Asymptotic behaviour of measures under automorphisms

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Various problems concerning probability measures on locally compact groups involve understanding what happens in the limit when a measure, or a sequence of measures, is operated upon by a sequence of automorphisms of the group. In the case of Lie groups it turns out that much of the thrust of the questions can be reduced to studying the behaviour of measures on Euclidean spaces under linear transformations. Our aim here is to describe some simple properties in this respect and their applications to various problems; convergence of types, concentration functions, factor compactness, Levy's measures are some of the topics to which applications will be made.

1 Asymptotics of linear automorphisms

Let V_1 and V_2 be finite-dimensional vector spaces over $\mathbb{R}$. Let $\mathrm{Hom}\,(V_1, V_2)$ denote the space of linear maps of V_1 to V_2. Let $\{\alpha_i\}$ be a sequence in $\mathrm{Hom}\,(V_1, V_2)$.

Proposition 1.1 *i) There exist a subspace U of V_1 and a subsequence $\{\alpha_{k_i}\}$ of $\{\alpha_i\}$ such that $\alpha_{k_i}(v)$ converges for all $v \in U$, and $\alpha_{k_i}(v) \to \infty$ for all $v \notin U$.*

ii) There exist a subspace W of V_1 and a subsequence $\{\alpha_{k_i}\}$ of $\{\alpha_i\}$ such that the following conditions hold: a) for every $w \in W$ there exists a sequence $\{v_i\}$ in V_1 such that $v_i \to w$ and $\{\alpha_{k_i}(v_i)\}$ is convergent, and b) $\alpha_{k_i}(v) \to \infty$ for all $v \in V_1$, uniformly for v in any given compact subset of $V_1 \backslash W$; that is, given a compact subset C of $V_1 \backslash W$ and a compact subset K of V_2 there exists i_0 such that $\alpha_{k_i}(v) \notin K$ for $i \geq i_0$ and $v \in C$.

Proof: i) Let $\mathcal{U}$ be the family of subspaces U' of V_1 for which there exists a subsequence $\{\alpha_{k_i}\}$ such that $\{\alpha_{k_i}(v)\}$ is convergent for all $v \in U'$. The family $\mathcal{U}$ is nonempty as it contains the zero subspace. By dimension considerations we see that $\mathcal{U}$ has a maximal element, say U. Then there exists a subsequence $\{\alpha_{k_i}\}$ such that $\{\alpha_{k_i}(v)\}$ converges for all $v \in U$. Now let $v \notin U$. We would like to show that $\alpha_{k_i}(v) \to \infty$. Suppose this is not true. Then there exists a subsequence $\{\alpha_{l_i}\}$ of $\{\alpha_{k_i}\}$ such that $\{\alpha_{l_i}(v)\}$ converges. Since $\{\alpha_{l_i}(u)\}$ also converges for all $u \in U$ it follows that $\{\alpha_{l_i}(u)\}$ converges for all u in the subspace spanned by v and U. But this contradicts the maximality of U in $\mathcal{U}$. Hence $\alpha_{k_i}(v) \to \infty$ for all $v \notin U$.

ii) The proof is similar to that of (i) so we shall be brief about some of the details. Let $\mathcal{W}$ be the family of subspaces W' of V_1 for which there exists a subsequence $\{\alpha_{k_i}\}$ such that any $w \in W'$ is the limit of a sequence $\{v_i\}$ in V_1 such that $\{\alpha_{k_i}(v_i)\}$ is convergent. The family is nonempty and has a maximal element, say W. Then clearly assertion a) holds, since $W \in \mathcal{W}$. Now let C be any compact subset of $V_1 \backslash W$ and suppose that α_{k_i} do not diverge to ∞ uniformly on C. This means that there exist a sequence $\{v_i\}$ in C and a subsequence $\{l_i\}$ of $\{k_i\}$ such that $\{\alpha_{l_i}(v_i)\}$ is bounded in V_2. Passing to subsequences we may assume that $\{v_i\}$ and $\{\alpha_i(v_i)\}$ are convergent. Let $v \in C \subset V_1 \backslash W$ be the limit of $\{v_i\}$. Then the subspace spanned by W and v is contained in $\mathcal{W}$, contradicting the maximality of W in $\mathcal{W}$. Therefore $\alpha_{k_i}(v) \to \infty$ uniformly on compact subsets of $V_1 \backslash W$. This proves the proposition. $\square$

Remark 1.2 In general given a sequence $\{\alpha_i\}$ the subspaces U and W as in the conclusion of Proposition 1.1 may not be unique; the conclusion may hold for different subspaces with respect to different subsequences; however, once a subsequence satisfying the condition is chosen, the corresponding subspaces are determined uniquely by the conditions to be met. Thus given a sequence we can pass to a subsequence such that for the new sequence both the subspaces U and W as in the conclusion of Proposition 1.1 are uniquely defined.

Example 1.3 Let $V = \mathbb{R}^n$, considered along with the standard basis which we shall denote by $\{e_1, \dots, e_n\}$. Let $\alpha_i : V \to V$, $i = 1, 2, \dots$ be the linear transformations corresponding (with respect to the basis as above) to the diagonal matrices $\mathrm{diag}\,(\lambda_i^{(1)}, \dots, \lambda_i^{(n)})$, where $\{\lambda_i\}$ are sequences in $\mathbb{R}^*$ (nonzero real numbers). Suppose that for each $k = 1, \dots, n$, $\{\lambda_i^{(k)}\}$ either converges (including possibly to 0) or $\lambda_i^{(k)} \to \infty$; if we start with an arbitrary sequence this always holds for some subsequence. It can be seen that when this holds, the conclusions as in Proposition 1.1 hold if we choose U and W to be the subspaces spanned by the e_k's such that the sequence $\{\lambda_i^{(k)}\}$ converges; in this case the two subspaces coincide.

Example 1.4 Suppose that $\{\alpha_i\}$ is a sequence for which there exists subspaces U and W such that for $v \in V_1$, $\{\alpha_i(v)\}$ converges if and only if $v \in U$, and v is a limit of a sequence $\{v_i\}$ with $\{\alpha_i(v_i)\}$ convergent if and only if $v \in W$. Then clearly U is contained in W. For sequences of diagonal matrices as in Example 1.3 we have seen that U equals W. However in general the two subspaces need not be the same. This can be seen from the following example. Let $V = I\!R^2$, with the standard basis $\{e_1, e_2\}$. Let $\rho \in (1, \infty)$ and $\beta_i : V \to V$, $i = 1, 2, \ldots$ be the linear transformations corresponding to the diagonal matrices diag (ρ^{2i}, ρ^{-2i}). Also for each i let θ_i be the transformation consisting of rotation of the plane by an angle π/ρ^i. Let $\alpha_i = \beta_i\theta_i$ for all i. Then a straightforward computation shows that (for any subsequence of $\{\alpha_i\}$) U is the zero subspace while W is the subspace spanned by e_2.

For any locally compact space X we shall henceforth denote by $P(X)$ the space of all probability measures on X, equipped with the usual weak* topology with respect to the space of bounded continuous functions on X; thus $\lambda_i \to \lambda$ for λ_i, $i = 1, 2, \ldots,$ and λ in $P(X)$ if $\int f \mathrm{d}(\lambda_i) \to \int f \mathrm{d}(\lambda)$ for all bounded continuous functions on X.

Corollary 1.5 *Let U and W be the subspaces of V_1 for which the conclusion as in Proposition 1.1 holds. If $\lambda \in P(V_1)$ is such that $\{\alpha_i(\lambda)\}$ converges then* supp λ *is contained in U. If $\{\lambda_i\}$ is a sequence in $P(V_1)$ such that $\lambda_i \to \lambda$ and $\{\alpha_i(\lambda_i)\}$ converges then* supp λ *is contained in W.*

Proof: Suppose that $\{\alpha_i(\lambda)\}$ converges to a measure $\mu \in P(V_2)$. By passing to a subsequence and modifying notation we shall assume, as we may, that $\{\alpha_i(v)\}$ converges for all $v \in U$, and $\alpha_i(v) \to \infty$ for all $v \notin U$. Let $f : V_2 \to [0, 1]$ be any continuous function with compact support. Then

$$\int_{V_2} f \mathrm{d}(\alpha_i(\lambda)) = \int_U f \circ \alpha_i \mathrm{d}\lambda + \int_{V_1 \backslash U} f \circ \alpha_i \mathrm{d}\lambda \leq \lambda(U) + \int_{V_1 \backslash U} f \circ \alpha_i \mathrm{d}\lambda.$$

Taking limit as $i \to \infty$ we get that $\int_{V_2} f \mathrm{d}\mu \leq \lambda(U)$. Since this holds for all continuous functions f with compact support it follows that $\lambda(U) = 1$, which proves the first assertion.

