Scattering properties of $\mathcal{PT}$-symmetric quantum systems

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Properties of the scattering matrix for 1-D $\mathcal{PT}$-symmetric systems have been analyzed. In the context of the continuum states, conservation properties are utilized to extract physical conclusions. Asymptotic states are shown to have necessarily broken $\mathcal{PT}$-symmetry, leading to restricted boundary conditions, some resembling the proposed $\mathcal{PT}$ CPA laser [1], with absorption/emission during the scattering process. It is noted that transmission is possible, only if incidence takes place from both sides of the system, which is unique to a $\mathcal{PT}$-symmetric quantum mechanical system. $\mathcal{PT}$-symmetric systems require additional conditions for scattering and transfer matrices, indicating a unique algebraic structure. The S-matrix is found to be *Hermitian*, instead of being unitary, and to satisfy a 'duality' condition, having known optical analogues [2]. Finally, a suggestive approach towards a 'physical' $\mathcal{PT}$-symmetric norm is made.

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Introduction

Due to the realness of the spectrum for certain range of parameters of the complex Hamiltonian (potential), and complex-conjugate pairs of eigenvalues appearing otherwise, $\mathcal{PT}$-symmetric quantum systems have been receiving considerable attention in recent times [3], leading to deterministic experimental realization of the same [4]. Owing to the non-Hermiticity of the system, a number of proposals for a redefined semi-positive-definite inner-product has been made [5], subjected to specific conditions. For the states of a $\mathcal{PT}$-symmetric system, generated by spectral operators, Mostafazadeh [6] showed that the system is pseudo-Hermitian, spanned on a bi-orthonormal basis, for which a complete prescription to obtain a semi-positive-definite inner-product was finally given by Das and Greenwood [7]. However, a general proof of equivalence of pseudo-Hermiticity and $\mathcal{PT}$-symmetry is still under speculation even for bounded operators, as has been shown for a more general case of bounded spectral operators [8]. Therefore, an alternative route of studying such systems is preferred, by utilizing conservation laws. With this motivation, the phenomena of scattering by a $\mathcal{PT}$-symmetric system is analyzed, following a number of attempts that have already been made [9].

The anti-linearity of the time-reversal ($\mathcal{T}$) operation causes difficulty in defining a 'suitable' norm for a $\mathcal{PT}$-symmetric system, as an extension to the usual Dirac-von-Neumann (L_2) norm, akin to the one achieved by Das and Greenwood, as the dual to the anti-linear operation is not well-defined in that formalism. As the left operation of such an operator is not defined in a linear vector (Hilbert) space, the dual space cannot be constructed in a straightforward manner, restricting the possibility of unitary evolution. However, for one-dimensional systems, the fact that parity (P) is an unambiguous discrete symmetry, can be utilized with great convenience to define a generalized L_2-like conserved '*scalar product*', bypassing the inconvenience due to anti-linearity of the $\mathcal{PT}$ operation. This is not the case if the one-dimensional system is obtained from higher-dimensional systems utilizing some symmetry, *e.g.*, spherical symmetry. The aforementioned scalar is **not** the new conserved norm subjected to conserved probability. Never the less, one can obtain a conservation law, on utilizing $\mathcal{PT}$-symmetry itself, thus enabling one to explain somewhat different asymptotic behavior for such systems. It can intuitively be understood that $\mathcal{PT}$-symmetry is necessarily broken for asymptotic states, thus hinting towards the non-existence of a local scalar, suitably interpretable as probability density. Never the less, stationarity is shown to be evident only for $\mathcal{PT}$-symmetric eigenstates, for which unitary time evolution must be excluded while defining $\mathcal{PT}$-transformation. They are also discrete, in terms of eigenvalues of the Hamiltonian, as evaluated directly in numerous examples [10], strongly suggesting towards the 'bound state' interpretation. In that sense, we arrive at a re-confirmation of the same, although through utilization of the basic symmetry of the system, which is *not* Hermitian. For the particular interest of comparison to real potentials, the complex potential is chosen

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to be asymptotically real (Hermitian). Subsequently, the symmetry algebra of the scattering process gets altered, leading to boundary conditions more constrained than the Hermitian case. It has been observed that transmission is possible only if plane-waves are incident from both directions. This is in agreement with the $\mathcal{PT}$ CPA laser [1], which stimulates emission, only with the incidence of two similar laser beams from opposite directions, owing to the inherent $\mathcal{PT}$-symmetry. Being a classical system, the $\mathcal{PT}$ CPA laser can be studied in terms of various matrix elements, which is not possible for the quantum mechanical analogue, due to the lack of a well-defined inner-product. Therefore, the results known for classical systems in terms of reflection and transmission coefficients, cannot be carried over naively. Additionally, the $\mathcal{PT}$ CPA laser fully absorbs incoming radiation for appropriate amplitude and phase relationship. This condition can be equivalently achieved in a CPA (Coherent Perfect Absorber), with only $\mathcal{T}$-symmetry, which has very recently been realized experimentally [11]. It is a perfectly absorbing system for coherent incidence. However a complete quantum mechanical analogy is yet to be achieved, for the absence of a generic physically measurable flux-density.

