of the form $q = [n+1/2-(n+k+2)][n+1/2-(n+k+1)]$ and these states have the same period $2K(4K)$ as the $n-k$ states when $n-k$ is odd (even). It turns out that the $n-k$ solutions are in fact common in both and so we only obtain the energy of the $n+k+2$ band edges and all of them have the same period $2K(4K)$ depending on if $n-k$ is odd (even) so that it is only a QES problem and not an exactly solvable problem as one is unable to obtain a single eigenstate with period $4K(2K)$ in case $n-k$ is odd (even).

We would like to point out some of the pecularities of the spectrum in these cases. For example, in case $(p,q) = (35/4, 3/4), (63/4, 3/4), (99/4, 3/4)$... then one finds that 3,4,5,... QES energy levels of period $4K, 2K, 4K, ...$ respectively are analytically known of which the one at the highest energy is doubly degenerate. As an illustration, in Table 7 we have given the 4 QES energy eigenstates all of period $2K$ for the $(63/4, 3/4)$ potential. The interesting point about this case is that the partner potentials $V_{\pm}(x)$ are not self-isospectral and hence one has discovered a new QES potential where 4 band edges of period $2K$ and the corresponding eigenfunctions are explicitly known. Of these, the one at $E = 16 + \frac{m}{4}$ is doubly degenerate, again due to crossover of energy levels. Using the ground state wave function, the superpotential is computed to be

$$W = \frac{3m}{2}\frac{\text{sn}(x)\text{cn}(x)}{\text{dn}(x)} - \frac{24m\text{sn}(x)\text{cn}(x)\text{dn}(x)}{[12m\text{sn}^2(x) - 2 - 5m - \sqrt{4 - 4m + 25m^2}]} \quad . \tag{46}$$

Using eqs. (12) and (13) the eigenstates of the SUSY partner potential V_+ are then determined.

6. Comments and Conclusions:

In this paper, we have discussed solutions of the type given in Table 3, which correspond to the parabolas of solvability shown in Fig. 5. Lamé potentials $(p, 0)$ with $p = a(a+1)$ and integer a, always have two parabolas of solvability passing through - one parabola gives all states of period $2K$ and the other gives all states of period $4K$. This provides a deeper understanding of why such Lamé potentials are fully solvable [12]. Similarly, we have obtained eigenstates for a large class of associated Lamé potentials (p, q). Further, using the formalism of supersymmetric quantum mechanics, we have been able to discover many new exactly solvable and quasi exactly solvable periodic potentials involving Jacobi elliptic functions. This is a very substantial improvement over the currently known small number of exactly solvable periodic problems.

Acknowledgements: One of us (US) acknowledges the hospitality of the Institute of Physics, Bhubaneswar and partial financial support from the U.S. Department of Energy. Both authors thank the Mehta Research Institute, Allahabad for hospitality.

References

[1] For the properties of Jacobi elliptic functions, see, for example, I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, 1980). The modulus parameter m is often called k^2 in the mathematics literature. The related complementary quantity $(1-m)$ is often called k'^2.

[2] F.M. Arscott, *Periodic Differential Equations* (Pergamon, Oxford, 1981); E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis* (Cambridge Univ. Press, Cambridge, 1980).

[3] W. Magnus and S. Winkler, *Hill's Equation* (Wiley, New York, 1966).

[4] See, for example, F. Cooper, A. Khare and U. P. Sukhatme, Phys. Rep. **251**, 267 (1995).

[5] G. Dunne and J. Feinberg, Phys. Rev. **D57**, 1271 (1998).

[6] G. Dunne and J. Mannix, Phys. Lett. **B428**, 115 (1998).

[7] H. Braden and A. Macfarlane, J. Phys. **A18**, 3151 (1995).

[8] Y. Brihaye and M. Godart, J. Math. Phys. **34**, 5283 (1993); Y. Brihaye and S. Braibant, J. Math. Phys. **34**, 2107 (1993).

[9] U. Sukhatme and A. Khare, E-print quant-ph/9902072 (1999).

[10] A. Turbiner, Comm. Math. Phys. **118**, 467 (1988).

[11] A.G. Ushveridze, *Quasi Exactly Solvable Models in Quantum Mechanics* (Institute of Physics, Bristol, 1993).

[12] Some isolated solutions of the Schrödinger equation for Lamé potentials with half-integer values of a are also known [2], but they have certain peculiar features which require further study.

Table Captions

Table 1: The eigenvalues and eigenfunctions for the 5 band edges corresponding to the $a = 2$ Lamé potential V_- which gives $(p,q) = (6,0)$ and its SUSY partner V_+. Here $B \equiv 1 + m + \delta$ and $\delta \equiv \sqrt{1 - m + m^2}$. The potentials $V_\pm$ have period $L = 2K(m)$ and their analytic forms are given by eqs. (14) and (17) respectively. The periods of various eigenfunctions and the number of nodes in the interval L are tabulated.

Table 2: The eigenvalues and eigenfunctions for the 7 band edges corresponding to the $a = 3$ Lamé potential V_- which gives $(p,q) = (12,0)$ and its SUSY partner V_+. Here $\delta_1 \equiv \sqrt{1 - m + 4m^2}$; $\delta_2 \equiv \sqrt{4 - m + m^2}$; $\delta_3 \equiv \sqrt{4 - 7m + 4m^2}$. The potentials $V_\pm$ have period $L = 2K(m)$ and their analytic forms are given by eqs. (19) and (20) respectively. The periods of various eigenfunctions and the number of nodes in the interval L are tabulated.

Table 3: Eigenvalues and eigenfunctions for various associated Lamé potentials (p,q) with $p = a(a+1)$ and $q = (a-n+1)(a-n)$ for $n = 1, 2, 3, ...$ The periods of various eigenfunctions and the number of nodes in the interval $2K(m)$ are tabulated. Here $\delta_4 \equiv \sqrt{1 - m + m^2(a-1)^2}$; $\delta_5 \equiv \sqrt{4 - 7m + 2ma + m^2(a-2)^2}$; $\delta_6 \equiv \sqrt{4 - m - 2ma + m^2(a-1)^2}$; $\delta_7 \equiv \sqrt{9 - 9m + m^2(a-2)^2}$.

Table 4: The five eigenvalues and eigenfunctions for the self-isospectral associated Lamé potential corresponding to $a = 2, b = 1$ which gives $(p,q) = (6,2)$. The potential is $V_-(x) = 6m\,\mathrm{sn}^2(x) + 2m\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} - 4m$, and has period $2K(m)$. The number of nodes in the interval $2K(m)$ is tabulated.

Table 5: The three eigenvalues and eigenfunctions for the associated Lamé potential corresponding to $a = b = 1$ which gives $(p,q) = (2,2)$. The potential has period $K(m)$ and the number of nodes in the interval $K(m)$ is tabulated.

Table 6: The five eigenvalues and eigenfunctions for the associated Lamé potential corresponding to $a = b = 2$ which gives $(p,q) = (6,6)$. Here $\delta_8 \equiv \sqrt{16 - 16m + m^2}$. The number of nodes in one period $K(m)$ of the potential is tabulated.

Table 7: Energy eigenvalues and eigenfunctions for the associated Lamé potential corresponding

to $a = 7/2, b = 1/2$ which gives $(p,q) = (63/4, 3/4)$. Here $\delta_9 \equiv \sqrt{4-4m+25m^2}$; $V_-(x) = \frac{63}{4}m\,\mathrm{sn}^2(x) + \frac{3}{4}m\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} - 2 - \frac{29m}{4} + \delta_9$. The last column gives the number of eigenfunction nodes in one period $2K(m)$ of the potential.

Figure Captions

Fig. 1: The (6,0) Lamé potential $V_-(x)$ corresponding to $a = 2$ [thick line] as given by eq. (14) and its supersymmetric partner potential $V_+(x)$ [thin line] as given by eq. (17) for three choices of m (a) 0.5 (b) 0.8 (c) 0.998.

Fig. 2: The (12,0) Lamé potential $V_-(x)$ corresponding to $a = 3$ [thick line] as given by eq. (19) and its supersymmetric partner potential $V_+(x)$ [thin line] as given by eq. (20) for three choices of m (a) 0.5 (b) 0.8 (c) 0.998.

Fig. 3: Band edge energies for the (12,0) Lamé potential corresponding to $a = 3$ as a function of the elliptic modulus parameter m. This figure is drawn using the eigenvalues given in Table 2.

Fig. 4: (a) Plots of the (p,q) associated Lamé potentials for $q = 2, m = 0.5$ and several values of p. (b) Plots of the (p,q) associated Lamé potentials for $p = 4, q = 2$ and several values of m.

Fig. 5: Parabolas of solvability. This figure illustrates all associated Lamé potentials (p,q) which are quasi solvable. Each parabola corresponds to a choice of q in Table 3. Parabola Pn is for $q = (a-n+1)(a-n)$ for $n = 1, 2, 3, ...$, and one knows n eigenstates for any point on it from Table 3.

Fig. 6: The (6,2) associated Lamé potential $V_-(x)$ [thick line] and its supersymmetric partner potential $V_+(x)$ [thin line] for three choices of m (a) 0.5 (b) 0.9 (c) 0.998.

Fig. 7: Band edge energies for the associated Lamé potential (6,2) as a function of the elliptic modulus parameter m. This figure corresponds to Table 4.

Fig. 8: The (6,6) associated Lamé potential $V_-(x)$ [thick line] as given by eq. (42) and its supersymmetric partner potential $V_+(x)$ [thin line] as given by eq. (45) for three choices of m (a) 0.5 (b) 0.9 (c) 0.998.

Fig. 9: Band edge energies for the associated Lamé potential (6,6) as a function of the elliptic modulus parameter m. This figure corresponds to Table 6.

Fig. 10: Energy gap $\Delta_2 \equiv |E_4 - E_3|$ as one moves along the parabola of solvability $P2$ corresponding to $q = (a-2)(a-3)$ and $p = a(a+1)$.