Now let $\{\lambda_i\}$ be a sequence in $P(V_1)$ such that $\lambda_i \to \lambda$ and $\{\alpha_i(\lambda_i)\}$ converges, to say a measure $\mu \in P(V_2)$. Passing to a subsequence we may assume that $\alpha_i(v) \to \infty$ uniformly on compact subsets of $V_1 \backslash W$. Suppose that λ is not supported on W. Then there exists a compact subset C of $V_1 \backslash W$ such that $\lambda(C) > 0$. Let $\delta = \lambda(C)/2 > 0$. Let K be a compact subset of V_2 such that $\mu(K) > 1 - \delta$. Let Ω be a compact neighbourhood of K. Since $\alpha_i(\lambda_i) \to \mu$ and $\mu(K) > 1 - \delta$ it follows that $\alpha_i(\lambda_i)(\Omega) > 1 - \delta$ for all large i. Let Θ be a compact neighbourhood of C in $V_1 \backslash W$. Since $\alpha_i(v) \to \infty$ uniformly on compact subsets of $V_1 \backslash W$ we get that $\alpha_i(\Theta)$ is disjoint from Ω for all large i. Hence for all large i, $\alpha_i(\lambda_i)(\alpha_i(\Theta)) < \delta$, and so $\lambda_i(\Theta) < \delta$. Since

$\lambda_i \to \lambda$ this implies that $\lambda(C) \leq \delta$, which is a contradiction since $\delta = \lambda(C)/2$. Hence λ is supported on W. $\square$

In the sequel we shall sometimes use the following terminology for subsets and sequences in locally compact spaces. A subset of a locally compact space X is said to be *bounded* if its closure X is compact. A sequence in a locally compact space X is said to be *divergent* if it has no convergent subsequence, or equivalently if any compact subset of X contains only finitely many elements of the sequence. (In the author's view these terms are intuitively more appealing than their counterparts, in the context of the properties discussed here).

Let V_1 and V_2 be two finite-dimensional vector spaces over $I\!R$, as before. Then $\mathrm{Hom}\,(V_1, V_2)$ is also a finite-dimensional vector space over $I\!R$; we shall consider it equipped with the topology as a real vector space.

Proposition 1.6 *If every $v \in V_1$ can be expressed as a limit of a sequence $\{v_i\}$ such that $\{\alpha_i(v_i)\}$ converges, then $\{\alpha_i\}$ is bounded in Hom (V_1, V_2).*

Proof: Suppose that for every $v \in V_1$ there exists a sequence $\{v_i\}$ such that $v_i \to v$ and $\{\alpha_i(v_i)\}$ converges. Let $\{v^{(1)}, v^{(2)}, \ldots, v^{(d)}\}$ be a basis of V_1, where d is the dimension. Let $\{v_i^{(j)}\}$, $j = 1, \ldots, d$, be sequences such that $v_i^{(j)} \to v^{(j)}$ and $\{\alpha_i(v_i^{(j)})\}$ converges, for each j. Let $\theta_i : V_1 \to V_1$ be the linear transformations defined by the conditions $\theta_i(v^{(j)}) = v_i^{(j)}$ for all $i = 1, 2, \ldots$, and $j = 1, 2, \ldots, d$. Then $\alpha_i \circ \theta_i \in \mathrm{Hom}\,(V_1, V_2)$ for all i. Also $\{\alpha_i \circ \theta_i(v^{(j)})\}$ converges for all $j = 1, 2, \ldots, d$, and therefore $\{\alpha_i \circ \theta_i\}$ is bounded in $\mathrm{Hom}\,(V_1, V_2)$. On the other hand $\{\theta_i\}$ converges to the identity transformation in $GL(V_1)$. Therefore it follows that $\{\alpha_i\}$ is bounded (one may consider for instance the norms of the transformations to see this). This proves the proposition. $\square$

Corollary 1.7 *Let $\lambda \in P(V_1)$ be such that* $\mathrm{supp}\,\lambda$ *spans V_1. If there exists a sequence $\{\lambda_i\}$ in $P(V_1)$ such that $\lambda_i \to \lambda$ and $\{\alpha_i(\lambda_i)\}$ converges then $\{\alpha_i\}$ is bounded in Hom (V_1, V_2).*

Proof: Suppose $\{\alpha_i\}$ is not bounded. Passing to a subsequence we may assume that it is divergent (namely has no convergent subsequence). Let $\{\alpha_{k_i}\}$ be a subsequence of $\{\alpha_i\}$ and W be a subspace of V_1 such that the conclusion as in Proposition 1.1 holds. Since $\lambda_{k_i} \to \lambda$ and $\{\alpha_{k_i}(\lambda_{k_i})\}$ converges, by Corollary 1.5 $\mathrm{supp}\,\lambda$ is contained in W. Since by hypothesis $\mathrm{supp}\,\lambda$ is spans V_1 it follows that $W = V_1$. Proposition 1.6 therefore implies that $\{\alpha_{k_i}\}$ is bounded in $\mathrm{Hom}\,(V_1, V_2)$. But this is a contradiction since $\{\alpha_i\}$ was arranged to be divergent. This shows that $\{\alpha_i\}$ is bounded, thus proving the corollary. $\square$

Example 1.8 Let $V_1 = V_2 = \mathbb{R}$. Let $\{\theta_i\}$ be a sequence of positive numbers such that $\theta_i \to \infty$, and for each i let $\alpha_i \in \mathrm{Hom}\,(V_1, V_2)$ be defined by $\alpha_i(t) = \theta_i t$ for all $t \in \mathbb{R}$. Then $\{\alpha_i\}$ is not bounded in $\mathrm{Hom}\,(V_1, V_2)$. However consider the measures λ_i, $i = 1, 2, \ldots$, defined by $\lambda_i(E) = \theta_i \ell(E \cap [0, \theta_i^{-1}])$ for all Borel subset E of $\mathbb{R}$, where ℓ is the Lebesgue measure on $\mathbb{R}$. Then $\lambda_i \to \delta_0$, the point mass at 0, while each $\alpha_i(\lambda_i)$ is the restriction of ℓ to $[0, 1]$. This shows that the condition in the corollary that $\mathrm{supp}\,\lambda$ spans V_1 can not be dropped.

Proposition 1.9 *Let V_1 and V_2 be finite-dimensional algebras over $\mathbb{R}$, and $\{\alpha_i\}$ be a sequence of algebra homomorphisms of V_1 into V_2. Let U and W be subspaces of V_1 such that the conclusion as in Proposition 1.1 holds (with respect to some subsequences). Then U and W are subalgebras of V_1.*

Proof: We shall show that W is a subalgebra. A similar argument shows that U is also a subalgebra. Let $v, v' \in W$ be given. Then there exist sequences $\{v_i\}$ and $\{v_i'\}$ such that $v_i \to v$, $v_i' \to v'$, and $\{\alpha_i(v_i)\}$ and $\{\alpha_i(v_i')\}$ are convergent. Then $v_i v_i' \to vv'$ and $\{\alpha_i(v_i v_i')\} = \{\alpha_i(v_i)\alpha_i(v_i')\}$ is convergent. Therefore $vv' \in W$. This shows that W is a subalgebra of V_1. $\square$

Corollary 1.10 *Let V_1 and V_2 be finite-dimensional algebras and let $\{\alpha_i\}$ be a sequence of algebra homomorphisms of V_1 into V_2. Let $\lambda \in P(V_1)$ be such that* $\mathrm{supp}\,\lambda$ *is not contained in any proper subalgebra of V_1. If there exists a sequence $\{\lambda_i\}$ in $P(V_1)$ such that $\lambda_i \to \lambda$ and $\{\alpha_i(\lambda_i)\}$ is convergent then $\{\alpha_i\}$ is bounded in Hom$\,(V_1, V_2)$.*

Proof: is similar to the proof of Corollary 1.7; we need only note that W as in the argument is now a subalgebra, by Proposition 1.9, and by hypothesis $\mathrm{supp}\,\lambda$ is not contained in any proper subalgebra of V_1. $\square$

In the following sections we shall apply these results and techniques to a variety of topics. It may be emphasized that the presentation here is not meant to give a comprehensive account of the work on the topic concerned. Rather, the aim is to describe a part of it, and this turns out to be substantial, that can be obtained using the above theme; we shall however mention along the way some key references through which the interested reader could pursue the topic further. In a similar spirit, wherever convenient (depending on the references cited for requisite results), we shall restrict ourselves to second countable (and hence separable metrizable) locally compact groups, even if the condition is not strictly needed, in the overall context of available literature.