As a simple 1-D model, we start with a straightforward calculation of reflection ($\Re$) and transmission ($\Im$) coefficients for the Scarf-II potential, suitably *complexified* to be $\mathcal{PT}$-symmetric, and compare it with the results for its real counterpart, to show that the flux is not conserved, under the Hermitian norm. Intuitively, this is expected, given the possibility of absorption/emission owing to imaginary part of the potential. However, it fails to account for the case of preserved $\mathcal{PT}$-symmetry, corresponding to no absorption/emission (real eigenvalues). Further, we will consider a generic 1-D quantum mechanical $\mathcal{PT}$-symmetric system. After obtaining specific relations for scattering, we will analyze the properties of the asymptotic states through the continuity equation, in order to obtain the $\mathcal{PT}$-symmetric boundary conditions. Finally, a few suggestive step towards a generic norm for the $\mathcal{PT}$-symmetric systems will be explicated.

I. SCATTERING PROPERTIES OF SCARF-II POTENTIAL

A $\mathcal{PT}$-symmetric potential can be constructed from a suitable real potential by proper complexification. On comparing reflection coefficient ($\Re$) and transmission coefficient ($\Im$) for both the real (read Hermitian) potential and its $\mathcal{PT}$-symmetric counterpart in line of ordinary Hermitian quantum mechanics, the need of a well-defined scalar product can be understood. Considering the example of the suitably parameterized Scarf-II potential:

$$V(x) = A^2 + (B^2 - A^2 - A\alpha)\frac{1}{\cosh^2(\alpha x)} + B(2A+\alpha)\frac{\tanh(\alpha x)}{\cosh(\alpha x)},$$

originating from a superpotential of the form, $W(x) = A\tanh(\alpha x) + B/\cosh(\alpha x)$, one obtains transmission and reflection coefficients as [12]:

$$\Im\left(k, A/\alpha, B/\alpha\right) = \frac{\Gamma\left[-A/\alpha - ik/\alpha\right]\Gamma\left[1 + A/\alpha - ik/\alpha\right]\Gamma\left[\frac{1}{2} + iB/\alpha - ik/\alpha\right]\Gamma\left[\frac{1}{2} - iB/\alpha - ik/\alpha\right]}{\Gamma\left(-ik/\alpha\right)\Gamma\left[1 + ik/\alpha\right]\Gamma^2\left[\frac{1}{2} - ik/\alpha\right]}, \tag{1}$$

and,

$$\begin{aligned}\Re\left(k, A/\alpha, B/\alpha\right) = \; & \Im\left(k, A/\alpha, B/\alpha\right)\left[\cos(\pi A/\alpha)\sinh(\pi B/\alpha)/\cosh(\pi k/\alpha)\right. \\ & \left. + i\sinh(\pi A/\alpha)\cosh(\pi B/\alpha)/\sinh(\pi k/\alpha)\right],\end{aligned} \tag{2}$$

respectively, which conserve the total flux as $|\Re|^2 + |\Im|^2 = 1$. Here $k = \frac{\alpha}{i}\left(n - A/\alpha\right)$, n being the label of the corresponding to normalized eigenstate. It turns out that k indeed is the asymptotic momentum of the state, and hence valid as same for scattering solutions. Further, the zeros of the scattering matrix can be interpreted as bound or resonance states, subjected to whether k is pure imaginary or complex, respectively. Real k yields scattering solutions, with non-trivial asymptotic behavior.

On considering the $\mathcal{PT}$-symmetric counterpart of the above superpotential, namely $W(x) = A\tanh(\alpha x) + iB/\cosh(\alpha x)$, which leads to the $\mathcal{PT}$-symmetric potential,

$$V(x) = A^2 - \left(A(A+\alpha) + B^2\right)\frac{1}{\cosh^2(\alpha x)} + iB(2A+\alpha)\frac{\tanh(\alpha x)}{\cosh(\alpha x)}, \tag{3}$$

one obtains the reflection and transmission coefficients as,

$$\Im\left(k, A/\alpha, iB/\alpha\right) = \frac{\Gamma\left[-A/\alpha - ik/\alpha\right]\Gamma\left[1+A/\alpha - ik/\alpha\right]\Gamma\left[\frac{1}{2} - B/\alpha - ik/\alpha\right]\Gamma\left[\frac{1}{2} + B/\alpha - ik/\alpha\right]}{\Gamma\left(-ik/\alpha\right)\Gamma\left[1+ik/\alpha\right]\Gamma^2\left[\frac{1}{2} - ik/\alpha\right]}, \quad (4)$$

and,

$$\Re\left(k, A/\alpha, iB/\alpha\right) = \Im\left(k, A/\alpha, iB/\alpha\right)\left[i\cos(\pi A/\alpha)\sin(\pi B/\alpha)/\cosh(\pi k/\alpha) + i\sinh(\pi A/\alpha)\cos(\pi B/\alpha)/\sinh(\pi k/\alpha)\right], \quad (5)$$