Table 1: Energy Eigenstates for $V_\pm$ Corresponding to $a = 2$

E	$\psi^{(-)}$	$[B - 3m\ \mathrm{sn}^2(x)]\psi^{(+)}$	Period	Nodes
0	$m + 1 + \delta - 3m\mathrm{sn}^2(x)$	1	$2K$	0
$2\delta - 1 - m$	$\mathrm{cn}(x)\mathrm{dn}(x)$	$\mathrm{sn}(x)[6m - (m+1)B + m\mathrm{sn}^2(x)(2B - 3 - 3m)]$	$4K$	1
$2\delta - 1 + 2m$	$\mathrm{sn}(x)\mathrm{dn}(x)$	$\mathrm{cn}(x)[B + m\mathrm{sn}^2(x)(3 - 2B)]$	$4K$	1
$2\delta + 2 - m$	$\mathrm{sn}(x)\mathrm{cn}(x)$	$\mathrm{dn}(x)[B + \mathrm{sn}^2(x)(3m - 2B)]$	$2K$	2
4δ	$m + 1 - \delta - 3m\mathrm{sn}^2(x)$	$\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x)$	$2K$	2

Table 2: Energy Eigenstates for $V_\pm$ Corresponding to $a = 3$

E	$\psi^{(-)}$	$\psi_0^- \psi^{(+)}$	Period	Nodes
0	$\mathrm{dn}(x)[1 + 2m + \delta_1 - 5m\mathrm{sn}^2(x)]$	1	$2K$	0
$3 - 3m + 2\delta_1 - 2\delta_2$	$\mathrm{cn}(x)[2 + m + \delta_2 - 5m\mathrm{sn}^2(x)]$	$10m(1 - m + \delta_2 - \delta_1)\mathrm{sn}(x)\mathrm{cn}^2(x)\mathrm{dn}^2(x)$ $-(1-m)\frac{\mathrm{sn}(x)\psi_0^-\psi^-}{\mathrm{cn}(x)\mathrm{dn}(x)}$	$4K$	1
$3 + 2\delta_1 - 2\delta_3$	$\mathrm{sn}(x)[2 + 2m + \delta_3 - 5m\mathrm{sn}^2(x)]$	$10m(1 + \delta_3 - \delta_1)\mathrm{cn}(x)\mathrm{sn}^2(x)\mathrm{dn}^2(x)$ $-(1 - 2m\mathrm{sn}^2(x))\frac{\mathrm{cn}(x)\psi_0^-\psi^-}{\mathrm{sn}(x)\mathrm{dn}(x)}$	$4K$	1
$2 - m + 2\delta_1$	$\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x)$	$\mathrm{dn}^3(x)[1 + 2m + \delta_1 + (m - 2 - 2\delta_1)\mathrm{sn}^2(x)]$	$2K$	2
$4\delta_1$	$\mathrm{dn}(x)[1 + 2m - \delta_1 - 5m\mathrm{sn}^2(x)]$	$\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}^3(x)$	2K	2
$3 - 3m + 2\delta_1 + 2\delta_2$	$\mathrm{cn}(x)[2 + m - \delta_2 - 5m\mathrm{sn}^2(x)]$	$10m(1 - m - \delta_2 - \delta_1)\mathrm{sn}(x)\mathrm{cn}^2(x)\mathrm{dn}^2(x)$ $-(1-m)\frac{\mathrm{sn}(x)\psi_0^-\psi^-}{\mathrm{cn}(x)\mathrm{dn}(x)}$	$4K$	3
$3 + 2\delta_1 + 2\delta_3$	$\mathrm{sn}(x)[2 + 2m - \delta_3 - 5m\mathrm{sn}^2(x)]$	$10m(1 - \delta_3 - \delta_1)\mathrm{cn}(x)\mathrm{sn}^2(x)\mathrm{dn}^2(x)$ $-(1 - 2m\mathrm{sn}^2(x))\frac{\mathrm{cn}(x)\psi_0^-\psi^-}{\mathrm{sn}(x)\mathrm{dn}(x)}$	$4K$	3

Table 3: Some Eigenstates for Various Associated Lamé Potentials

q	E	$\mathrm{dn}^{-a}(x)\psi$	Period	Nodes
$a(a-1)$	ma^2	1	$2K$	0
$(a-1)(a-2)$	$1+m(a-1)^2$	$\frac{\mathrm{cn}(x)}{\mathrm{dn}(x)}$	$4K$	1
$(a-1)(a-2)$	$1+ma^2$	$\frac{\mathrm{sn}(x)}{\mathrm{dn}(x)}$	$4K$	1
$(a-2)(a-3)$	$2+m(a^2-2a+2)\pm 2\delta_4$	$\frac{[m(2a-1)\mathrm{sn}^2(x)-1+m-ma\pm\delta_4]}{\mathrm{dn}^2(x)}$	$2K$	$2,0$
$(a-2)(a-3)$	$4+m(a-1)^2$	$\frac{\mathrm{sn}(x)\mathrm{cn}(x)}{\mathrm{dn}^2(x)}$	$2K$	2
$(a-3)(a-4)$	$5+m(a^2-4a+5)\pm 2\delta_5$	$\frac{\mathrm{cn}(x)[m(2a-1)\mathrm{sn}^2(x)-2+2m-ma\pm\delta_5]}{\mathrm{dn}^3(x)}$	$4K$	$3,1$
$(a-3)(a-4)$	$5+m(a^2-2a+2)\pm 2\delta_6$	$\frac{\mathrm{sn}(x)[m(2a-1)\mathrm{sn}^2(x)-2+m-ma\pm\delta_6]}{\mathrm{dn}^3(x)}$	$4K$	$3,1$
$(a-4)(a-5)$	$10+m(a^2-4a+5)\pm 2\delta_7$	$\frac{\mathrm{sn}(x)\mathrm{cn}(x)[m(2a-1)\mathrm{sn}^2(x)-3+2m-ma\pm\delta_7]}{\mathrm{dn}^4(x)}$	$2K$	$4,2$

Table 4: Energy Eigenstates for the $(6,2)$ Potential

E	$\psi^{(-)}$	Period	Nodes
0	$\mathrm{dn}^2(x)$	$2K$	0
$5-3m-2\sqrt{4-3m}$	$\frac{\mathrm{cn}(x)}{\mathrm{dn}(x)}[3m\mathrm{sn}^2(x)-2-\sqrt{4-3m}]$	$4K$	1
$5-2m-2\sqrt{4-5m+m^2}$	$\frac{\mathrm{sn}(x)}{\mathrm{dn}(x)}[3m\mathrm{sn}^2(x)-2-m-\sqrt{4-5m+m^2}]$	$4K$	1
$5-2m+2\sqrt{4-5m+m^2}$	$\frac{\mathrm{sn}(x)}{\mathrm{dn}(x)}[3m\mathrm{sn}^2(x)-2-m+\sqrt{4-5m+m^2}]$	$4K$	3
$5-3m+2\sqrt{4-3m}$	$\frac{\mathrm{cn}(x)}{\mathrm{dn}(x)}[3m\mathrm{sn}^2(x)-2+\sqrt{4-3m}]$	$4K$	3

Table 5: Energy Eigenstates for the $(2,2)$ Potential

E	$\mathrm{dn}(x)\psi^{(-)}$	Period	Nodes
0	$\mathrm{dn}^2(x)+\sqrt{1-m}$	K	0
$4\sqrt{1-m}$	$\mathrm{dn}^2(x)-\sqrt{1-m}$	$2K$	1
$2-m+2\sqrt{1-m}$	$\mathrm{sn}(x)\mathrm{cn}(x)$	$2K$	1

Table 6: Energy Eigenstates for the $(6, 6)$ Potential

E	$\mathrm{dn}^2(x)\psi^{(-)}$	Period	Nodes
0	$1-(4-m-\delta_8)\mathrm{sn}^2(x)+(4-2m-\delta_8)\mathrm{sn}^4(x)$	K	0
$-4+2m+2\delta_8$	$1-2\mathrm{sn}^2(x)+m\mathrm{sn}^4(x)$	$2K$	1
$2-m-6\sqrt{1-m}+2\delta_8$	$\mathrm{sn}(x)\mathrm{cn}(x)[1-(1-\sqrt{1-m})\mathrm{sn}^2(x)]$	$2K$	1
$2-m+6\sqrt{1-m}+2\delta_8$	$\mathrm{sn}(x)\mathrm{cn}(x)[1-(1+\sqrt{1-m})\mathrm{sn}^2(x)]$	K	2
$4\delta_8$	$1-(4-m+\delta_8)\mathrm{sn}^2(x)+(4-2m+\delta_8)\mathrm{sn}^4(x)$	K	2

Table 7: Energy Eigenstates for the $(63/4, 3/4)$ Potential

E	$\mathrm{dn}^{1/2}(x)\psi^{(-)}$	Period	Nodes
0	$[12m\mathrm{sn}^2(x)-2-5m-\delta_9]\mathrm{dn}^2(x)$	$2K$	0
$2-m+\delta_9$	$\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}^2(x)$	$2K$	2
$2\delta_9$	$[12m\mathrm{sn}^2(x)-2-5m+\delta_9]\mathrm{dn}^2(x)$	$2K$	2
$14-7m+\delta_9$	$\mathrm{sn}(x)\mathrm{cn}(x)[1-2\mathrm{sn}^2(x)]$	$2K$	4
$14-7m+\delta_9$	$[1-8\mathrm{sn}^2(x)\mathrm{cn}^2(x)]$	$2K$	4

Figure 1(a)

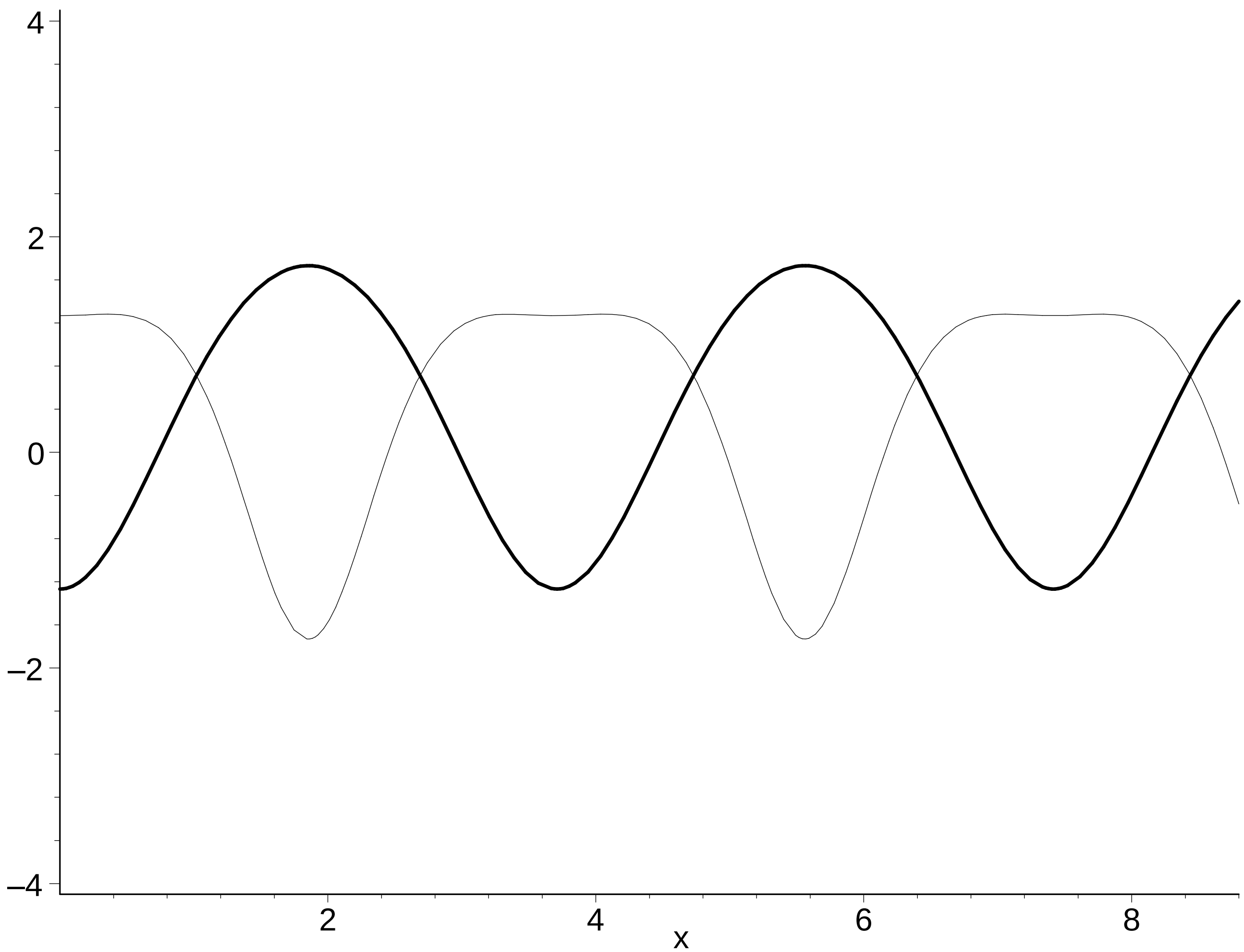


Figure 1(b)