2 Convergence of types and invariance groups

Specialising the results of the last sections to the case when the domain and range spaces are the same, leads to various theorems on what is called 'convergence of types'.

Let V be a finite-dimensional real vector space. A probability measure $\lambda \in P(V)$ is said to be *full* if the support of λ spans V (namely it is not contained in any proper subspace of V). If V is an algebra then we say that $\lambda \in P(V)$ is *a-full* if the support of λ is not contained in any proper subalgebra of V. The results above readily yield the following theorem.

Theorem 2.1 (Convergence of types theorem) *Let V be a finite-dimensional vector space over $\mathbb{R}$ and $\{\alpha_i\}$ be a sequence in $GL(V)$. Let $\lambda, \mu \in P(V)$ and suppose that there exists a sequence $\{\lambda_i\}$ in $P(V)$ such that $\lambda_i \to \lambda$ and $\alpha_i(\lambda_i) \to \mu$. Suppose also that one of the following conditions holds:*

i) λ and μ are full measures on V, or

ii) V is an algebra, each α_i is an algebra homomorphism of V, and λ and μ are a-full measures on V.

Then $\{\alpha_i\}$ is bounded (relatively compact) in $GL(V)$.

Proof: By Corollaries 1.7 and 1.10 the conditions $\lambda_i \to \lambda$ and $\alpha_i(\lambda_i) \to \mu$ imply together with i) and ii) as above that $\{\alpha_i\}$ is bounded in $\mathrm{Hom}\,(V, V)$. For all i let $\mu_i = \alpha_i(\lambda_i)$. Then the conditions in the hypothesis also imply that $\mu_i \to \mu$ and $\alpha_i^{-1}(\mu_i) \to \lambda$, and hence by Corollaries 1.7 and 1.10 $\{\alpha_i^{-1}\}$ is also bounded in $\mathrm{Hom}\,(V, V)$; we note in this respect that if $\alpha \in GL(V)$ is an algebra homomorphism then so is α^{-1}. Together, the conclusions mean that $\{\alpha_i\}$ is a bounded in $GL(V)$. □

The name 'convergence of types' signifies the following. Two probability measures $\lambda, \lambda' \in P(V)$ are said to be of the same 'type', if there exists $\alpha \in GL(V)$ such that $\alpha(\lambda) = \lambda'$. Theorem 2.1(i) shows that if $\{\lambda_i\}$ and $\{\lambda'_i\}$ are convergent sequences in $P(V)$ such that for each i the measures λ_i and λ'_i are of the same type and if the limits of $\{\lambda_i\}$ and $\{\lambda'_i\}$ are full measures, then they are of the same type.

For any $\lambda \in P(V)$ we denote by $I(\lambda)$ and $J(\lambda)$ the subgroups of $GL(V)$ defined by

$$I(\lambda) = \{\gamma \in GL(V) \mid \gamma(\lambda) = \lambda\} \text{ and}$$
$$J(\lambda) = \{\gamma \in GL(V) \mid \gamma(v) = v \text{ for all } v \in \mathrm{supp}\,\lambda\}.$$

It is easy to see that $I(\lambda)$ and $J(\lambda)$ are closed subgroups of $GL(V)$, and that $J(\lambda)$ is a normal subgroup of $I(\lambda)$. We note the following:

Corollary 2.2 *$I(\lambda)/J(\lambda)$ is compact. In particular if the support of λ spans V then $I(\lambda)$ is compact.*

Proof: Let W be the subspace spanned by the support of λ. Let G be the subgroup of $GL(V)$ consisting of all $\alpha \in GL(V)$ such that $\alpha(W) = W$. If $\alpha \in I(\lambda)$ then the support of λ is invariant under the action of α, and hence $\alpha(W) = W$. Hence $I(\lambda)$ is contained in G. Let $\eta : G \to GL(W)$ be the map defined by setting, for any $\alpha \in G$, $\eta(\alpha)$ to be the restriction of α to W. Clearly η is a continuous surjective homomorphism. The kernel of η is the subgroup consisting of all α such that $\alpha(w) = w$ for all $w \in W$. Since $\operatorname{supp}\lambda$ spans W the subgroup is the same as $J(\lambda)$. This yields a continuous bijective map $\overline{\eta} : G/J(\lambda) \to GL(W)$, defined by $\overline{\eta}(gJ(\lambda)) = \eta(g)$ for all $g \in G$. Since $I(\lambda)$ is a closed subgroup of G containing $J(\lambda)$, to show that $I(\lambda)/J(\lambda)$ is compact it now suffices to show that $\eta(I(\lambda))$ is compact. Let λ' be the measure on W given by restriction of the measure λ on V (recall that λ is supported on W). Clearly $\eta(I(\lambda))$ consists precisely of all $\beta \in GL(W)$ which leave invariant the measure λ', namely it is the subgroup $I(\lambda')$ of $GL(W)$. Thus it now suffices to prove the corollary in the special case when λ is a full measure on V. Now if λ is a full measure then by Theorem 2.1 every sequence $\{\alpha_i\}$ in $I(\lambda)$ is bounded in $GL(V)$. This shows that $I(\lambda)$ is compact in this case. □

Remark 2.3 We note also, in the converse direction, that given a compact subgroup K of $GL(V)$ there exists $\lambda \in P(V)$ such that $\operatorname{supp}\lambda$ spans V and λ is K-invariant. Furthermore, it may be noted that any compact subgroup of $GL(V)$ is a group of isometries with respect to an inner product on V. Thus, for a probability measure λ whose support spans V the linear automorphisms leaving λ invariant are isometries with respect to an inner product.

The study of the asymptotics of measures on algebras as above can be applied to the question of convergence of types of measures on Lie groups. Let G be a connected Lie group and $\mathrm{Aut}\,(G)$ be the group of continuous automorphisms of G; (we consider the latter with its usual structure as a Lie group - see [18]). Then the question of convergence of types may be (informally) posed as follows: Let $\{\lambda_i\}$ be a sequence in $P(G)$ and $\{\alpha_i\}$ be a sequence in $\mathrm{Aut}\,(G)$ such that $\lambda_i \to \lambda$ and $\alpha_i(\lambda_i) \to \mu$, where $\lambda, \mu \in P(G)$. Then under what further conditions on λ and μ, by way of 'genericity' of the measures which would correspond to the fullness in the classical case of $\mathbb{R}^n$, $n \geq 1$, can we conclude that $\{\alpha_i\}$ is bounded (relatively compact) in $\mathrm{Aut}\,(G)$?

Suppose that the Lie group G can be realised as a subset of a real vector space V in such a way that the automorphisms α_i are restrictions of nonsingular linear transformations of V, namely elements of $GL(V)$. Now if the limit measures λ and

μ as above are generic in the sense that their supports (individually), considered as subsets of V, span V as a vector space, then under the convergence conditions as above Theorem 2.1 implies that $\{\alpha_i\}$ is bounded, when viewed as a sequence in $GL(V)$, and hence in $\mathrm{Aut}\,(G)$. If, furthermore, V can be chosen to be an algebra such that α_i are algebra automorphisms, then whenever the supports of λ and μ are known to be not contained a proper subalgebra of V, we can conclude that $\{\alpha_i\}$ is bounded. Variations of this strategy can be used together with Lie group-theoretic techniques to prove various convergence of types theorems for measures on Lie groups. We shall not go into the technical details here but rather content ourselves noting some results that can be obtained in this way; the interested reader is referred to [8] and [10] for details.