in the saame way as before. Subsequently,

$$|\Re|^2 + |\Im|^2 = 1 + \left[\frac{2\cos^2(\pi A/\alpha)\sin^2(\pi B/\alpha)\sinh^2(\pi k/\alpha) + \sin(2\pi A/\alpha)\sin(2\pi B/\alpha)\sinh 2\pi k/\alpha}{\left(\sinh^2(\pi k/\alpha) + \sin^2(\pi A/\alpha)\cos^2(\pi B/\alpha)\right)\cosh^2(\pi k/\alpha) - \cos^2(\pi A/\alpha)\sin^2(\pi B/\alpha)\sinh^2(\pi k/\alpha)}\right], \quad (6)$$

and hence, the flux is not conserved! Physically one would expect this to be the case in presence of the absorbing/emitting imaginary part of the potential. However, when $\mathcal{PT}$-symmetry is unbroken, there must not be any absorption/emission as the eigenvalues are necessarily real, which has been experimentally established [4]. The deviation term in square bracket in Eq.6 does not vanish for preserved $\mathcal{PT}$-symmetry in general [14]. This is precisely the situation generically proposed earlier [15] for $\mathcal{PT}$-symmetric systems. It can be easily verified that the extra term vanishes for $B \to \pm iB$, which is obvious, as the system becomes Hermitian.

It is well-known that pseudo-Hermitian systems require the generalization of the scalar product [7]. $\mathcal{PT}$-symmetric systems can be shown to be pseudo-Hermitian for discrete (bound or resonance) states, with mandatory existence of a by-orthonormal basis (rigged Hilbert space) [6, 16]. In general, this similarity is still under speculation [16]. Therefore, in absence of the definition of norm, the direct verification of flux conservation for a generic $\mathcal{PT}$-symmetric system is ambiguous. $|\Im|^2$ and $|\Re|^2$ are to be re-defined suitably under that norm, to see whether the flux is conserved under $\mathcal{PT}$-symmetry or whether there is absorption/emission. Owing to the realness of the discrete spectrum for unbroken $\mathcal{PT}$-symmetry, it is expected that the probability flux can be conserved, while scattered by such $\mathcal{PT}$-symmetric potential, albeit subjected to proper re-definition.

II. S-MATRIX FOR $\mathcal{PT}$-SYMMETRIC SYSTEM AND CONTINUITY EQUATION

Operator action, in quantum mechanics, can be defined without the help of a well-defined norm, so long as expectation values are not summoned into the picture, given a right-operation (or left, but obviously not both) is defined. Although the matrix elements can be evaluated only after fixing a norm, algebraic conditions can still be obtained. In the absence of a physical norm for states of a $\mathcal{PT}$-symmetric system, one can still obtain conserved bilinear, from a generic conservation law in spirit of the equation of continuity [13]. To obtain the same, we consider a generic one-dimensional system defined by a $\mathcal{PT}$-symmetric non-Hermitian potential, which is asymptotically Hermitian, as it converges to a unique real constant as $x \longrightarrow \pm\infty$. Then the general asymptotic solution can always be written as a linear superposition of incoming and outgoing plane waves as,

$$\psi(x) \longrightarrow \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if} x \longrightarrow -\infty \\ Ce^{ikx} + De^{-ikx} & \text{if } x \longrightarrow \infty \end{cases}, \quad (7)$$

with A, B, C, D being complex (C) numbers.

In the asymptotic treatment of real (read Hermitian) converging potential, the complex co-efficients are linked as,

$$\begin{pmatrix} A \\ B \end{pmatrix} = M \begin{pmatrix} C \\ D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C \\ B \end{pmatrix} = S \begin{pmatrix} A \\ D \end{pmatrix}, \quad (8)$$

where M and S are transfer and scattering matrices respectively, linking left-right and incoming-outgoing states. The fact that one can define a conserved current: $j(x,t) = \frac{\hbar}{i2\pi}\left[\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t) - \psi(x,t)\frac{\partial}{\partial x}\psi^*(x,t)\right]$, leads to the unitarity of the S-matrix, and transfer matrix satisfies the condition,

$$M^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9)$$

In case of a $\mathcal{PT}$-symmetric system, the usual definition of the probability current cannot be arrived at. This can be understood by looking at the 1-D Schrödinger equation: $-\frac{\partial^2}{\partial x^2}\psi(x,t)+V(x)\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$. In the Hermitian case, we arrive at the equation of continuity by removing the potential term from the equation, by considering together the complex-conjugate of the Schrödinger equation. But if the potential is $\mathcal{PT}$-symmetric, as $V(x) \longrightarrow V^*(x)$, we are to take a $\mathcal{PT}$-transformation of the equation [13], in order to obtain the $\mathcal{PT}$-version of the equation of continuity:

$$\frac{\hbar}{i2\pi}\frac{\partial}{\partial x}\left(\psi(x,t)\frac{\partial}{\partial x}\psi^*(-x,t) - \psi^*(-x,t)\frac{\partial}{\partial x}\psi(x,t)\right) = i\frac{\partial}{\partial t}\left(\psi^*(-x,t)\psi(x,t)\right). \tag{10}$$