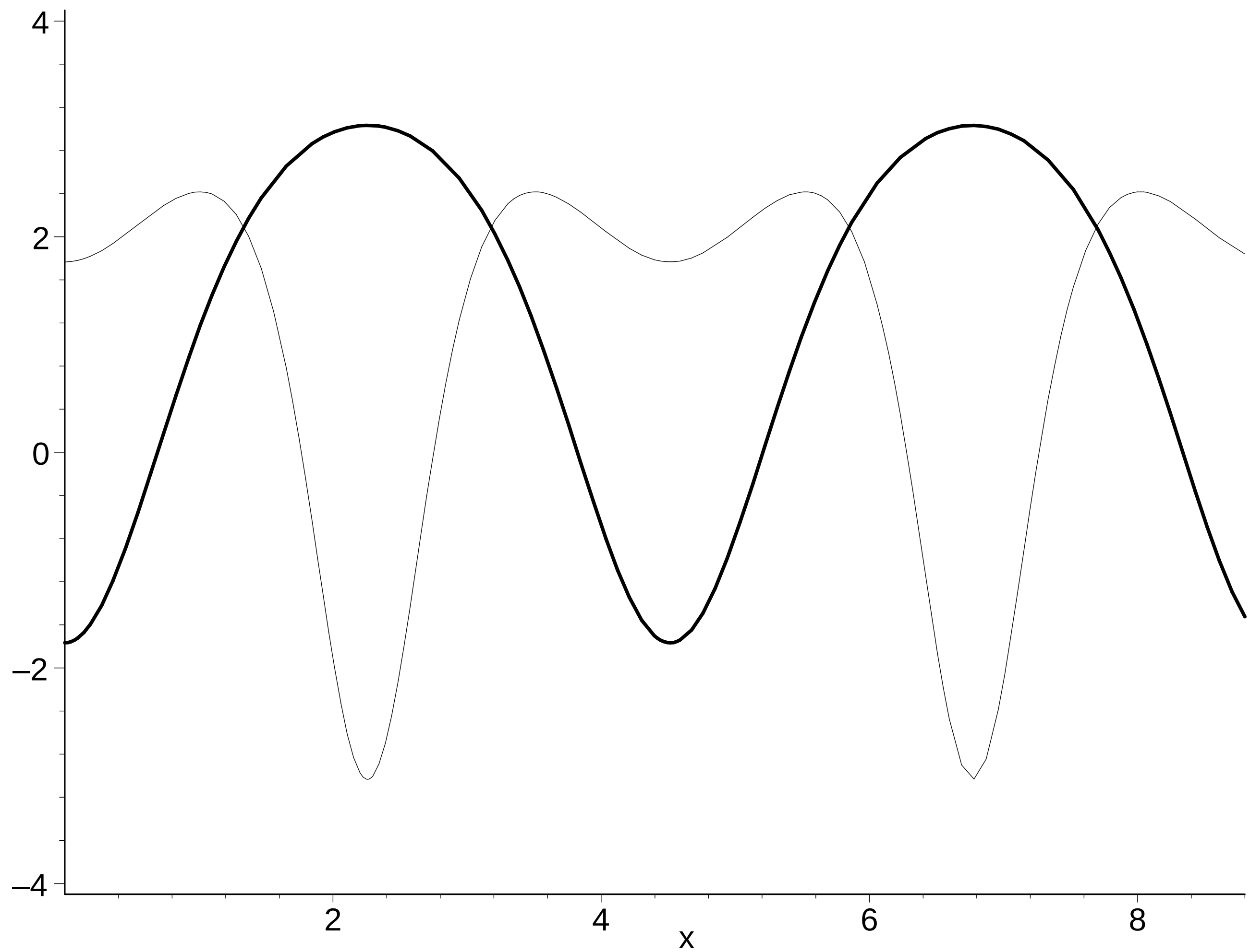


Figure 1(c)

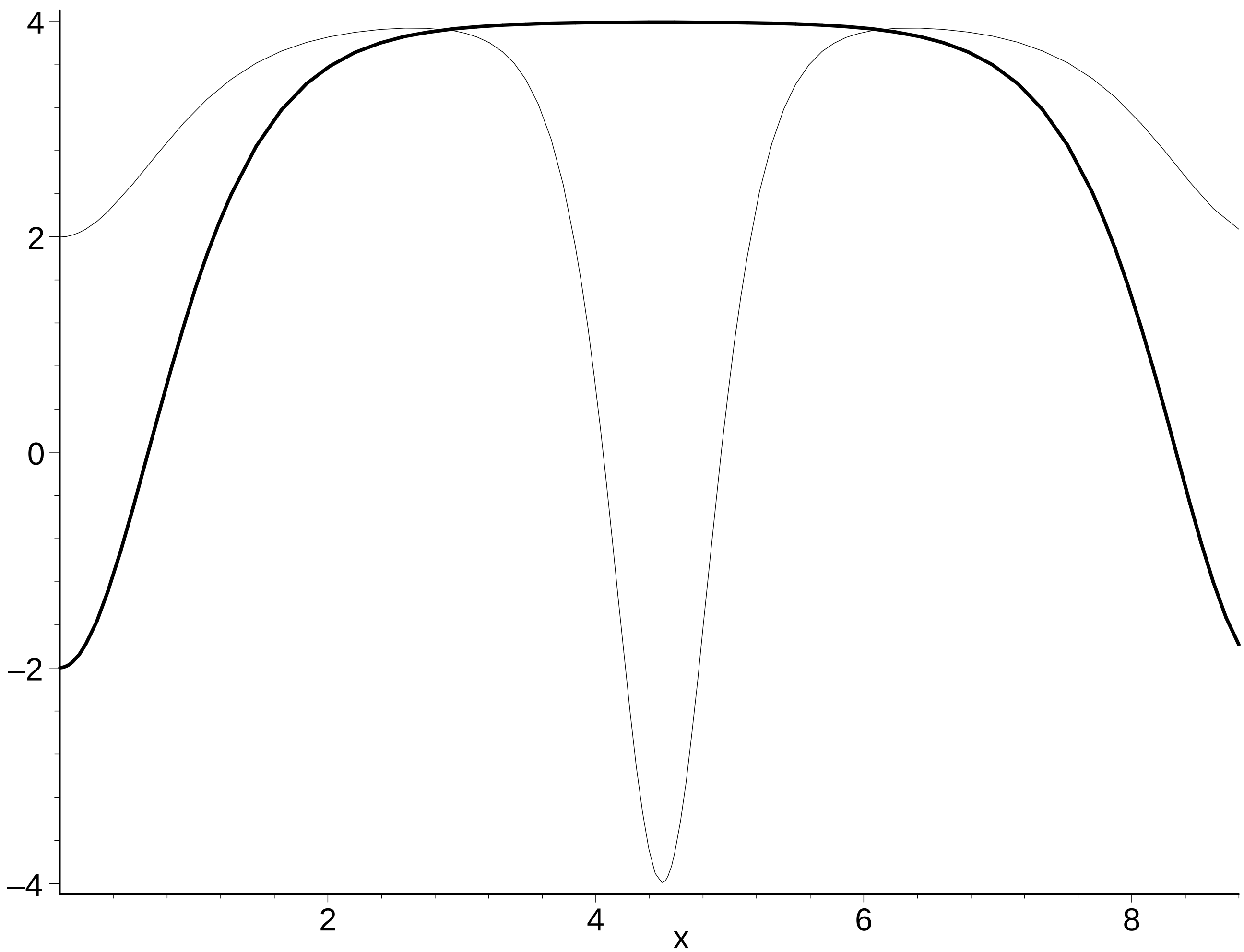


Figure 2(a)

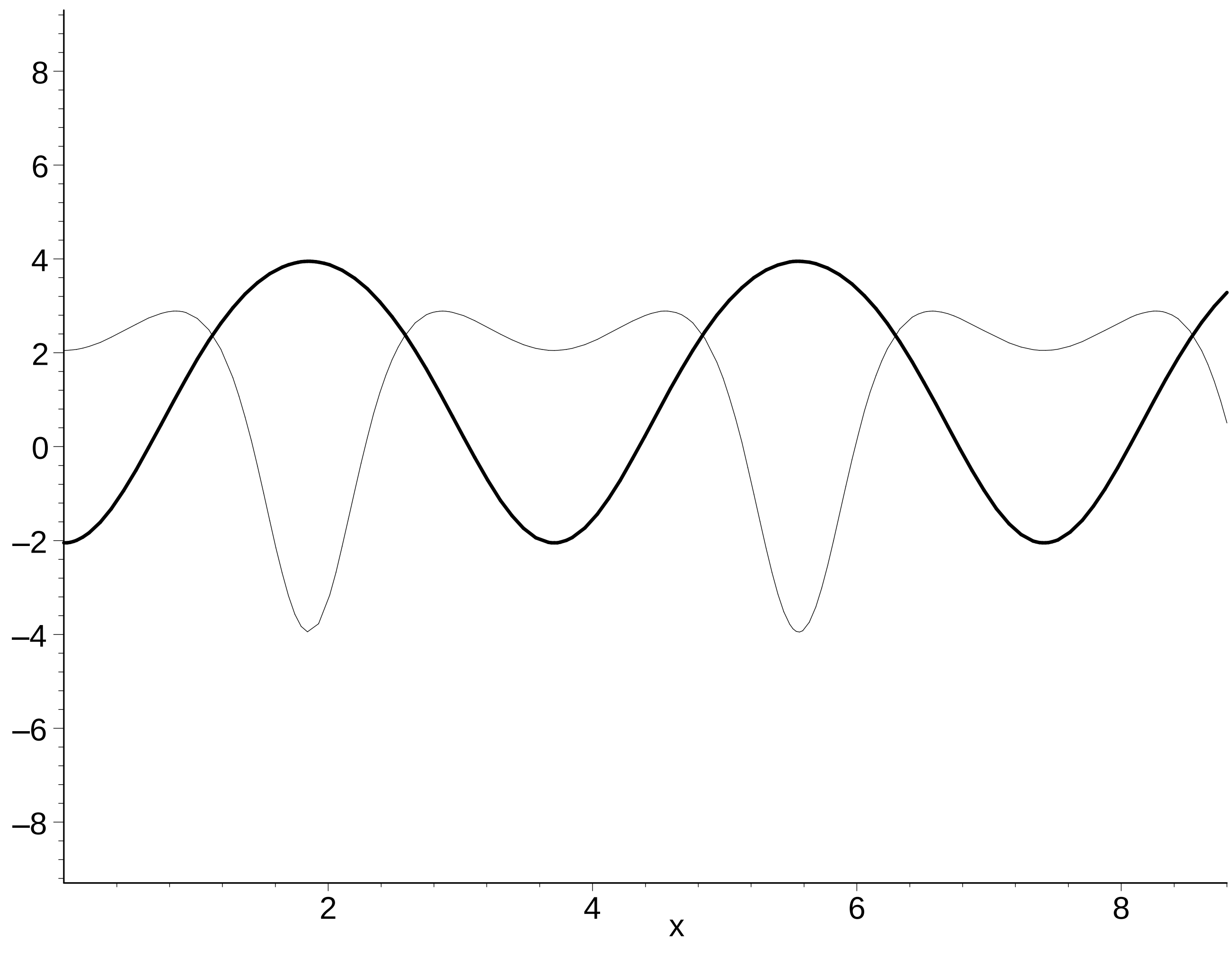


Figure 2(b)