It may be observed that for the case when G is the n-dimensional torus with $n \geq 2$, $\mathrm{Aut}\,(G)$ is an infinite discrete group, while $P(G)$ is compact, so no convergence of types kind of assertion can be expected for these groups. In the light of these examples we assume that the center of G has no compact subgroup of positive dimension; this condition also ensures that the automorphism group has only finitely many connected components (see [6]). Under this condition we have the following.

Theorem 2.4 (see [10], Corollary 3.3) *Let G be a connected Lie group such that the center of G contains no compact subgroup of positive dimension. Let $\mathcal{G}$ be the Lie algebra of G, and $\rho : G \to GL(\mathcal{G})$ be the adjoint representation of G. Let $\lambda, \mu \in P(G)$ be such that every subalgebra of $End\,\mathcal{G}$ which contains either* $\mathrm{supp}\,\lambda$ *or* $\mathrm{supp}\,\mu$ *also contains $\rho(G)$. Suppose also that the support of λ is not contained in any closed normal subgroup H such that G/H is a vector group (topologically isomorphic to $\mathbb{R}^n$ for some n). Let $\{\alpha_i\}$ be a sequence of continuous automorphisms of G. Suppose that there exists a sequence $\{\lambda_i\}$ in $P(G)$ such that $\lambda_i \to \lambda$ and $\alpha_i(\lambda_i) \to \mu$. Then $\{\alpha_i\}$ is bounded.*

In the case of simply connected nilpotent Lie groups the convergence of types theorem takes a simple form. The following result, which was proved earlier by Hazod and Nobel, can be obtained as a special case of the above theorem (actually in this case the technical issues involved are substantially simpler); see [8] for details.

Theorem 2.5 *Let G be a simply connected nilpotent Lie group. Let $\lambda, \mu \in P(G)$ be such that neither* $\mathrm{supp}\,\lambda$ *nor* $\mathrm{supp}\,\mu$ *is contained in a (closed) connected Lie subgroup of G. Let $\{\alpha_i\}$ be a sequence of continuous automorphisms of G. If there exists a sequence $\{\lambda_i\}$ in $P(G)$ such that $\lambda_i \to \lambda$ and $\alpha_i(\lambda_i) \to \mu$ then $\{\alpha_i\}$ is bounded.*

Some other results of Hazod on convergence of types have also been also deduced from Theorem 2.4 in [10].

We next describe an analogue of Corollary 2.2; (see [8], Theorem 1.8).

Corollary 2.6 *Let G be a connected Lie group such that the center of G has no compact subgroup of positive dimension. Let $\lambda \in P(G)$ and let*

$$I(\lambda) = \{\alpha \in \mathrm{Aut}\,(G) \mid \alpha(\lambda) = \lambda\} \quad \textit{and}$$

$$J(\lambda) = \{\alpha \in \mathrm{Aut}\,(G) \mid \alpha(g) = g \textit{ for all } g \in \textit{supp}\,\lambda\}.$$

Then $I(\lambda)/J(\lambda)$ is compact. In particular if the support of λ is not contained in any proper closed subgroup of G then $I(\lambda)$ is compact.

3 Concentration functions

In this section we apply the results on asymptotics of measures to the study of what are called concentration functions of probability measures. By and large we follow the argument as in [9]; there are however some differences towards the end.

We begin by recalling various definitions and facts. Let G be a locally compact group. For any $\mu \in P(G)$ and any compact subset K of G let

$$c_K(\mu) = \sup_{x\in G} \mu(Kx);$$

it is called the *concentration function* of μ; calling it a 'function' may seem confusing to some readers as the implicit domain for such a function is the rather out of the way set of all compact subsets of G; the nomenclature is however traditional, and it may be noted in this respect that for the classical case of $G = I\!R$ it suffices to consider compact sets of the form $[-t, t]$, on which it corresponds to a function of a real variable. We note that if μ is 'concentrated in a small part' of the group, in the intuitive sense, then $c_K(\mu)$ is close to 1, while if it is 'thinly spread' then $c_K(\mu)$ is small.

Now, given a probability measure μ on G consider its convolution powers μ^n, for $n = 1, 2, \ldots$. How does the 'concentration' behave as n goes to infinity? The vague intuitive expectation would be that if the group is noncompact the convolution powers would 'scatter', so their concentration function would go to 0, for every fixed compact subset K.

We say that $\mu \in P(G)$ is *scattering* if for every compact subset K of G, $c_K(\mu^n) \to 0$ as $n \to \infty$.

For $\mu \in P(G)$ we denote by $G(\mu)$ be the smallest closed subgroup of G containing the support of μ. To begin with we note the following.

Lemma 3.1 *Let G be a locally compact group and $\mu \in P(G)$. Then μ is scattering as a measure on G if and only if it is scattering as a measure on $G(\mu)$.*

Proof: Let K be any compact subset of G. Let $x \in G$ and $n \geq 1$ be such that $\mu^n(Kx) > 0$. Let $g \in Kx \cap G(\mu)$ (the preceding condition implies in particular that this set is nonempty). Then $x \in K^{-1}g$ and hence

$$\mu^n(Kx) = \mu^n(Kx \cap G(\mu)) \leq \mu^n(KK^{-1}g \cap G(\mu)) = \mu((KK^{-1} \cap G(\mu))g).$$

As this holds for all $x \in G$ and $n \geq 1$ be such that $\mu^n(Kx) > 0$ it follows that for all n the value of the concentration function of μ^n with respect to G, for the compact set K is majorised by that with respect to concentration in concentration in $G(\mu)$ for the compact set $KK^{-1} \cap G(\mu)$. This shows that if μ is scattering as a measure on $G(\mu)$ it is scattering as a measure on G. It is straightforward to verify that the other way assertion also holds. This proves the lemma. $\square$

The following is a similar general fact which is easy to prove.

Lemma 3.2 *Let G be a locally compact group and $\mu \in P(G)$. Let H be a closed normal subgroup of G and let $\eta : G \to G/H$ be the quotient homomorphism. If $\eta(\mu)$ is scattering then so is μ.*

Remark 3.3 Let G be a locally compact group and $\mu \in P(G)$. If $G(\mu)$ has a compact normal subgroup H such that $\mu(Hx) = 1$ for some $x \in G$, then μ is not scattering; in this case clearly $c_H(\mu^n) \geq \mu^n(Hx^n) = 1$ for all n. The condition in the hypothesis of the theorem below may be compared with this situation, except that the subgroup H there need not be compact. See the remarks at the end of the section for a more complete picture.

The following is the main result to be proved in this section.

Theorem 3.4 *Let V be a finite-dimensional vector space over $\mathbb{R}$ and let μ be a probability measure on $GL(V)$. Suppose that $G(\mu)$ is noncompact. Then either μ is scattering or there exists an open normal subgroup H of $G(\mu)$ such that $G(\mu)/H$ is infinite and $\mu(Hx) = 1$ for some $x \notin H$.*

The theorem enables us to conclude scattering property under various topological and algebraic conditions. We note some consequences of the theorem before going over to the proof. (See also Corollary 3.12 below for application to more general situation).

Corollary 3.5 *Let V be a finite-dimensional vector space over $\mathbb{R}$ and μ be a probability measure on $GL(V)$ such that $G(\mu)$ is noncompact. Suppose also that one of the following conditions is satisfied:*

i) $G(\mu)$ has only finitely many connected components;

ii) there is no continuous surjective homomorphism of $G(\mu)$ onto $\mathbb{Z}$;

iii) μ is symmetric (that is, $\mu(\{x^{-1} \mid x \in E\}) = \mu(E)$ for all Borel subsets E of G).

Then μ is scattering.

Proof: If μ is not scattering then by Theorem 3.4 there exists an open normal subgroup H of $G(\mu)$ such that $G(\mu)/H$ is infinite and $\mu(H) = 1$; in this case clearly $G(\mu)/H$ is isomorphic to $\mathbb{Z}$ and μ is not symmetric. This shows that the assertion as in the corollary holds under conditions ii) or iii). Condition i) is a particular case of ii). This proves the corollary. □

We begin with some preliminary general results. The essential ideas involved in the proofs of the next two Propositions, which play a crucial role in studying the question of scattering, go back to Csiszár [5].