Thus, one arrives at a new definition of flux, which is conserved. This is achieved at the expense of real positive-definite norm of the Hermitian theory, as evident from the time-derivative part of the above equation. This further leads to re-interpretation of the scattering process, which will be discussed a little later. On following the conservation of this new 'flux' over scattering, where we consider asymptotic solutions given in Eq.7, one finds,

$$AB^* - BA^* = CD^* - DC^*. \tag{11}$$

Following the above, the transfer matrix obeys,

$$M^\dagger \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12}$$

As for the scattering matrix, the same conservation leads to Hermiticity, $S^\dagger = S$, but not unitarity! The notion of Hermitian conjugate here is purely mathematical. Hermitian conjugate of any matrix Λ is taken to be the matrix that results into the dual of any vector $Y = \Lambda X$ by left-operating on the dual of vector X. The mere structure of Eq.11 allows this construction, and there is no attempt to extract any physical interpretation for this Hermitian conjugation, unlike in usual quantum mechanics. But even then, the defining meanings of M and S holds, owing to boundary conditions. A re-defined physical norm must only affect the elements of these matrices, but not their definitions.

As an interesting remark, it is straightforward to realize that the scattering matrix cannot be unitary for pseudo-Hermitian Hamiltonians. This is relevant as $\mathcal{PT}$-symmetric systems conditionally behave as pseudo-Hermitian [6], and scattering matrix can be defined for such states. Let us consider a pseudo-Hermitian system with a generalized norm $< n|\eta|m >$, with η being the 'norm operator'. From the definition $|out >= S|in >$ and from flux conservation, we have,

$$\begin{aligned} < in|\eta|in > &= < out|\eta|out >=< in|S^\dagger\eta S|in > \\ \text{or,} \quad \eta &= S^\dagger\eta S \\ \text{or,} \quad S^{-1}\eta &= S^\dagger\eta. \end{aligned} \tag{13}$$

Therefore, S cannot be unitary for a non-trivial η. But as stated before, one cannot carry-over the prescription of pseudo-Hermitian norm to $\mathcal{PT}$-symmetric scattering states. Subsequently, even for scattering states, the conserved quantity for $\mathcal{PT}$-symmetric systems (Eq.10) cannot be considered as a norm. The Hermiticity of the scattering matrix is a consequence of anti-linearity of the $\mathcal{PT}$-transformation, which only arises from the consideration of asymptotic states. Therefore, the scattering matrix is not evidently Hermitian for pseudo-Hermitian systems. In the next section, we will explicate the construction of a pseudo-Hermitian norm, with accordance to Das and Greenwood [7], for the case of $\mathcal{PT}$-symmetry, and discuss the difficulties that arise.

On appreciating the fact that in general, the $\mathcal{PT}$-symmetric potential has the form,

$$V(x) = V_{even}(x) + iV_{odd}(x), \tag{14}$$

where the suffixes mention respective parity of the functional parts of the potential which are *real*. Then, clearly, $H^\dagger\left(V_{odd}(x)\right) = H\left(-V_{odd}(x)\right)$. Let $\psi_\pm(x,t)$ be solutions to $H\left(\pm V_{odd}(x)\right)$. The corresponding S-matrices, $S_\pm$, are Hermitian. On considering the time-independent scenario, if $V_{odd}(x \to \infty) \longrightarrow 0$ and if $V_{even}(x)$ is also well-behaved asymptotically, the asymptotic momenta $K_\pm$ will be related as $K^*_\pm = K_\mp$. It is clearly seen that $\psi^*_+(x,t)$ and $\psi_-(x,t)$ are the eigenfunctions to $H\left(-V_{odd}(x)\right)$, whereas $\psi^*_-(x,t)$ and $\psi_+(x,t)$ are the eigenfunctions to $H\left(V_{odd}(x)\right)$, with different eigenvalues. But, as both the Hamiltonians asymptotically converge to that of a free particle, these solutions must be the same, as there is no degeneracy in the 1-D case. Same can be argued about the corresponding eigenvalues; the asymptotic coefficients for both the systems are to satisfy,

$$
\begin{aligned}
A_+^* = B_-, \quad B_+^* = A_-, \\
C_+^* = D_- \text{ and } G_+^* = F_-.
\end{aligned}
$$

Then the definition of S-matrix yields,

$$
\begin{aligned}
\begin{pmatrix} B_+^* & C_+^* \end{pmatrix} &= \begin{pmatrix} A_+^* & D_+^* \end{pmatrix} S^\dagger(V_{odd}) \\
\text{or,} \quad \begin{pmatrix} B_+^* & C_+^* \end{pmatrix} \begin{pmatrix} B_- \\ C_- \end{pmatrix} &= \begin{pmatrix} A_+^* & D_+^* \end{pmatrix} S^\dagger(V_{odd}) \begin{pmatrix} B_- \\ C_- \end{pmatrix} \\
\text{or,} \quad \begin{pmatrix} A_- & D_- \end{pmatrix} \begin{pmatrix} B_- \\ C_- \end{pmatrix} &= \begin{pmatrix} B_- & C_- \end{pmatrix} S^\dagger(V_{odd}) S(-V_{odd}) \begin{pmatrix} A_- \\ D_- \end{pmatrix} \\
&= \begin{pmatrix} A_- & D_- \end{pmatrix} S^\tau(-V_{odd}) S^*(V_{odd}) \begin{pmatrix} B_- \\ F_- \end{pmatrix} \\
\text{or,} \quad S^\tau(-V_{odd}) S^*(V_{odd}) &= I \\
\text{or,} \quad S^\dagger(V_{odd})\, S(-V_{odd}) &= I,
\end{aligned}
\tag{15}
$$