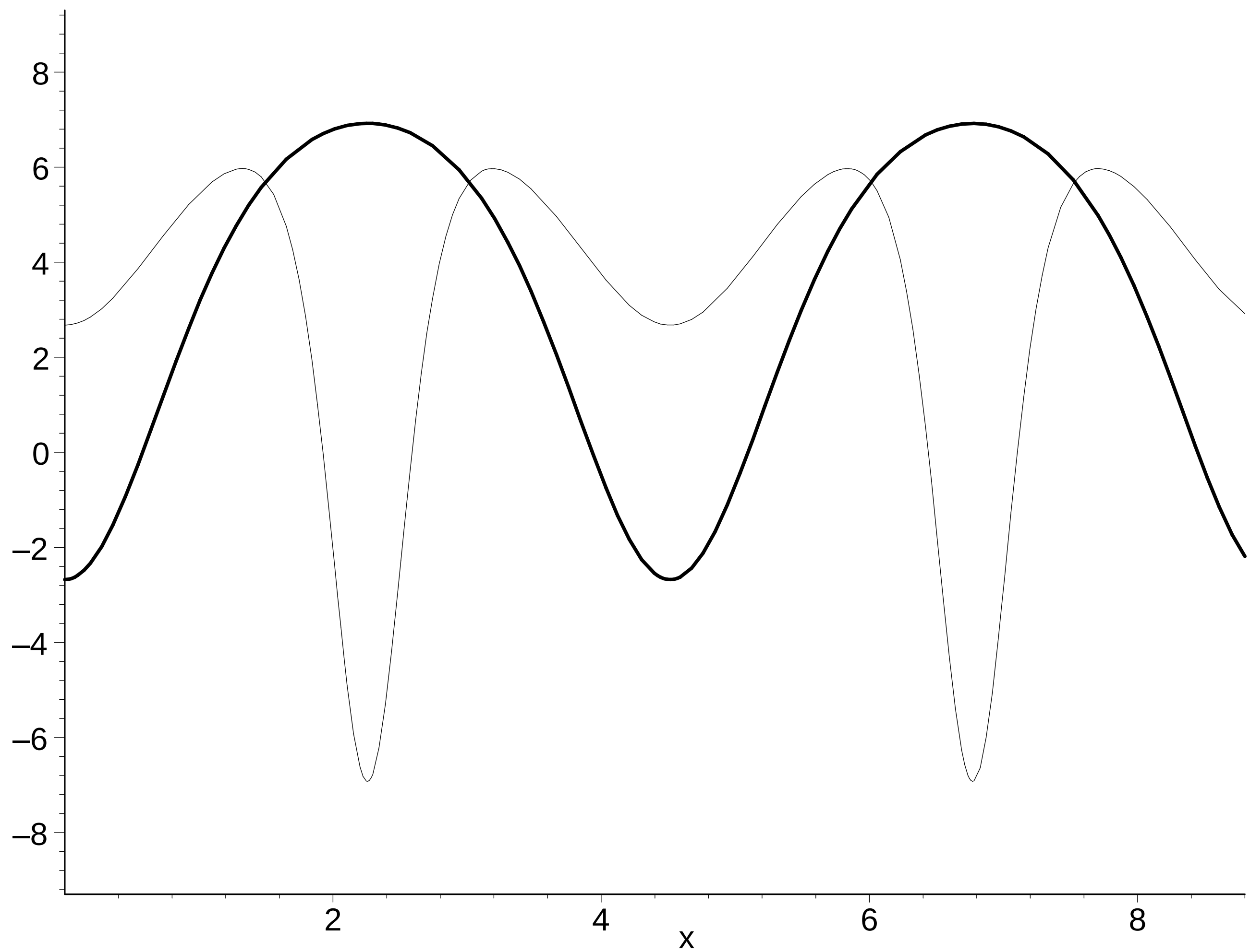


Figure 2(c)

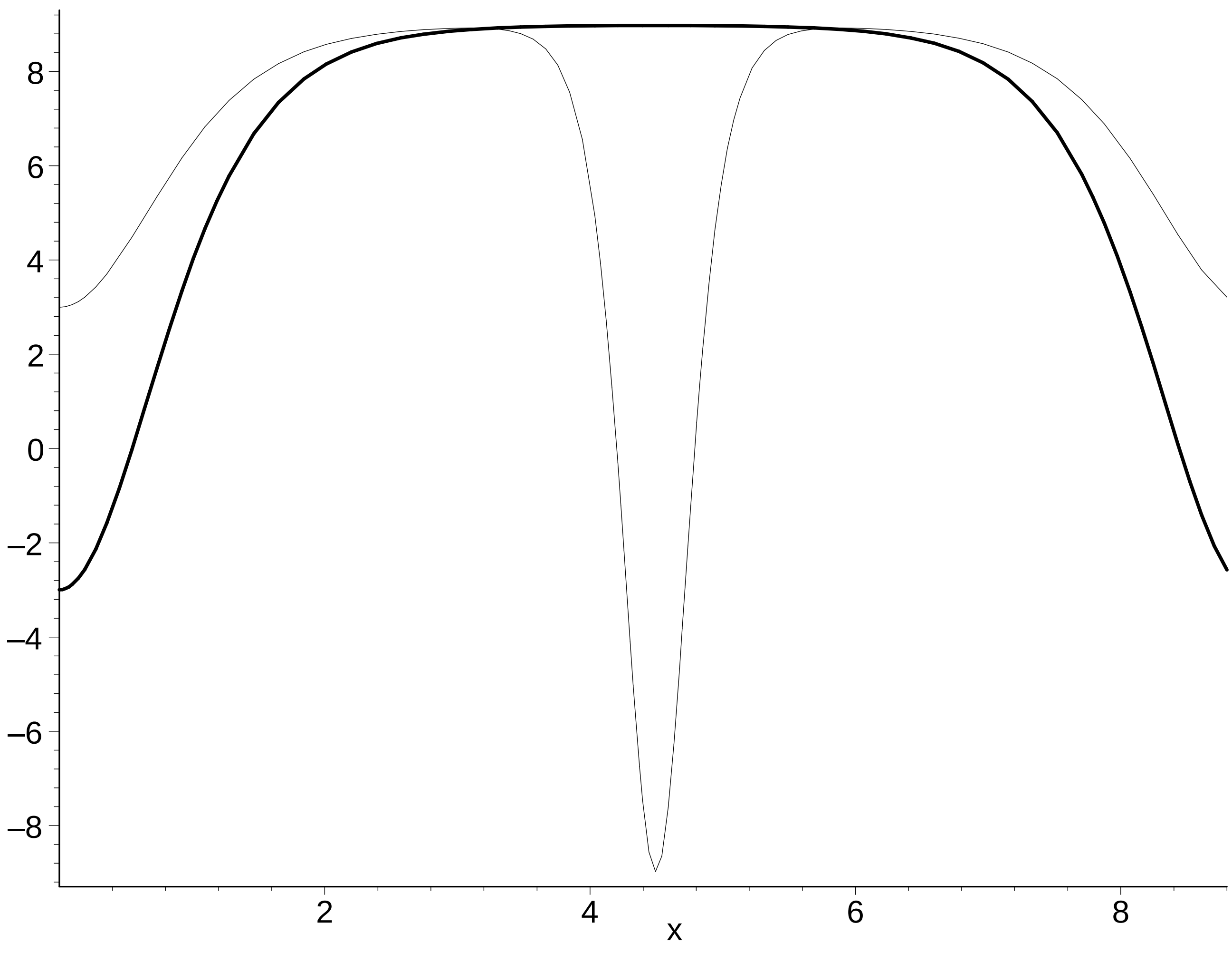


Figure 3

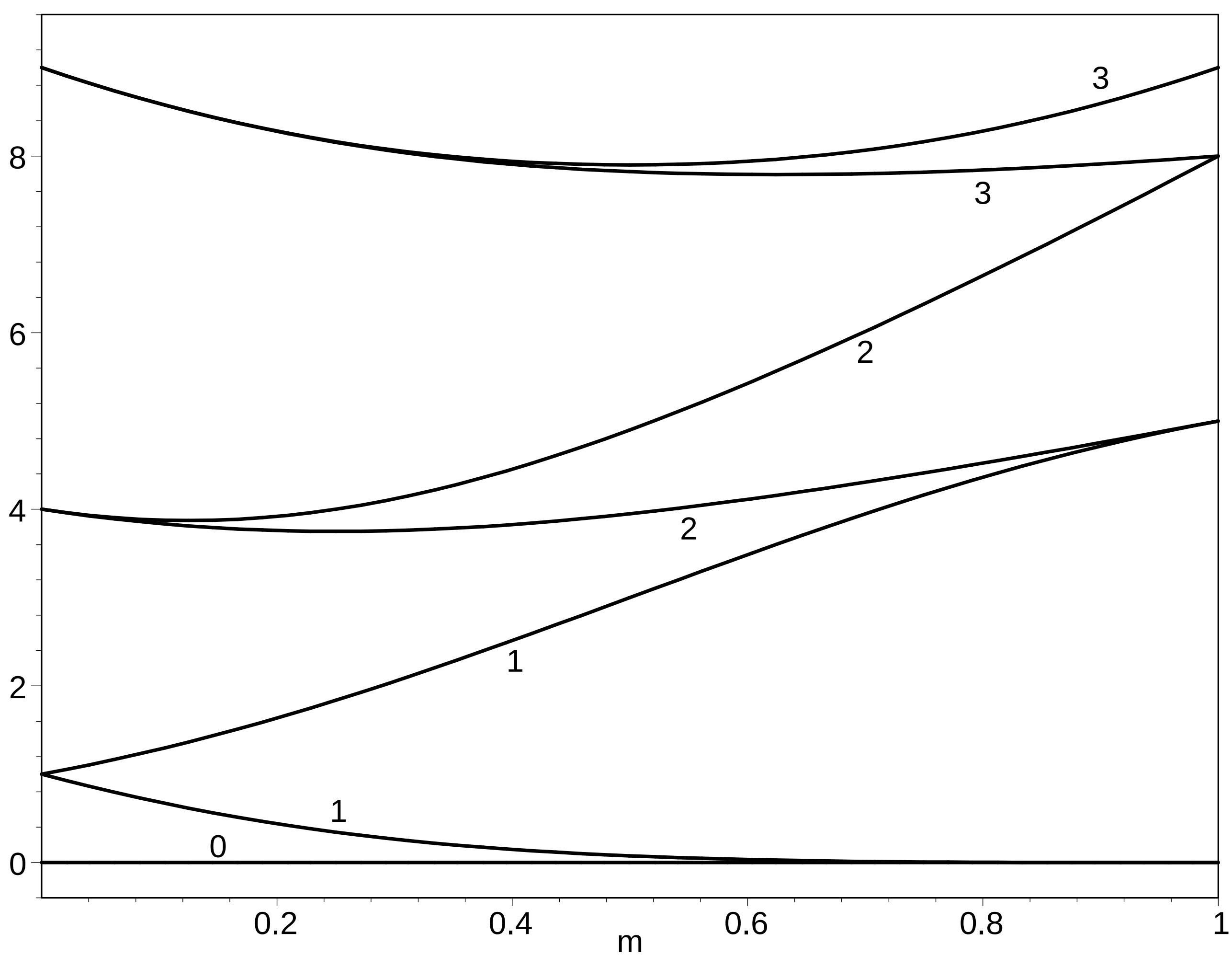


Figure 4(a)