Proposition 3.6 *Let G be a locally compact group and $\mu \in P(G)$. Then for any compact subset K of G, the sequence $\{c_K(\mu^n)\}$ is nonincreasing. Furthermore if l_K is the limit of the sequence, then either $l_K = 0$ for all compact subsets K, or for every $\epsilon > 0$ there exists a compact subset $K(\epsilon)$ such that $l_{K(\epsilon)} > 1 - \epsilon$.*

Proof: Let K be any compact subset of G. For any $n > m$ and $x \in G$ we have

$$\mu^n(Kx) = \int_G \mu^m(Kxy^{-1})\mathrm{d}\mu^{n-m}(y) \leq c_K(\mu^m),$$

and hence the sequence $\{c_K(\mu^n)\}$ is nonincreasing. For any compact subset K of G let l_K denote the limit of the sequence $\{c_K(\mu^n)\}$. Suppose that there exists a compact subset K of G such that $l_K > 0$. Let $\epsilon > 0$ be given, and let $\delta = l_K\epsilon/2$. There exists m such that $c_K(\mu^m) \leq l_K + \delta$. Let C be a compact symmetric subset of G such that $\mu^m(C) > 1 - \delta$. Let $n > m$ and $x \in G$ be arbitrary. Then

$$\mu^n(Kx) = \int_{CKx} \mu^m(Kxy^{-1})\mathrm{d}\mu^{n-m}(y) + \int_{G\setminus CKx} \mu^m(Kxy^{-1})\mathrm{d}\mu^{n-m}(y).$$

For $y \notin CKx$ the sets Kxy^{-1} and C are disjoint and hence $\mu^m(Kxy^{-1}) \leq \delta$, and hence the second term above is at most δ. On the other hand the first term is at most

$c_K(\mu^m)\mu^{n-m}(CKx) \le (l_K+\delta)\mu^{n-m}(CKx)$. Thus $\mu^n(Kx) \le (l_K+\delta)\mu^{n-m}(CKx)+\delta$. Since this holds for all $x \in G$ we get that $c_K(\mu^n) \le (l_K+\delta)c_{CK}(\mu^{n-m})+\delta$. Passing to limit as $n \to \infty$ we get that $l_K \le (l_K+\delta)l_{CK}+\delta$. Hence $l_{CK} \ge (l_K-\delta)/(l_K+\delta) > 1-\epsilon$. This proves the proposition. □

Proposition 3.7 *Let G be a locally compact second countable group and $\mu \in P(G)$. If μ is not scattering then there exists a sequence $\{g_n\}$ in $G(\mu)$ such that $\{\mu^n g_n^{-1}\}$ is relatively compact.*

Proof: In view of Proposition 3.1 we may assume that $G = G(\mu)$. Suppose μ is not scattering. Then by Proposition 3.6 there exists a compact subset K of G such that $c_K(\mu^n) > \frac{1}{2}$ for all n. Therefore there exists $g_n \in G$, for each n, such that $\mu^n(Kg_n) > \frac{1}{2}$.

Similarly for any $t > 0$ there exist a compact subset K_t and a sequence $\{g_n^{(t)}\}$ such that $\mu^n(K_t g_n^{(t)}) > 1-t$ for all n. Then for $t < \frac{1}{2}$ and any n, $K_t g_n^{(t)} \cap K g_n$ is nonempty; this implies that $g_n^{(t)} \in K_t^{-1}Kg_n$, and in turn that $K_t g_n^{(t)}$ is contained in $K_t K_t^{-1} K g_n$. Thus for every $t > 0$ there exists a compact subset C_t, namely $C_t = K_t K_t^{-1} K$, such that $(\mu^n g_n^{-1})(C_t) = \mu^n(C_t g_n) > 1-t$. This shows that $\{\mu^n g_n^{-1}\}$ is relatively compact (see [26], Chapter II, Theorem 6.7). This proves the proposition. □

We next note the following observation due to Riddhi Shah.

Lemma 3.8 *Let G be a locally compact second countable group and $\mu \in P(G)$. Let $\{g_n\}$ be a sequence in G such that $\{\mu^n g_n^{-1}\}$ is relatively compact. Then $\{g_n \mu g_n^{-1}\}$ is relatively compact.*

Proof: We have $\mu^{n+1}g_n^{-1} = (\mu^n g_n^{-1})(g_n \mu g_n^{-1})$ for all n. Since $\{\mu^{n+1}g_n^{-1}\}$ and $\{\mu^n g_n^{-1}\}$ are relatively compact the preceding equation implies that $\{g_n \mu g_n^{-1}\}$ is relatively compact (see [26], Chapter III, Theorem 2.1). □

We now recall the following result due to A. Mukherjea [24]; a proof starting from basic principles may be found in [9], Appendix; (though in [9] the group is assumed to be second countable the assumption is not needed in the proof).

Proposition 3.9 *Let G be a locally compact group. Let $\mu \in P(G)$ be such that $\{\mu^n\}$ is relatively compact. Then $G(\mu)$ is compact.*

The following Proposition, first noted in [4], settles the questions of decay of concentration functions for the case of abelian groups.

Proposition 3.10 *Let G be a locally compact second countable abelian group and $\mu \in P(G)$. Then μ is scattering if and only if there does not exist any compact open subgroup K of $G(\mu)$ such that $\mu(gK) = 1$ for some $g \in G(\mu)$.*

Proof: The 'only if' part is immediate from Remark 3.3. We now prove the 'if' part. In view of Proposition 3.1 without loss of generality we may assume $G = G(\mu)$. Suppose that μ is not scattering. Then by Proposition 3.7 there exists a sequence $\{g_n\}$ such that $\{\mu^n g_n\}$ is relatively compact. Let $\widehat{G}$ be the dual group of G, and $f = \widehat{\mu}$, the Fourier transform of μ. Then the Fourier transform of $\mu^n g_n$ is $\widehat{g_n} f^n$, for any n, $\widehat{g_n}$ being the evaluation character corresponding to g_n defined by $\widehat{g_n}(\chi) = \chi(g_n)$ for all $\chi \in \widehat{G}$. If λ is a limit point of $\{\mu^n g_n\}$ then for any $\chi \in \widehat{G}$, $\widehat{\lambda}(\chi)$ is a limit of a subsequence of $\widehat{g_n} f^n(\chi)$. Since $|\widehat{g_n} f^n| = |f^n|$ this implies that for any $\chi \in \widehat{G}$, $|\widehat{\lambda}(\chi)|$ is either 0 or 1; furthermore, if $|f(\chi)| < 1$ then $\widehat{\lambda}(\chi) = 0$, and since $|f(\chi)| \leq 1$ it follows that $|f(\chi)| = 1$ whenever $|\widehat{\lambda}(\chi)| = 1$. As $|\widehat{\lambda}|$ is a continuous function this implies that the set $\Omega = \{\chi \in \widehat{G} \mid |\widehat{\lambda}(\chi)| = 1\}$ is an open and closed neighbourhood of the identity in $\widehat{G}$, and $|f(\chi)| = 1$ for all $\chi \in \Omega$. Being an open and closed neighbourhood of the identity, Ω contains an open subgroup of $\widehat{G}$, say Ψ. Let H be the annihilator of Ψ in G. As Ψ is open in $\widehat{G}$, H is a compact subgroup, and Ψ is the dual group of G/H. The restriction of f to Ψ is the Fourier transform of the image of μ on G/H. Since the restriction has absolute value 1 identically, it follows that the image of μ on G/H is a point mass. Thus there exists $g \in G$ such that $\mu(gH) = 1$. Since $G = G(\mu)$, it follows that the subgroup generated by H and g is dense in G. Thus H is a compact subgroup and G/H is monothetic. Therefore it follows that there exists a compact open subgroup K of G containing H. Clearly $\mu(gK) = 1$; this proves the proposition. $\square$

We next prove the following result which covers a special case of Theorem 3.4, with a stronger conclusion.