where the relation between the coefficients have been used. Here τ signifies transposition. The final result is the *Duality* condition already well-appreciated in the optical analogues of $\mathcal{PT}$-symmetric systems [2]. Eq.15 ensures Hermiticity, and subsequent unitarity, of the system for $V_{odd} \rightarrow 0$. This result is intuitively linked with the fact that $V_{odd} \rightarrow -V_{odd}$ essentially is the Hermitian conjugation. In the above derivation, the asymptotic behavior of the $\mathcal{PT}$-symmetric system being Hermitian, has been utilized extensively. Also, the free particle solution having a unique momentum is a key fact here.

The $\mathcal{PT}$-symmetric conservation relation in Eq.10 allows some valuable insights. It yields Eq.11, with the critical observation that if A and C are considered to be the respective amplitudes of the fluxes from $\pm\infty$, then the absence of either to begin with makes the two amplitudes on the other side complex-conjugates. Also, it does not mean that the outgoing flux on the previous side vanishes, although it actually does, as we will see later from symmetry arguments. Moreover, as only the cross-terms appear in Eq.11, incident, reflected or transmitted fluxes are not intuitively separable, unlike in a Hermitian system, thus preventing any hint regarding the physical norm. Further, the conserved scalar not being real, the unitary evolutions are no more trivial (unitarity itself has to be re-defined as the norm is to be re-defined). This also restricts it from being a physical observable. We, for now, do not intend to surpass these difficulties, but try to extract as much information as possible from the available structure. In the next section, we try to approach a possible norm for $\mathcal{PT}$-symmetric systems.

In absence of a physical scalar product, the equation of continuity can be utilized to study the system, as the inherent symmetry of the system is incorporated within it. From Eq.10, if $\mathcal{PT}$-symmetry is unbroken, *i.e.*, if $\psi^*(-x,t) \equiv \psi(x,t)$, the 'current' itself vanishes:

$$
\frac{\partial}{\partial t}\psi^2(x,t) = 0. \tag{16}
$$

Although the wave-function is still complex in general, it is explicitly time-independent and hence, physically corresponds to a stationary state. Though this is justifiable at the level of equation of motion (time-dependent Schrödinger equation), it does not allow proper eigenvalue equation (The time-independent Schrödinger equation) for stationary states. In other words, the bound 'energy' is always zero!, which is incorrect, even experimentally [4]. This contradiction is straightforward to arrive at for any state preserving the symmetry of the corresponding system. Instead of looking for a solution $\psi(x,t)$, which explicitly depends on time despite of $\psi^2(x,t)$ being time-independent, we observe that in Hermitian systems , the 'stationary' n-th bound-state eigenfunction is 'real' only up to a multiplicative unitary term, $\exp(-iE_n t)$, and thus satisfies proper eigenvalue equation $i\hbar\frac{\partial}{\partial t}\psi_n(x,t) = E_n\psi_n(x,t)$. Such states preserve the 'Hermitian symmetry', $\psi^*(x,t) \equiv \psi(x,t)$, corresponding to zero current. The corresponding physical conserved quantity $|\psi(x,t)|^2$ allows unitary time-dependence of $\psi(x,t)$, satisfying the eigenequation, which stays *implicit* while the stationary state is deemed real. Thus, the 'realness' of the wave-functions strictly implies to the space part only. Therefore, while defining $\mathcal{PT}$-symmetry of a eigenfunction, the 'energy-phase' is always to be isolated in order to satisfy the eigenequation of *any* $\mathcal{PT}$-symmetric system. It amounts to considering $\psi(x,t) \rightarrow \exp(-iE_n t)\psi(x,t)$, with the exponent to be left-out while defining the $\mathcal{PT}$-symmetry of the eigenfunction, similar to defining realness for eigenfunction of a Hermitian system. Physically, this is because the conserved 'charge' must not be time-dependent, yet there must be unitary time-dependence in the eigenfunction in order to yield correct energy eigenvalues. Conserved charge of the form $\psi(x,t)\psi^*(-x,t)$ allows unitary time-dependence just like the form $|\psi(x,t)|^2$. Incidentally, non-$\mathcal{PT}$-symmetric states, with complex-conjugate eigenvalues, correspond to non-vanishing current, thus being non-trivially time-dependent. This is expected, as they are interpreted as resonances.