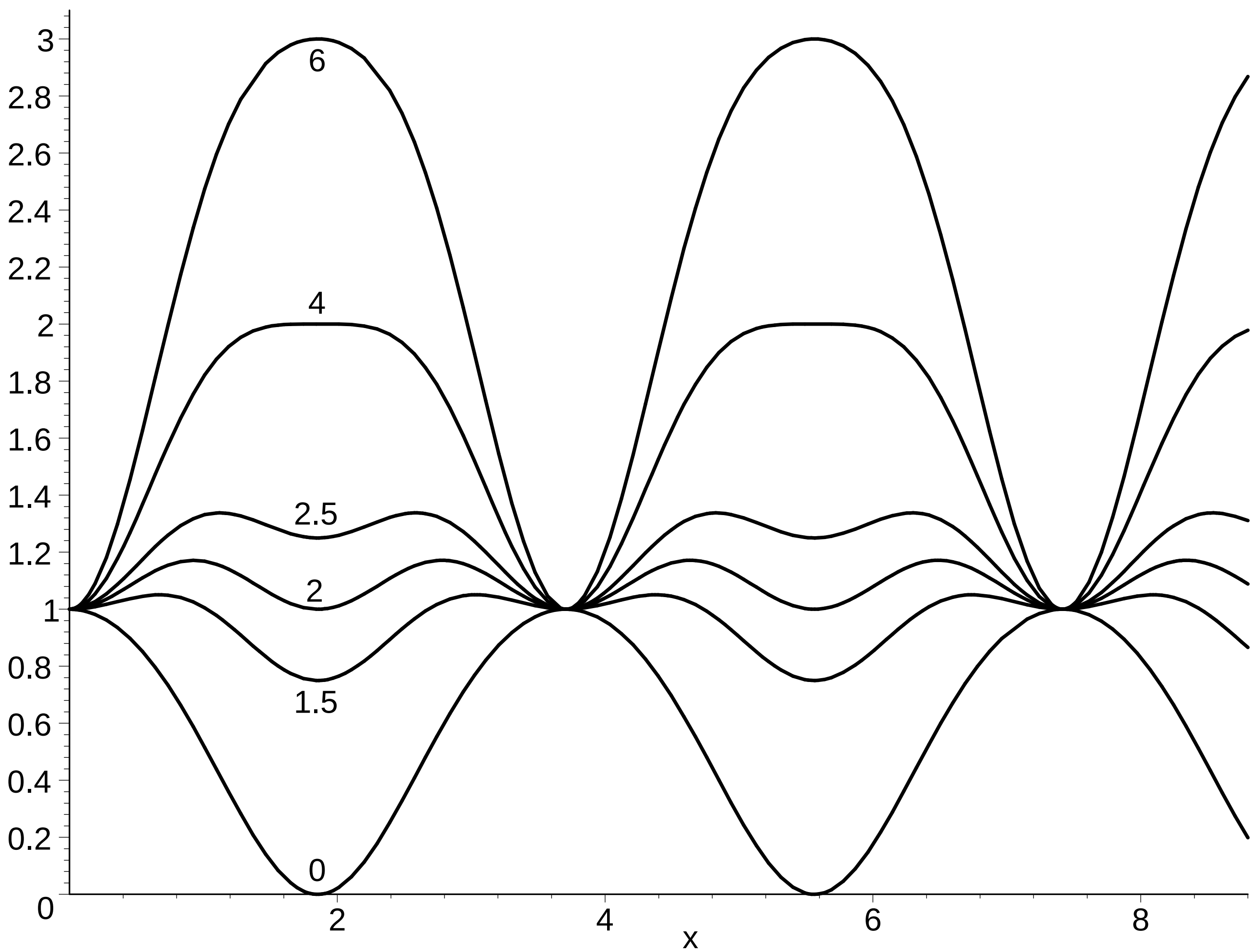


Figure 4(b)

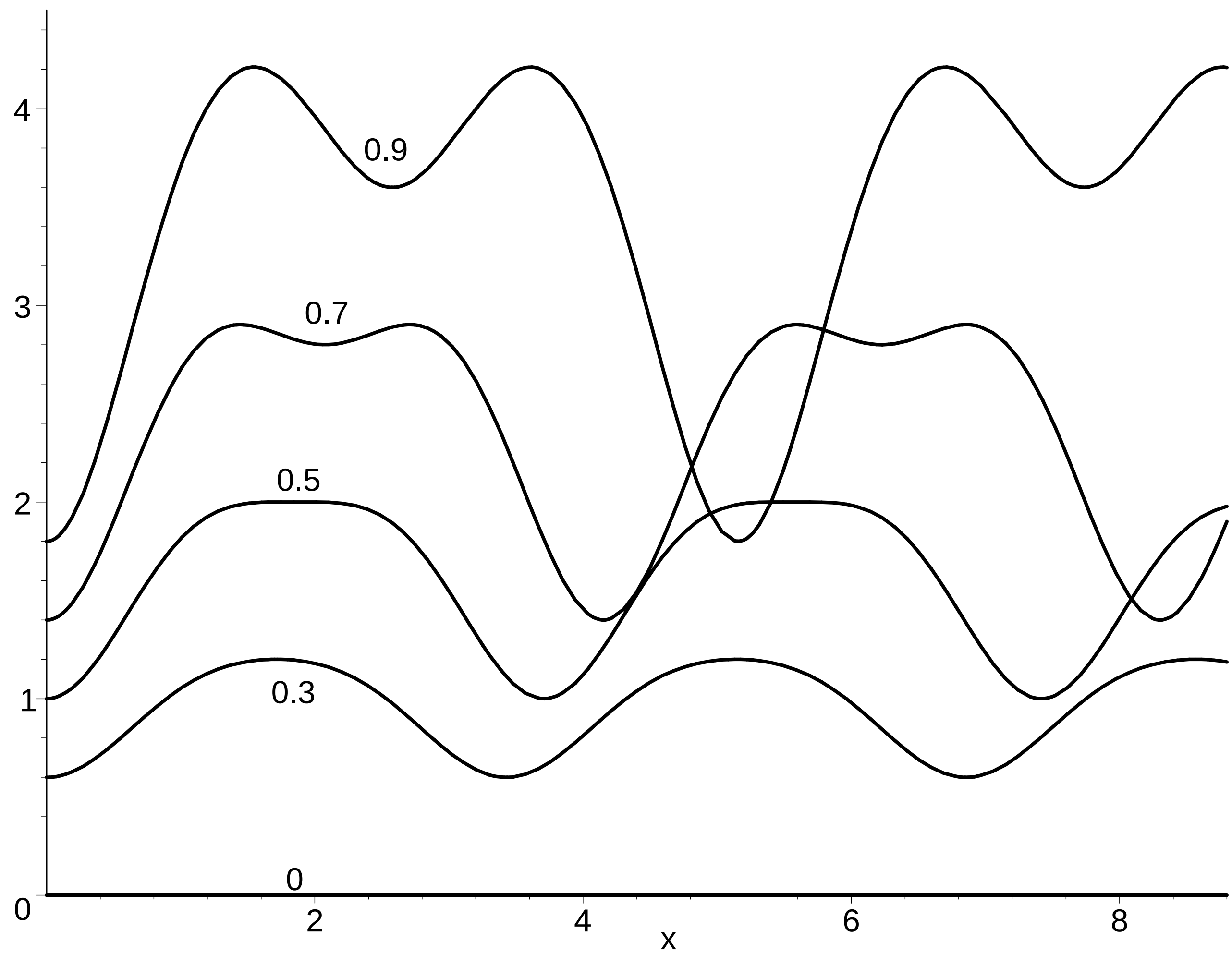


Figure 5

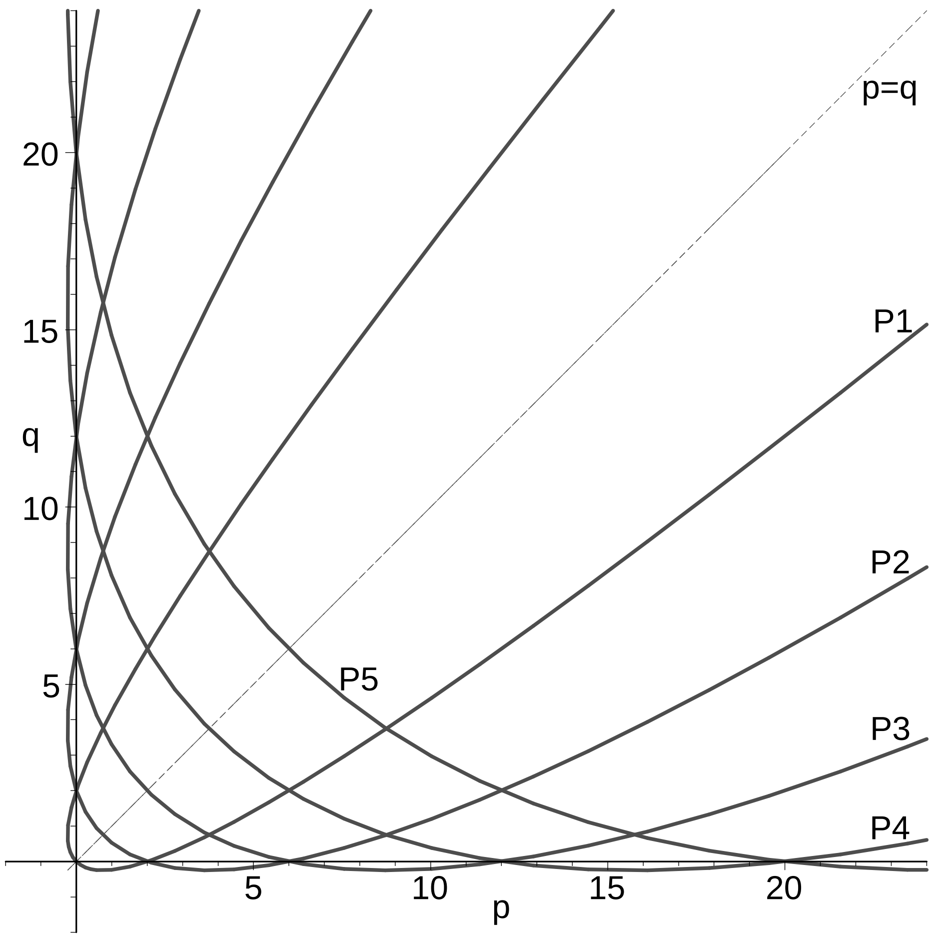


Figure 6(a)