Proposition 3.11 *Let V be a finite-dimensional vector space over $\mathbb{R}$. Let W be a closed subgroup of $GL(V)$ topologically isomorphic to $\mathbb{R}^d$ for some $d \geq 0$, and let C be a compact subgroup of $GL(V)$ normalising W. Let μ be a nonscattering probability measure on $GL(V)$ such that $G(\mu)$ is contained in the closed subgroup CW. Then there exist a compact open normal subgroup K of $G(\mu)$ and $x \in G(\mu)$ such that $\mu(Kx) = 1$.*

Proof: To begin with we note that by replacing C by a smaller subgroup if necessary, we may assume that $G(\mu)W$ is dense in CW. We now proceed by induction on d. If $d = 0$ the assertion is obvious. We may assume that the contention holds for lower values of d than the dimension of W under consideration. We view W as a vector space and considering the action of C on W by conjugation, express W as a direct sum $W_0 \oplus W_1$, where W_0 is the vector subspace consisting of all points fixed under the action of C, and W_1 is a C-invariant vector subspace containing no nonzero fixed point. Then CW is a direct product of W_0 and CW_1. Let p_0 and p_1 be the projection homomorphisms onto W_0 and CW_1 respectively. Suppose that W_0 is of positive dimension. So W_1 is of lower dimension than W, and hence by the induction hypothesis the contention of the theorem holds for CW_1. Since μ is nonscattering so is $p_1(\mu)$, and hence $G(p_1(\mu))$ contains a compact open normal subgroup K such that $p_1(\mu)$ is supported on a coset of K. Since W_0 is a vector space and μ is nonscattering by Proposition 3.10 $p_0(\mu)$ is a point measure. Now let $x \in \operatorname{supp} \mu$. Considering its projections under p_0 and p_1 we see that the set $(\operatorname{supp} \mu)x^{-1}$ is contained in K. Hence $\mu(Kx) = 1$. This proves the proposition in the case at hand. We may therefore suppose that W_0 is trivial; that is, the C-action on W has no nonzero fixed points.

Since μ is nonscattering, by Propositions 3.7 there exists a sequence $\{g_n\}$ in CW such that $\{\mu^n g_n^{-1}\}$ is relatively compact. Hence by Lemma 3.8 $\{g_n \mu g_n^{-1}\}$ is relatively compact. If $G(\mu)$ is compact then the assertion in the proposition is obvious. Now suppose that $G(\mu)$ is noncompact. Then by Proposition 3.9 $\{g_n\}$ is unbounded. By passing to a subsequence of $\{g_n\}$ we get a divergent sequence $\{g_i\}$ in CW such that $\{g_i \mu g_i^{-1}\}$ converges. Since C is compact we may also assume $\{g_i\}$ to be contained in W.

Now let $E = \operatorname{Hom}(V, V)$ and consider the maps $x \mapsto g_i x g_i^{-1}$ of E into itself. Since $\{g_i \mu g_i^{-1}\}$ converges, by Corollary 1.5 $\operatorname{supp} \mu$ is contained in the subspace E_1 of E consisting of all x such that $\{g_i x g_i^{-1}\}$ is converges. For $c \in C$, let $\alpha_c : W \to W$ be the map defined by $\alpha_c(y) = (c^{-1}yc)y^{-1}$ for all $y \in W$. Since $g_i \in W$ for all i, for $x \in cW$, where $c \in C$, we have $g_i x g_i^{-1} = g_i c g_i^{-1} = c\alpha_c(g_i)$.

Now consider any $x \in cW \cap (\operatorname{supp} \mu)$, where $c \in C$. Then $\{g_i x g_i^{-1}\}$ is convergent, and hence so is $\{\alpha_c(g_i)\}$. In any vector space W given a subset S there exists a smallest vector subspace U such that the image of S in W/U is relatively compact. Let U be the smallest subspace of W as above such that the image of $\{g_i\}$ in W/U is relatively compact; since $\{g_i\}$ is unbounded the subspace U is nontrivial. As α_c is a linear map on W and $\{\alpha_c(g_i)\}$ is convergent it follows that U is contained in the kernel of α_c. Therefore U is centralised by c, and hence also by x. As this holds for all x in the support of μ it follows that $G(\mu)$ centralises U. Since $U \subset W$ it follows that U is centralised by $G(\mu)W$. Since by our assumption in the beginning of the proof $G(\mu)W$ is dense in CW, it follows that U is centralised by CW. Therefore, in particular, U is pointwise fixed under the conjugation action of C. However this is a contradiction since U is a nonzero subspace of W, and by our choice the latter has no nonzero fixed point. This shows that $G(\mu)$ has to be compact, in the case at hand. This proves the proposition. □

Proof of Theorem 3.4: We shall proceed by induction on the dimension of V. If the dimension is 1 then $GL(V)$ is abelian and the theorem follows from Proposition 3.10. Now let V be of higher dimension and suppose that the contention of the theorem holds for all vector spaces of dimension less than that of V. Let $GL_1(V)$ be the subgroup of $GL(V)$ consisting of elements of determinant ± 1. It is a closed normal subgroup and the quotient is topologically isomorphic to $\mathbb{R}$. Consider the image of μ in the quotient, say $\overline{\mu}$. If $\overline{\mu}$ is not a point mass then by Proposition 3.10 $\overline{\mu}$ is scattering, and hence by Lemma 3.2 so is μ. Now suppose that $\overline{\mu}$ is a point mass. If it is a point mass at a nonzero point then it can be verified that the contention of the theorem holds if we choose H to be $GL_1(V) \cap G(\mu)$. Therefore we may assume that the image is the point mass at the zero in the quotient group, which means that $G(\mu)$ is contained in $GL_1(V)$.

Now suppose that μ is not scattering. Then by Proposition 3.7 there exists a sequence $\{g_n\}$ in $G(\mu)$ such that $\{\mu^n g_n^{-1}\}$ is relatively compact. Since by hypothesis $G(\mu)$ is noncompact, by Proposition 3.9 it follows that $\{g_n\}$ is not relatively compact. Also, by Lemma 3.8 it follows that $\{g_n \mu g_n^{-1}\}$ is relatively compact. Let $\{e_1, \ldots, e_d\}$, where d is the dimension of V, be a basis of V, and for each n let $g_n = \theta_n' a_n \theta_n$ be a polar decomposition, where θ_n' and θ_n are contained in the orthogonal group (corresponding to an inner product) and a_n are represented (with respect to the basis $\{e_1, \ldots, e_d\}$) by diagonal matrices with positive entries in the increasing order, as $\mathrm{diag}\,(\lambda_n^{(1)}, \ldots, \lambda_n^{(d)})$; thus $\{\lambda_n^{(k)}\}$, $k = 1, \ldots, d$, are sequences of positive numbers such that $\lambda_n^{(1)} \leq \lambda_n^{(2)} \leq \cdots \leq \lambda_n^{(d)}$ for each n. We note that since $\{g_n \mu g_n^{-1}\}$ is relatively compact $\{a_n \theta_n \mu \theta_n^{-1} a_n^{-1}\}$ is also relatively compact.

We can choose an increasing sequence $\{n_i\}$ of natural numbers such that $\{g_{n_i}\}$ is divergent, $\{\theta_{n_i}\}$ is convergent, $\{a_{n_i} \theta_{n_i} \mu \theta_{n_i}^{-1} a_{n_i}^{-1}\}$ is convergent, and for each $k = 1, \ldots, d$, $\lambda_{n_i}^{(k)}$ either converges or diverges, as $i \to \infty$. We note that since $\{g_{n_i}\}$ is divergent and the determinants of g_n are ± 1, $\lambda_{n_i}^{(1)} \to 0$ and $\lambda_{n_i}^{(d)} \to \infty$, as $i \to \infty$. Let θ denote the limit of $\{\theta_{n_i}\}$.

Now for each i let $\mu_i = \theta_{n_i} \mu \theta_{n_i}^{-1}$. Then $\{a_{n_i} \mu_i a_{n_i}^{-1}\}$ is convergent. We view μ and μ_i as measures on $E = \mathrm{Hom}\,(V, V)$. For any $g \in GL(V)$ the map $x \mapsto gxg^{-1}$ for all $x \in E$ is a linear transformation of E. For all $1 \leq p \leq d$, and $1 \leq q \leq d$ let $e_{pq} \in E$ be the transformation corresponding to the matrix whose (p, q) entry is 1 and all other entries are 0. Then $\{e_{pq} \mid p, q = 1, \ldots, d\}$ is a basis of E. For $a = \mathrm{diag}\,(\lambda^{(1)}, \ldots, \lambda^{(d)}) \in GL(V)$ the corresponding map $x \mapsto axa^{-1}$ for all $x \in E$ is represented by a diagonal matrix with respect to the basis $\{e_{pq} \mid p, q = 1, \ldots, d\}$, the eigenvalue corresponding to e_{pq} being $\lambda^{(p)}/\lambda^{(q)}$. Since $\mu_i \to \theta \mu \theta^{-1}$ and $\{a_{n_i} \mu_i a_{n_i}^{-1}\}$ is convergent, by Corollary 1.5 (see also Example 1.3) this implies that $\theta \mu \theta^{-1}$ is supported on the subspace of E spanned by the e_{pq}'s such that $\{\lambda_{n_i}^{(p)}/\lambda_{n_i}^{(q)}\}$ is convergent.