One cannot directly call a $\mathcal{PT}$-symmetric state a bound one, as the conserved 'current' is not yet interpretable as corresponding to any physical probability. But stationarity of the eigenstates itself, at least, hint towards it, as the status of the wave-function as complete representation of the physical state of the system is still preserved. Further, it has been explicitly shown that the norm operator η for such systems will necessarily be stationary [17]. Then one can demand on physical grounds that the conserved charge corresponding to scattering states cannot be stationary. Following Eq.10, this means that the corresponding eigenfunction must be non-trivially time-dependent. Then the corresponding 'current' must have a non-zero space derivative. It cannot be identically zero, which is the condition for bound states, and hence the corresponding eigenfunction *cannot* be $\mathcal{PT}$-symmetric. This conclusion prevents us from considering the scattering solutions for a $\mathcal{PT}$-symmetric system to be superpositions of $\mathcal{PT}$-symmetric/anti-symmetric functions, specifically plane waves with *real* or *pure imaginary* coefficients. It also cannot be a *single* plane wave, even with complex co-efficient. This automatically rules out particular boundary conditions, *e.g.*, only incoming flux in either side of the potential. One can have a situation like incidence from left; which will result into reflection back, but *no* transmission. Additionally, as the scattering states are not $\mathcal{PT}$-symmetric, the corresponding eigenvalues cannot be real, and this physically means absorption/emission.

As an example of the above conclusion, we reconsider the potential in Eq.3. As obtained in [12], it has two independent solutions, which, under suitable complexification of the potential, asymptotically have the forms,

$$F_{1,2}(x;A,B,\alpha,k) \longrightarrow \begin{cases} A_{1,2}\exp(ikx) + B_{1,2}\exp(-ikx) & \text{if } x \longrightarrow -\infty \\ C_{1,2}\exp(ikx) + D_{1,2}\exp(-ikx) & \text{if } x \longrightarrow \infty \end{cases}, \tag{17}$$

where,

$$\begin{aligned}
A_1 &= \frac{\Gamma\left(-B/\alpha - A/\alpha + \frac{1}{2}\right)\Gamma\left(-2ik/\alpha\right)}{\Gamma\left(-A/\alpha - ik/\alpha\right)\Gamma\left(-B/\alpha + \frac{1}{2} - ik/\alpha\right)} e^{\pi(k/\alpha + iB/\alpha + iA/\alpha)} - A/\alpha + 2ik/\alpha, \\
B_1 &= \frac{\Gamma\left(-B/\alpha - A/\alpha + \frac{1}{2}\right)\Gamma\left(2ik/\alpha\right)}{\Gamma\left(-A/\alpha + ik/\alpha\right)\Gamma\left(-B/\alpha + \frac{1}{2} + ik/\alpha\right)} e^{\pi(-k/\alpha + iB/\alpha + iA/\alpha)} - A/\alpha - 2ik/\alpha, \\
C_1 &= \frac{\Gamma\left(-B/\alpha - A/\alpha + \frac{1}{2}\right)\Gamma\left(2ik/\alpha\right)}{\Gamma\left(-A/\alpha + ik/\alpha\right)\Gamma\left(-B/\alpha + \frac{1}{2} + ik/\alpha\right)} e^{\frac{\pi}{2}(k/\alpha - iB/\alpha - iA/\alpha)} - A/\alpha - 2ik/\alpha, \\
D_1 &= \frac{\Gamma\left(-B/\alpha - A/\alpha + \frac{1}{2}\right)\Gamma\left(-2ik/\alpha\right)}{\Gamma\left(-A/\alpha - ik/\alpha\right)\Gamma\left(-B/\alpha + \frac{1}{2} - ik/\alpha\right)} e^{\frac{\pi}{2}(-k/\alpha - iB/\alpha - iA/\alpha)} - A/\alpha + 2ik/\alpha, \\
A_2 &= -i\frac{\Gamma\left(\frac{3}{2} + A/\alpha + B/\alpha\right)\Gamma\left(-2ik/\alpha\right)}{\Gamma\left(\frac{1}{2} + B/\alpha + ik/\alpha\right)\Gamma\left(1 + A/\alpha - 1k/\alpha\right)} e^{\pi(k/\alpha - iB/\alpha - iA/\alpha)} - A/\alpha + 2ik/\alpha, \\
B_2 &= -i\frac{\Gamma\left(\frac{3}{2} + A/\alpha + B/\alpha\right)\Gamma\left(-2ik/\alpha\right)}{\Gamma\left(\frac{1}{2} + B/\alpha + ik/\alpha\right)\Gamma\left(1 + A/\alpha + 1k/\alpha\right)} e^{\pi(k/\alpha + iB/\alpha + iA/\alpha)} - A/\alpha - 2ik/\alpha, \\
C_2 &= i\frac{\Gamma\left(\frac{3}{2} + A/\alpha + B/\alpha\right)\Gamma\left(-2ik/\alpha\right)}{\Gamma\left(\frac{1}{2} + B/\alpha + ik/\alpha\right)\Gamma\left(1 + A/\alpha + 1k/\alpha\right)} e^{\frac{\pi}{2}(-k/\alpha + iB/\alpha + iA/\alpha)} - A/\alpha - 2ik/\alpha, \\
D_2 &= i\frac{\Gamma\left(\frac{3}{2} + A/\alpha + B/\alpha\right)\Gamma\left(-2ik/\alpha\right)}{\Gamma\left(\frac{1}{2} + B/\alpha + ik/\alpha\right)\Gamma\left(1 + A/\alpha - 1k/\alpha\right)} e^{\frac{\pi}{2}(-k/\alpha + iB/\alpha + iA/\alpha)} - A/\alpha + 2ik/\alpha.
\end{aligned} \tag{18}$$

These coefficients are all complex for arbitrary momentum k, and do not vanish in general. The phase factor in each of them carries a term linear in iB, the parameter signifying $\mathcal{PT}$-symmetry, ensuring the overall complexity of the coefficients.