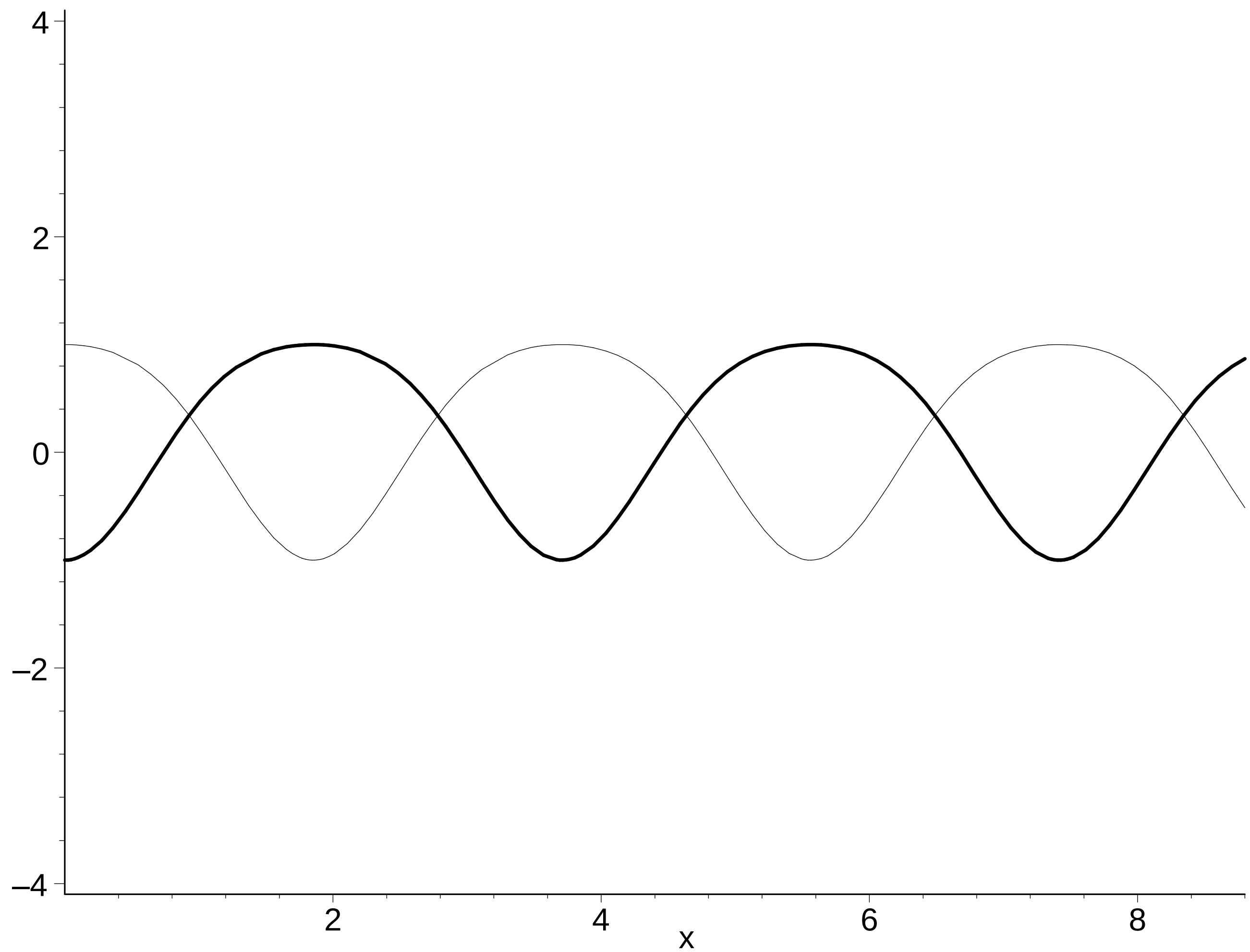


Figure 6(b)

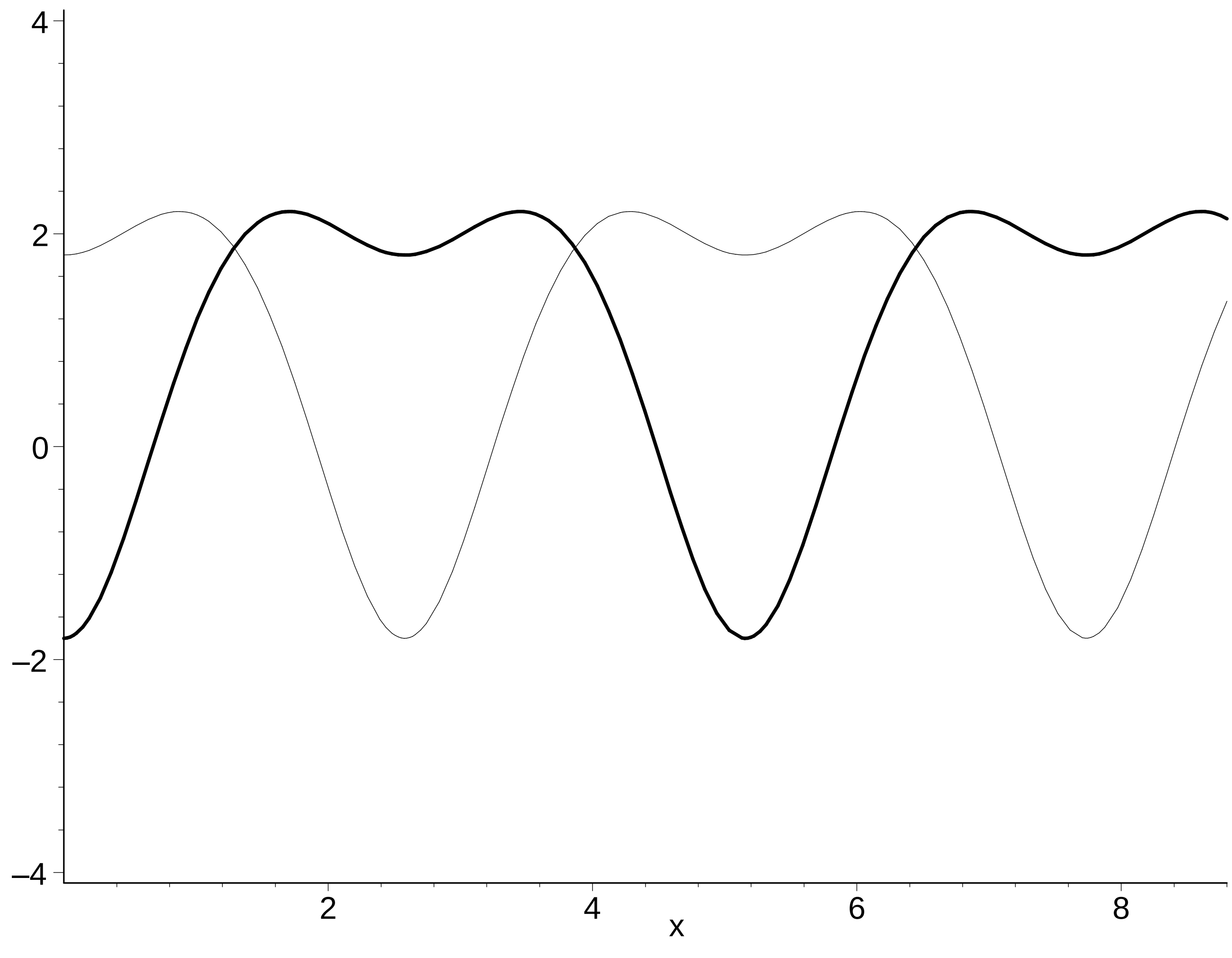


Figure 6(c)