Since $\lambda_n^{(1)} \leq \lambda_n^{(2)} \leq \cdots \leq \lambda_n^{(d)}$ for each n, $\lambda_{n_i}^{(1)} \to 0$ and $\lambda_{n_i}^{(d)} \to \infty$, it follows that there exists a k, $1 \leq k < d$ such that $\lambda_{n_i}^{(p)}/\lambda_{n_i}^{(q)} \to \infty$ whenever $p > k$ and $q \leq k$. Thus every element of the support of $\theta\mu\theta^{-1}$ leaves invariant the subspace spanned by $\{e_1, \ldots, e_k\}$. Then $x(V_1) \subset V_1$ for all x in the support of μ, where V_1 is the subspaces of V spanned by $\{\theta^{-1}e_1, \ldots, \theta^{-1}e_k\}$. Hence $G(\mu)$ is contained in the subgroup $P = \{g \in GL_1(V) \mid g(V_1) = V_1\}$. By modifying the initial choice of the basis we may assume that V_1 is the subspace spanned by $\{e_1, \ldots, e_k\}$; this is strictly not necessary for the following argument, but it will be convenient in following the proof.

Now let V_2 be a subspace complementary to V_1 in V, and let $\psi_1 : P \to GL(V_1)$ and $\psi_2 : P \to GL(V_2)$ be the maps defined by setting $\psi_1(g)$ to be the restriction of g to V_1 and $\psi_2(g)$ the factor of g on V_2. Since by assumption μ is not scattering, $\psi_1(\mu)$ and $\psi_2(\mu)$ are non-scattering. Hence by the induction hypothesis for each of $j = 1$ and 2, either $G(\psi_j(\mu))$ is compact or there exists an open normal subgroup H_j of $G(\psi_j(\mu))$ such that $G(\psi_j(\mu))/H_j$ is infinite and $\psi_j(\mu)(H_j x_j) = 1$ for some $x_j \in G(\psi_j(\mu))\backslash H_j$. If the latter condition holds for either $j = 1$ or 2 then there exists an open normal subgroup H of $G(\mu)$ such that $\mu(Hx) = 1$ for some $x \notin H$, and the contention of the theorem holds in this case. Therefore we may assume that $G(\psi_1(\mu))$ and $G(\psi_2(\mu))$ are both compact. Now let $W = \ker\psi_1 \cap \ker\psi_2$, namely the subgroup of P consisting of all g such that the restriction to V_1 is the identity and the factor on V_2 is the identity (these transformations are represented by upper triangular rectangular block matrices if V_1 and V_2 are subspaces spanned by $\{e_1, \ldots, e_k\}$ and $\{e_{k+1}, \ldots, e_d\}$ respectively). Let C be the subgroup of $G(\psi_1(\mu)) \times G(\psi_2(\mu))$, the latter is a subgroup of $GL(V_1) \times GL(V_2)$ which we consider canonically as a subgroup of $GL(V)$, via the decomposition of V as $V_1 \oplus V_2$. Then C normalises W and $G(\mu)$ is contained in CW.

Observe that W is a vector group; in fact it is canonically isomorphic to $\mathrm{Hom}\,(V_2, V_1)$. Since μ is not scattering and $G(\mu)$ is contained in CW, by Proposition 3.11 it follows that there exist a compact open normal subgroup H and $x \in G(\mu)$ such that $\mu(Hx) = 1$. Also, since $G(\mu)$ is noncompact, $G(\mu)/H$ is infinite and $x \notin H$. This proves the theorem. $\square$

A locally compact group G is said to be *almost connected* if G/G^0 is compact, where G^0 denotes the connected component of the identity in G. From the case of subgroups of linear groups we now deduce the following result, using some general facts.

Corollary 3.12 *Let G be any almost connected locally compact noncompact group and μ be a probability measure on G such that the support of μ is not contained in any proper closed subgroup of G. Then μ is scattering.*

Proof: Any G as in the hypothesis admits a compact normal subgroup M such that G/M is a noncompact Lie group ([23], § 4.6). By Lemma 3.2 it suffices to show that the image of μ on G/M is scattering, and hence we may without loss of generality assume that G is a noncompact Lie group with finitely many connected components. To prove the corollary it would therefore suffice, in view of Corollary 3.5(i), to know that any such Lie group admits a representation $\rho : G \to GL(V)$ over a finite-dimensional vector space V, such that $\overline{\rho(G)}$ is noncompact. This can be seen to be the case, using some structure theory for Lie groups. Let G^0 be the connected component of the identity in G. If the image of G^0 under its adjoint representation has noncompact closure, then the assertion is obvious, since the representation extends to G. If not, then G^0 is a direct product of a compact group with an abelian group, and passing to quotient modulo a compact normal subgroup we may assume G^0 to be $\mathbb{R}^d$ for some d. Then a quotient of G by a finite normal subgroup is a group of euclidean motions, and admits a faithful representation in which the image is closed. This proves the corollary. $\square$

The following simple example shows that $G(\mu)$ being noncompact does not imply that μ is scattering, even for measures on linear groups.

Example 3.13 Let $\alpha \in (0,1)$ and $G = L \cdot C$ be the semidirect product where $L = \mathbb{R}$ and $C = \mathbb{Z}$ and the action of C on L is given by $ntn^{-1} = \alpha^n t$ for all $n \in C = \mathbb{Z}$ and $t \in L = \mathbb{R}$. Let $I = [0,1]$ and ℓ denote the Lebesgue measure on I, viewed as a measure on $\mathbb{R}$. Let μ be the measure on G defined by $\mu(E \times \{1\}) = \ell(E)$, and $\mu(E \times \{k\}) = 0$ for $k \neq 1$, for any Borel subset E of $\mathbb{R}$. Then $G(\mu) = G$ which is noncompact. We write μ as λg, where λ is supported on L and g is the generator of C corresponding to $1 \in \mathbb{Z}$. Then for $n \geq 2$ we have $\mu^n = (\lambda g) \cdots (\lambda g) = \lambda(g\lambda g^{-1})(g^2\lambda g^{-2}) \cdots (g^{n-1}\lambda g^{-(n-1)})g^n$. It can be seen that for all n the measure $\lambda(g\lambda g^{-1})(g^2\lambda g^{-2}) \cdots (g^{n-1}\lambda g^{-(n-1)})$ is supported on the subset, say K, of L which corresponds to the interval $[0, (1-\alpha)^{-1}]$ in $\mathbb{R}$, under the identification of L with $\mathbb{R}$. It follows that $c_K(\mu^n) = 1$ for all n. Hence μ is not scattering.

Corollary 3.12 was proved, for a connected locally compact group, in [14], via a stronger notion of 'collapsible measures'. The study of collapsible measures, which we shall not go into here, also depends on techniques analogous to those discussed here.

A more general result on concentration functions, valid for all locally compact groups, was proved around the same time in [21]. A complete characterisation of the scattering condition for measures on locally compact groups has been given by Jaworski; see [19] and [20]. It is shown in particular that if G is a noncompact locally compact group and there exists $\mu \in P(G)$ such that $G(\mu) = G$ which is not scattering then G is a semidirect product of $\mathbb{Z}$ with a subgroup N of G which is 'contractive modulo a compact subgroup' under the action of a generator of $\mathbb{Z}$ and μ is supported on a nontrivial coset of N in G. It is also shown that if G is unimodular then $\mu \in P(G)$ such that $G(\mu) = G$ is nonscattering if and only if it is supported on nontrivial coset of a compact subgroup.