Clearly, the system is asymptotically Hermitian, and the flux attenuation/enhancement takes place locally. Thus the conclusion from Eq.6, which is asymptotically valid, is justified. The $\mathcal{PT}$-symmetric states are 'bound', corresponding to discrete and real eigenvalues. Only states without $\mathcal{PT}$-symmetry can be realized asymptotically, which always posses complex eigenvalues. Although we do not have an inner-product to define $\Re$ and $\Im$ locally, we have their expressions at the boundaries, to be considered as proper limits. Same is true for the inner product. This is to be kept in mind throughout that these arguments are provided strictly for a 1-D system, and by flux, nothing more than the 'current' appearing in LHS of Eq.10 is meant. But the interpretation of the eigenfunction of a $\mathcal{PT}$-symmetric Hamiltonian as the complete local representation of the state of the system is still valid, just like that for a Hermitian system.

Despite the system being asymptotically Hermitian, the scattered particle 'carries' the memory of the local symmetry of the Hamiltonian in terms of the constraints on the co-efficients (of the plane-wave modes), which restricts our choice. Thus we can justify Eq.11, and interpret the asymptotic behavior of a $\mathcal{PT}$-symmetric system as the manifestation of non-stationarity of a scattering state, subjected to the intrinsic symmetry of the system. Eq.11 can be physically understood to represent the situation where plane waves, arriving from both sides of the potential, get reflected back. One can read from Eq.11, that a particular 'measure' remains *conserved* in both the sides, which is,

$$BA^* - AB^* = \begin{pmatrix} A^* & B^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \tag{19}$$

The corresponding quantity for a Hermitian system is $|A|^2 - |B|^2$, up to an overall sign, interpreted as the difference between incident and reflected probability densities. For $\mathcal{PT}$-symmetric systems, terms like AB^* can be interpreted as $\psi_{in} PT \psi_{out}$. This is with accordance to the RHS of Eq.10, and is analogus to correlation/overlap function, rather than probability density, of a Hermitian system. Moreover, $\mathcal{PT}$-symmetry can be unbroken only in the case of stationary states, just as the case for Hermitian symmetry. Broken $\mathcal{PT}$-symmetry is known to generate complex-conjugate eigenvalues, interpretable in terms of resonances, which are non-stationary with accordance to the present arguments.

Recently, the CPA, or anti-laser, system found experimental verification [11]. Two coherent beams of laser were incident on a sample with an optical profile respecting $\mathcal{T}$-symmetry, which when is unbroken, both $\Re$ and $\Im$ were observed to vanish. This system was later shown to be a special case of the $\mathcal{PT}$ CPA laser [1], which can generate stimulated emission while shone with coherent radiation under suitable boundary (initial) conditions. It can also completely absorb that radiation for appropriate amplitude and phase relationship, which precisely is the CPA system. As coherent radiation is essentially classical in nature, the evaluation of $\Re$ and $\Im$ is straight-forward. Here we have a quantum mechanical analogue, but utilizing plane waves, instead of coherent radiation. This is analogous to the treatment of [1], where plane waves are considered, representing individual Fourier components of laser. $\mathcal{PT}$-symmetry results in specific relations between the transfer matrix (M) elements, subsequently making the material a perfect absorber or emitter, subjected to suitable boundary conditions. For quantum systems, an extra input, the well-defined norm, is necessary to deal with matrix elements physically for scattering states. Still, from our study of the boundary conditions, the equivalence is obvious. We suggest that experimental realization of the $\mathcal{PT}$ CPA laser will shed much light on the structure of an appropriate inner product for $\mathcal{PT}$-symmetric systems. As $\mathcal{PT}$-symmetry is broken for asymptotic states, by definition, one can expect absorption/emission, and hence, finite loss/gain in the system.

III. A LOOK FOR THE GENERIC NORM IN 1-D $\mathcal{PT}$-SYMMETRIC SYSTEMS

While dealing with scattering states of a $\mathcal{PT}$-symmetric complex potential, if the imaginary part of the potential vanishes at $\pm\infty$, asymptotically the system becomes Hermitian. Therefore, it needs to be studied carefully that how the appropriate conserved scalar product continually goes into the Hermitian norm. Here we try to discuss a $\mathcal{PT}$-symmetric norm in line of that for pseudo-Hermitian systems [7]. A $\mathcal{PT}$-symmetric Hamiltonian is defined as:

$$\begin{aligned} H = (PT)H(PT)^{-1} &\equiv PTHPT \\ &= P(THT)P = PH^*P \\ &= P(H^\dagger)^\tau P = P\tau H^\dagger \tau^{-1} P \\ &= (P\tau)H^\dagger (P\tau)^{-1}, \end{aligned} \tag{20}$$