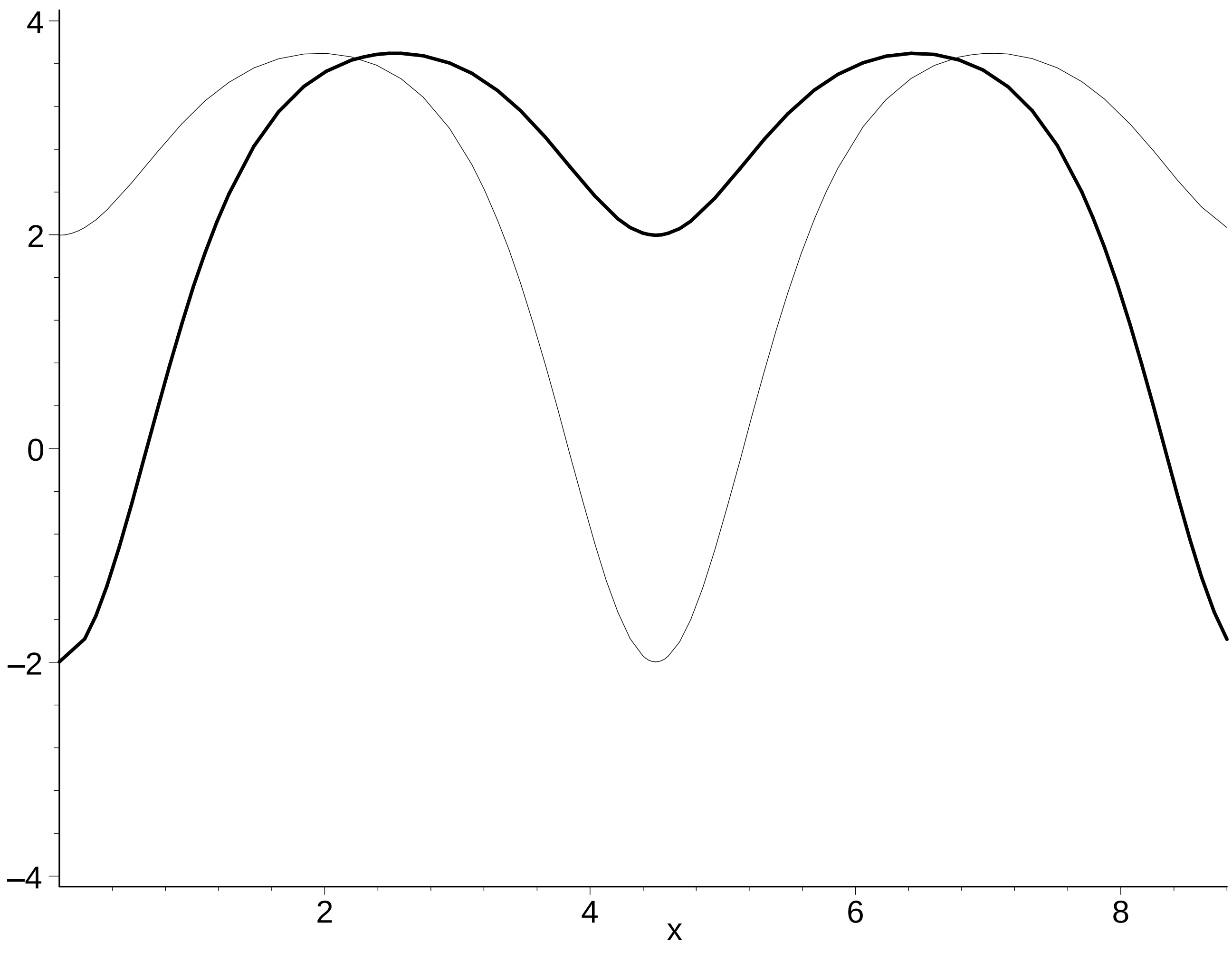


Figure 7

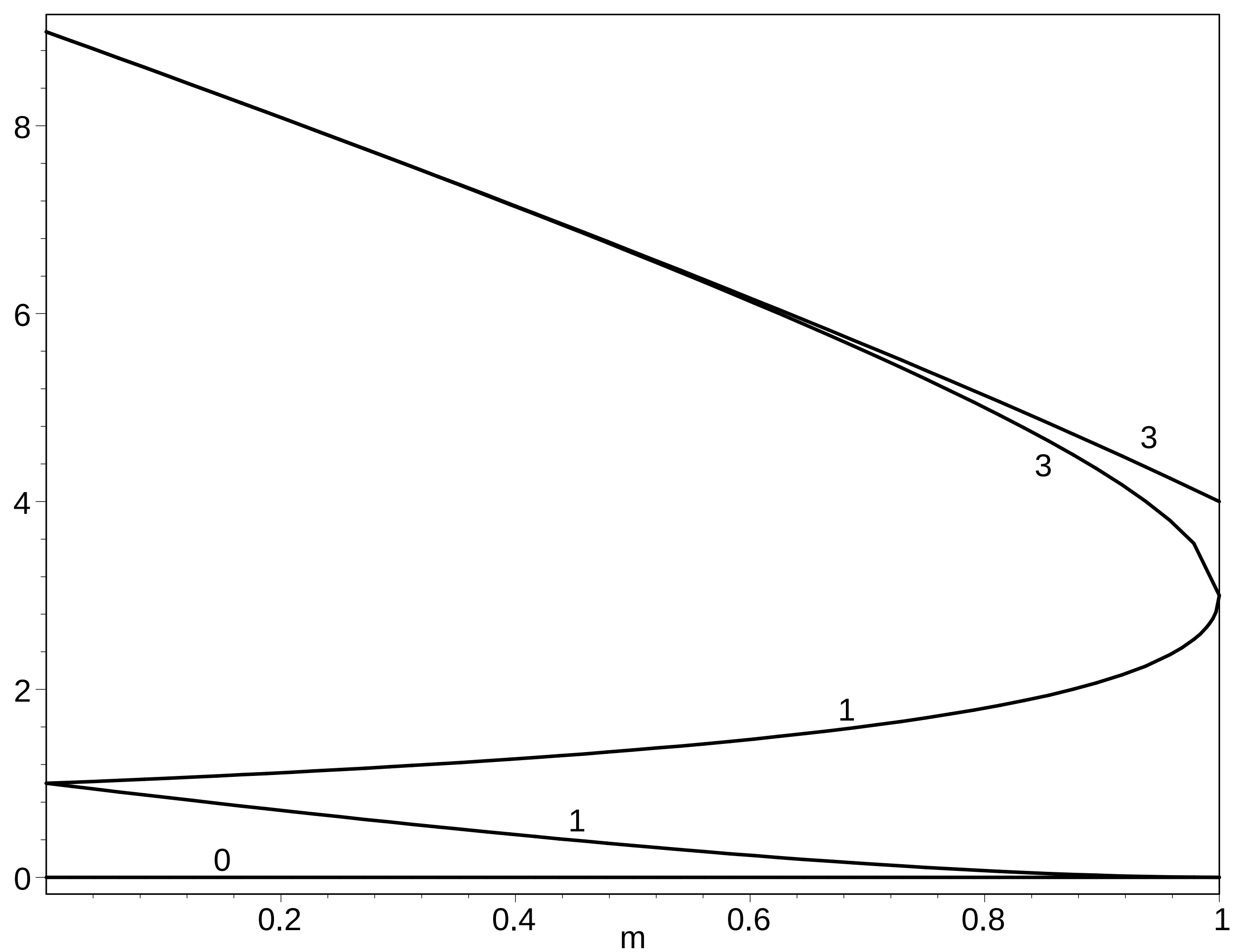


Figure 8(a)

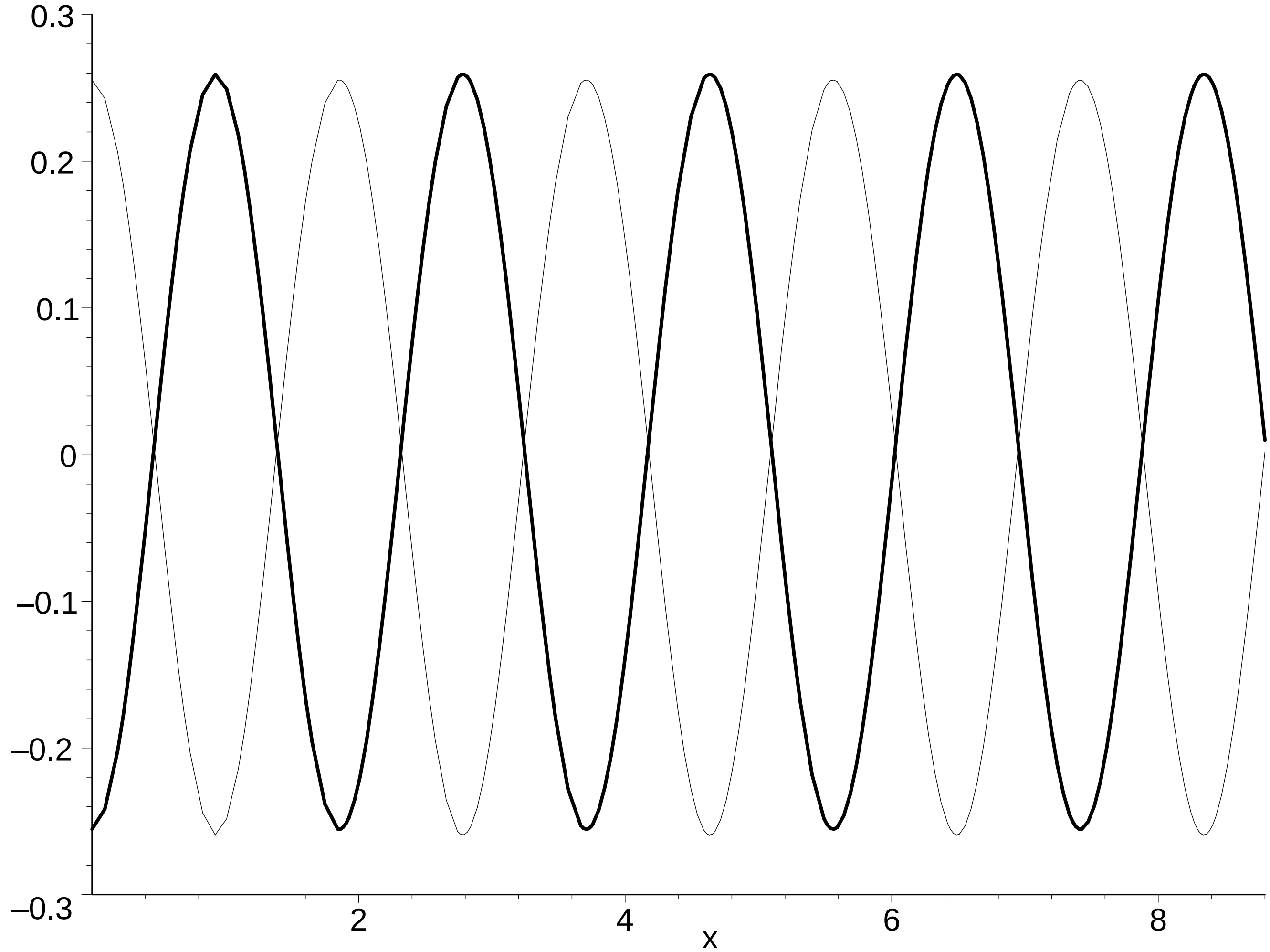


Figure 8(b)

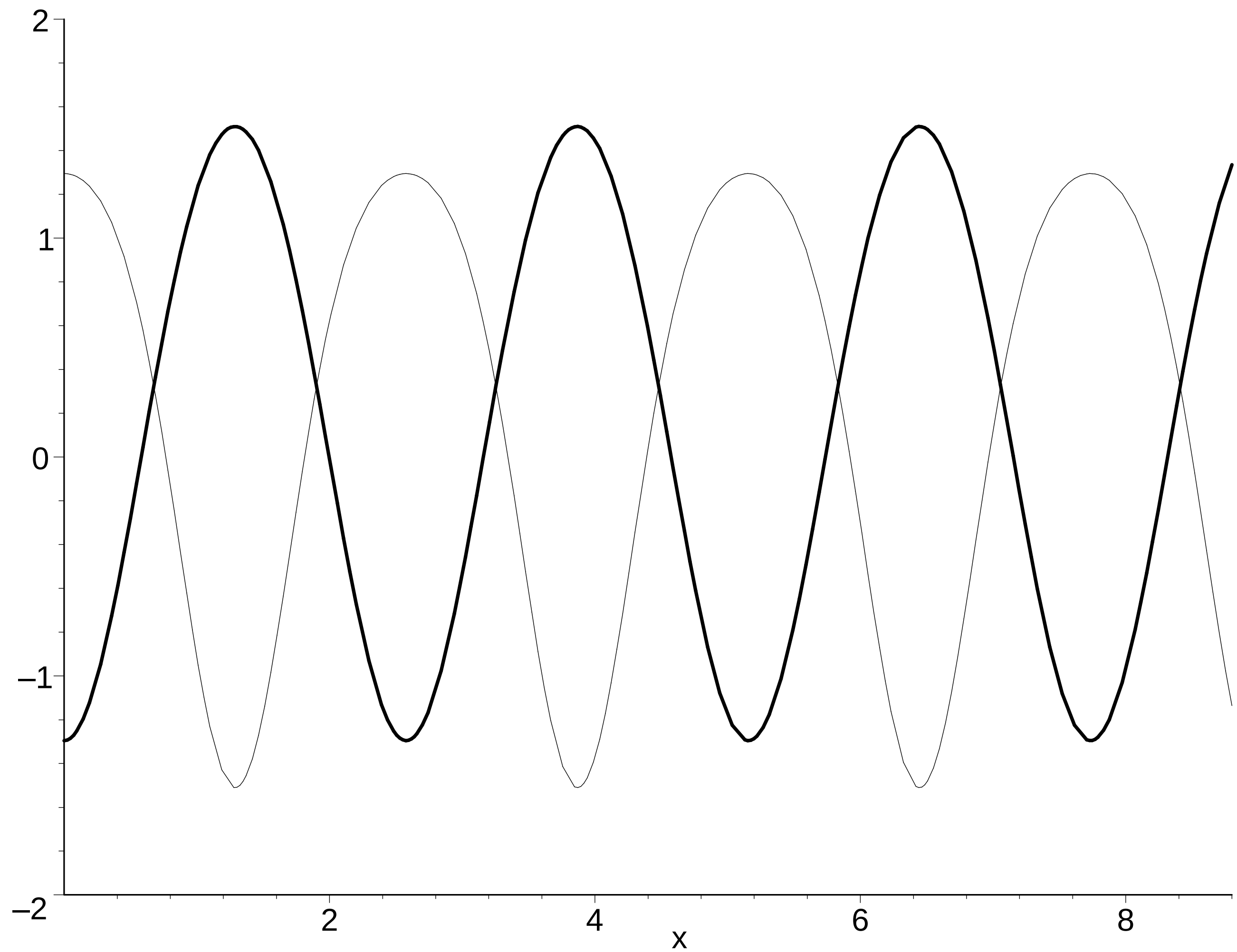


Figure 8(c)