where τ is the *transposition operator*, defined to relate a particular matrix into its transpose through similarity transformation. It is easy to see that τ depends on the particular matrix and it's representation in the vector space (choice of basis). Using the Schrödinger representation, the relation $THT = H^*$ can be realized as follows:

$$PTHPT = H$$

$$
\begin{aligned}
\text{or,} \qquad THT &= PHP \\
\text{or,} \quad < m|THT|n > &= < m|PHP|n > \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy < m|x >< x|PHP|y >< y|n > \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy < m|x >< -x|H| - y >< y|n > \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy < m| - x > \delta(x-y) H(x) < -y|n > \\
&= \int_{-\infty}^{\infty} dx < m|x > H(-x) < x|n > . \qquad (21)
\end{aligned}
$$

As for a $\mathcal{PT}$-symmetric Hamiltonian, $H(-x) \equiv H^*(x)$, one can identify the last term of the above equation as $< m|H^*|n >$, justifying our earlier identification. The last expression of Eq.20 looks similar to the RHS of the definition of pseudo-Hermitian Hamiltonian, $H = S^{-1}H^\dagger S$ [7], subjected to the identification $(P\tau)^{-1} = S$. It is defined for the Hamiltonian operator and in general to be considered implying transposition only for that operator. Now, one can naively arrive at the prescribed norm:

$$
< \phi|(P\tau)^{-1}|\psi >=< \phi|\tau^{-1}P|\psi >=< \phi|\tau P|\psi >, \qquad (22)
$$

as the transpose of a transposed operator is the operator itself.

The operator τ preserves the antilinear nature, as it has appeared through the utilization of the time-reversal operator. To see that, one can re-consider the conserved 'charge' from Eq.10 as [13]:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(x,t) PT\psi(x,t) dx &= \int_{-\infty}^{\infty} \phi(x,t)\psi^*(-x,t) dx \\
&= \int_{-\infty}^{\infty} \psi^*(x,t)\phi(-x,t) dx \\
&= \int_{-\infty}^{\infty} \psi^*(x,t) P\phi(x,t) dx, \qquad (23)
\end{aligned}
$$

The last result appears as a generalization of the Dirac-von Neuman norm, which has already been proposed extensively [18], with the eigenfunctions ϕ and ψ getting interchanged as an extra feature, which is solely due to the role of parity in 1-D. This will not be the case for a 1-D system achieved through symmetry reduction from some higher dimensional system, *e.g.*, a 3-D system with spherical symmetry. The interchange of ϕ and ψ is in same spirit of anti-unitary operation [19], where for any $|\alpha^\star >= \Theta|\alpha >$, Θ being anti-unitary, one has:

$$
< \alpha^\star|\beta^\star >=< \beta|\alpha > . \qquad (24)
$$

Here, the anti-unitary operator Θ is a generalization of the anti-linear operator T. From Eqs 22 and 23, we conclude that constructing a pseudo-Hermitian norm for $\mathcal{PT}$-symmetric systems, at least for discrete states, necessarily incorporates anti-linearity through operator τ. That such an operator is representation-dependent is duly justified, as the form of norm-operator η always depends on the pseudo-Hermitian system itself. However, the fact that transposition necessarily requires a predefined scalar product, actually makes the norm in the first prescription ill-defined, in the sense of a cyclical construction logic. The second prescription albeit gives us a well-defined conserved scalar product, but it does not qualify as the one leading to the norm, as positive definiteness is not ensured. Also, which state is to be chosen for right-operation is not clear, if one naively starts with this prescription. These inadequacies extend to our earlier difficulty for calculating $|\Re|^2$ and $|\Im|^2$ for a $\mathcal{PT}$-symmetric system in absence of a proper scaler product. There have been prescriptions to extend the conserved scalar product of Eq.23 to be positive-definite [18], but only for systems with finite Hilbert spaces. Systems with non-compact algebraic structure are yet to be tackled, not to mention the already stated difficulty of generic bounded spectral operators. Complex systems with the imaginary part of the potential vanishing at $\pm\infty$ are asymptotically Hermitian, corresponding to a scalar product with suitable asymptotic behavior, coinciding with the Hermitian norm. In this regard, the prescription in Eq.22 looks more suitable of the two, as the task boils down to generalizing τP to obtain a proper pseudo-Hermitian norm, corresponding to $\eta(x \to \pm\infty) \longrightarrow I$.

IV. CONCLUSION

In conclusion, suitable conserved current can be utilized for $\mathcal{PT}$-symmetric systems to obtain valuable informations about the nature of scattering states. It is observed that, these systems necessarily require the presence of both incoming and outgoing states for the asymptotic case. This type of boundary condition has exact analogues for $\mathcal{PT}$ CPA laser and other optical systems. The lack of a proper norm for the generic scattering case precludes the proper extraction of reflection and transmission coefficients, in order to analyze flux conservation. Interestingly, equation of continuity yields invariants, which point towards a different symmetry structure for the scattering states.

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