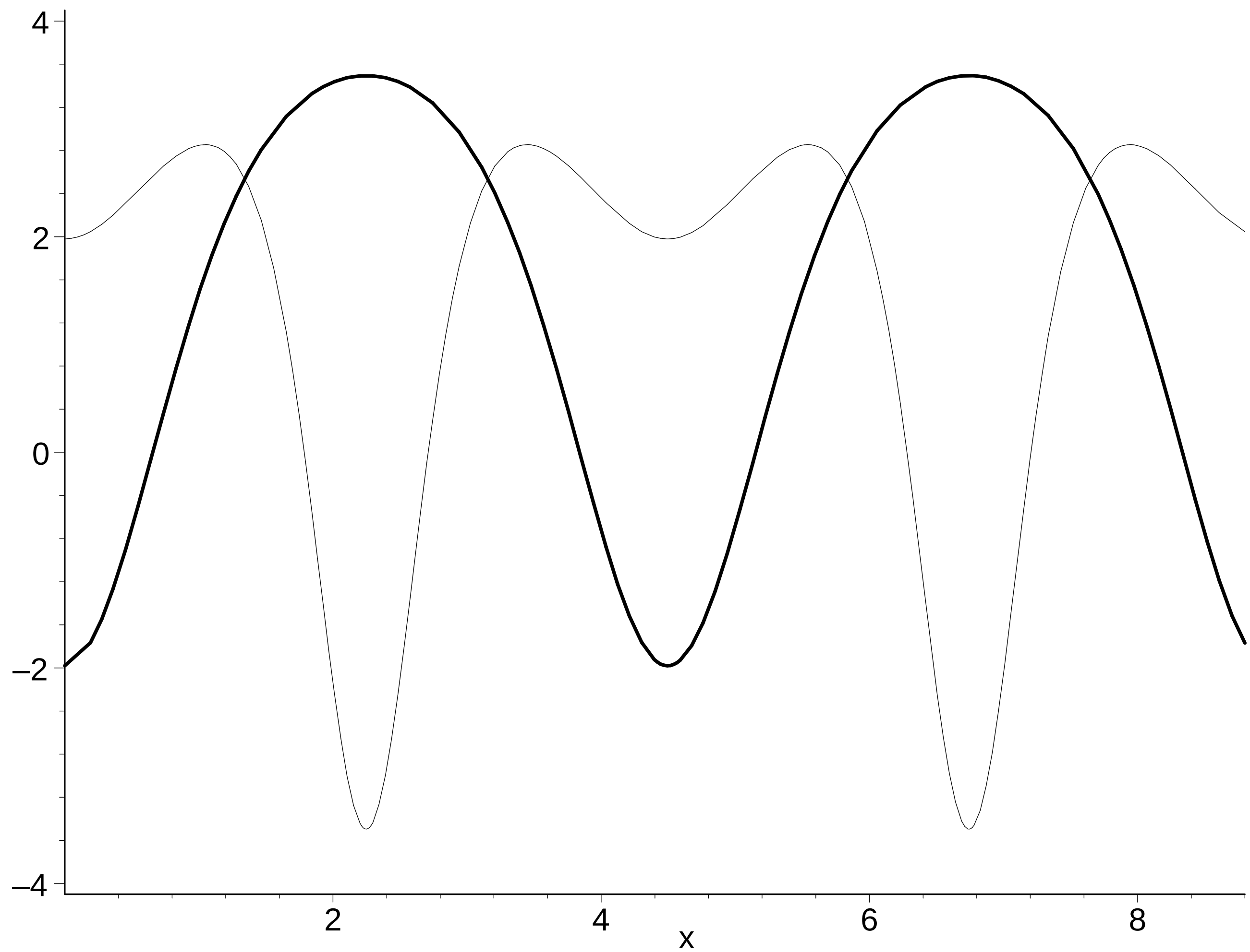


Figure 9

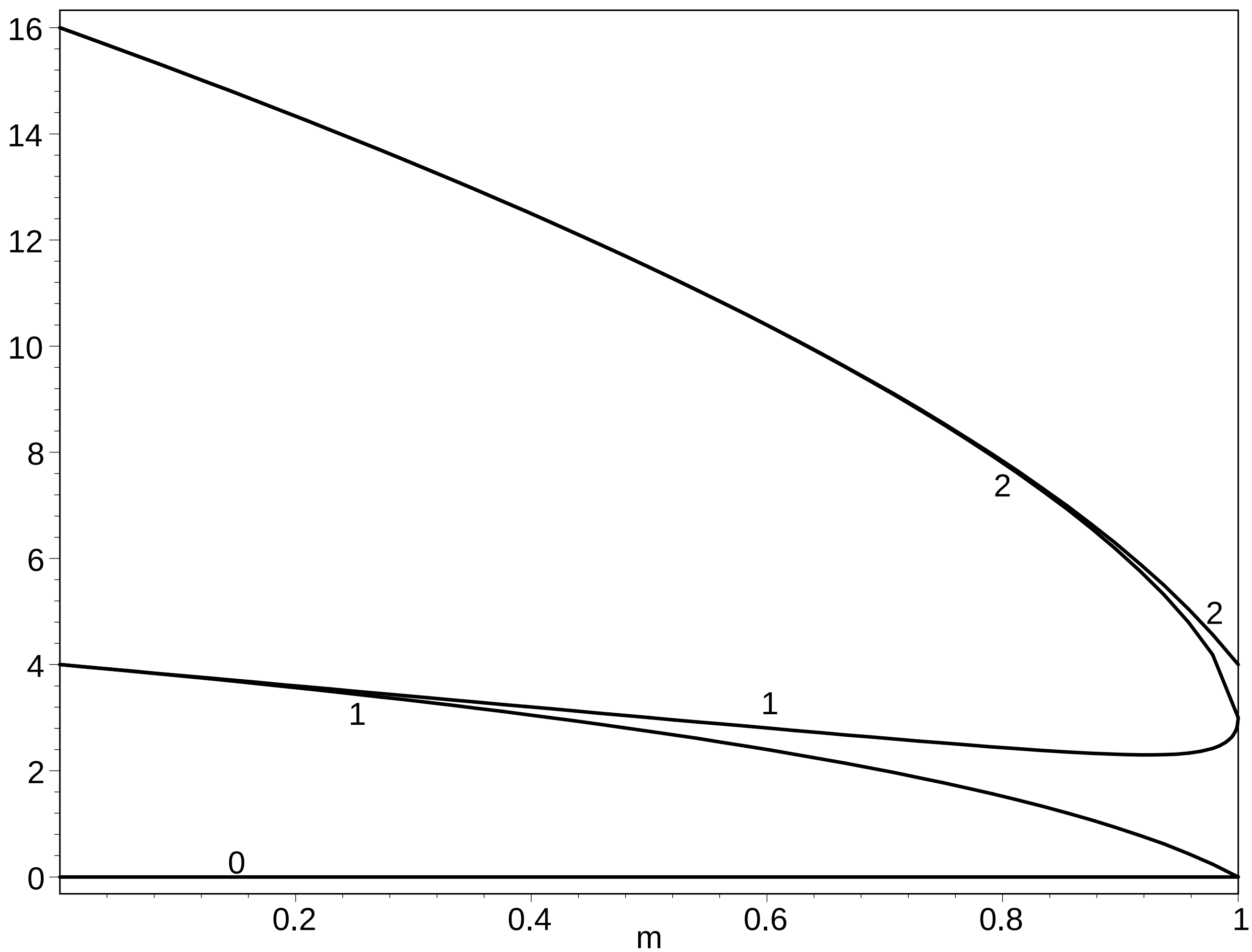
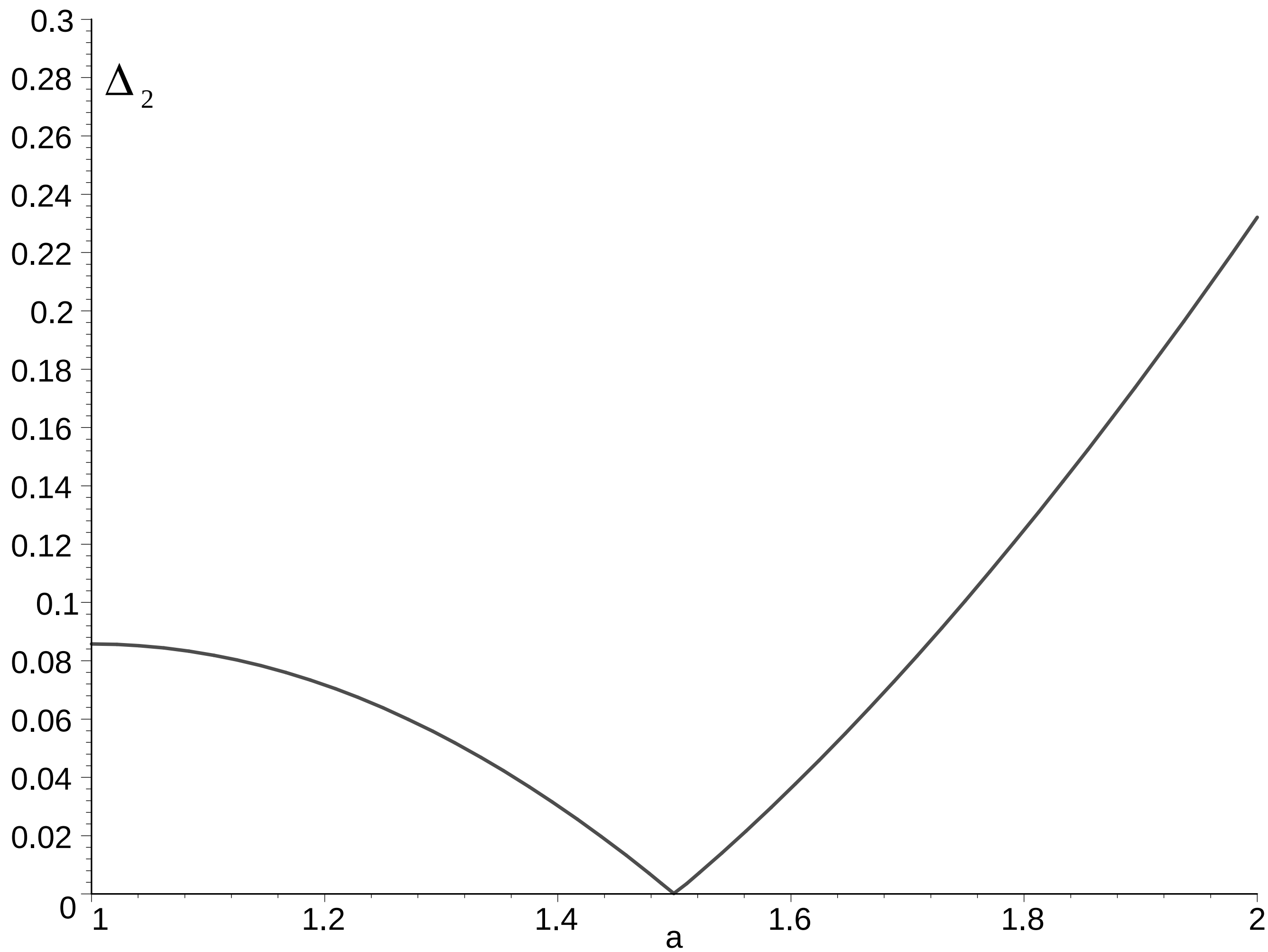


Figure 10



0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
Δ 2
1
1.2
1.4
1.6
1.8
2
a