4 Factors and factor compactness

In this section we apply the study of asymptotics of measures to 'factor compactness'. The main ideas involved are taken from [11] and [12].

Let G be a locally compact group and let μ be a probability measure on G. A measure $\lambda \in P(G)$ is called a *factor* of μ if there exists a $\nu \in P(G)$ such that $\mu = \lambda\nu = \nu\lambda$; sometimes a factor in this sense is called a two-sided factor to distinguish from the notion of a one-sided factor, but we will be concerned with only one notion of factors, as above. We shall denote by $F(\mu)$ the set of all factors of μ. Recall that $G(\mu)$ denotes the smallest closed subgroup of G containing the support of μ. We shall denote by $N(\mu)$ the normaliser of $G(\mu)$ in G, namely $N(\mu) = \{g \in G \mid gG(\mu)g^{-1} = G(\mu)\}$. We first note the following.

Proposition 4.1 *For any $\lambda \in F(\mu)$ the support of λ is contained in $N(\mu)$.*

Proof: Let $\nu \in P(G)$ be such that $\mu = \lambda\nu = \nu\lambda$. Then $(\operatorname{supp}\lambda)(\operatorname{supp}\nu)$ and $(\operatorname{supp}\nu)(\operatorname{supp}\lambda)$ are dense subsets of $\operatorname{supp}\mu$. Let $g \in \operatorname{supp}\lambda$ be given. Let $x \in \operatorname{supp}\nu$ and $y \in \operatorname{supp}\lambda$ be arbitrary. Then we have $g(xy)g^{-1} = (gx)(yx)(gx)^{-1}$, and by the observations above gx, yx and gx are contained $\operatorname{supp}\mu$, so $gxyg^{-1}$ is contained in $G(\mu)$. As this holds for all $x \in \operatorname{supp}\nu$, $y \in \operatorname{supp}\lambda$ and $(\operatorname{supp}\nu)(\operatorname{supp}\lambda)$ is dense in $\operatorname{supp}\mu$ it follows that $gzg^{-1} \in G(\mu)$ for all $z \in \operatorname{supp}\mu$ and hence for all $z \in G(\mu)$. Similarly, using the relation $g^{-1}(yx)g = (xg)^{-1}(xy)(xg)$ we conclude that $g^{-1}zg \in G(\mu)$ for all $z \in G(\mu)$. This shows that $g \in N(\mu)$, thus proving the proposition. $\square$

Let $Z(\mu)$ be the centraliser of $\operatorname{supp}\mu$ in G, namely the subgroup of G consisting of all g in G such that $gx = xg$ for all $x \in \operatorname{supp}\mu$. We note that for any $\lambda \in F(\mu)$ and $g \in Z(\mu)$ we have $\lambda g \in F(\mu)$, since if ν is such that $\mu = \lambda\nu = \nu\lambda$ then $\mu = (\lambda g)(g^{-1}\nu) = (g^{-1}\nu)(\lambda g)$; more generally this shows also that if $g \in G$ is such that $g\mu g^{-1} = \mu$ then for any $\lambda \in F(\mu)$, $\lambda g \in F(\mu)$.

One question of interest is whether modulo the translations by elements of $Z(\mu)$ the set of factors is compact, and in particular whether $F(\mu)$ is compact when $Z(\mu)$ is compact.

Proposition 4.2 *Let G be a locally compact group and let $\mu \in P(G)$. Let $\{\lambda_n\}$ be a sequence of factors of μ. Then there exists a sequence $\{g_n\}$ in $N(\mu)$ such that the following conditions are satisfied.*

i) the sequences $\{g_n \mu g_n^{-1}\}$ and $\{g_n^{-1} \mu g_n\}$ are relatively compact;

ii) if F is a closed normal subgroup of G and $\eta : G \to G/F$ is the natural quotient map, then $\{\eta(g_n)\}$ is a relatively compact subset of G/F if and only if there exists a sequence $\{x_n\}$ in F such that $\{\lambda_n x_n\}$ is relatively compact in $P(G)$.

Proof: For each n let $\nu_n \in P(G)$ be such that $\mu = \lambda_n \nu_n = \nu_n \lambda_n$. Let $\{K_m\}$ be a sequence of compact subsets of G such that $\mu(K_m) > 1 - 4^{-(m+1)}$ for all m. For all m, n let

$$E_n^m = \{g \in G \mid \lambda_n(K_m g^{-1}) > 1 - 2^{-m}\},$$

and let $F_n = \cap_{m=1}^{\infty} E_n^m$ for all n. Then we have

$$\mu(K_m) = (\lambda_n \nu_n)(K_m) = \int_{E_n^m} \lambda_n(K_m g^{-1}) \mathrm{d}\nu_n(g) + \int_{G \backslash E_n^m} \lambda_n(K_m g^{-1}) \mathrm{d}\nu_n(g)$$

and therefore

$$1 - 4^{-(m+1)} \leq \mu(K_m) \leq \nu_n(E_n^m) + (1 - 2^{-m})\nu_n(G \backslash E_n^m) = 1 - 2^{-m}\nu_n(G \backslash E_n^m).$$

Hence $\nu_n(G \backslash E_n^m) \leq 2^{-(m+2)}$ for all m and n, and in turn

$$\nu_n(G \backslash F_n) \leq \Sigma_m 2^{-(m+2)} = 1/4,$$

for all n. Similarly we put

$$B_n^m = \{g \in G \mid \lambda_n(g^{-1} K_m) > 1 - 2^{-m}\},$$

for all m and n, and $C_n = \cap_{m=1}^{\infty} B_n^m$, for all n, and conclude that $\nu_n(G \backslash C_n) \leq 1/4$. It follows that $\nu_n(F_n \cap C_n) \geq 1/2$. Therefore we can pick $g_n \in F_n \cap C_n$, which in view of Proposition 4.1 may further be assumed to be contained in $N(\mu)$. Let $\{g_n\}$ be a sequence so formed. From the definition of F_n it follows that $\lambda_n(K_m g_n^{-1}) > 1 - 2^{-m}$ for all m and n, which shows that the $\{\lambda_n g_n\}$ is relatively compact (see [26], Chapter II, Theorem 6.7). Since $\mu = \lambda_n \nu_n = (\lambda_n g_n)(g_n^{-1} \nu_n)$ for all n, the preceding conclusion implies also that $\{g_n^{-1} \nu_n\}$ is relatively compact (see [26], Chapter III, Theorem 2.1). Similarly, using that $g_n \in C_n$ and $\mu = \nu_n \lambda_n$ for all n we conclude that $\{g_n \lambda_n\}$ and $\{\nu_n g_n^{-1}\}$ are relatively compact. Now, $g_n \mu g_n^{-1} = (g_n \lambda_n)(\nu_n g_n^{-1})$, and $g_n^{-1} \mu g_n = (g_n^{-1} \nu_n)(\lambda_n g_n)$ for all n, and hence it follows that $\{g_n \mu g_n^{-1}\}$ and $\{g_n^{-1} \mu g_n\}$ are relatively compact.

ii) Now let F be a closed normal subgroup of G, $\eta : G \to G/F$ be the quotient map, and suppose that $\{\eta(g_n)\}$ is relatively compact in G/F. Then there exists a sequence $\{x_n\}$ in F such that $\{x_n^{-1}g_n\}$ is relatively compact. Now, $\lambda_n x_n = (\lambda_n g_n)(x_n^{-1}g_n)^{-1}$ for all n, and since $\{\lambda_n g_n\}$ and $\{x_n^{-1}g_n\}$ are relatively compact this implies that $\{\lambda_n x_n\}$ is relatively compact. This proves the 'only if' part of ii). Conversely suppose that there exists a sequence $\{x_n\}$ in F such that $\{\lambda_n x_n\}$ is relatively compact. By the choice of the g_n's $\{\lambda_n g_n\}$ is also relatively compact, and hence we get that $\{x_n^{-1}g_n\}$ is relatively compact. Therefore $\{\eta(g_n)\}$ is relatively compact. This proves ii), and completes the proof of the proposition. □

We shall now discuss factor compactness of measures on linear groups. Let V be a finite-dimensional vector space over $\mathbb{R}$. A subgroup G of $GL(V)$ is said to be *algebraic* if there exists a polynomial function on $GL(V)$ (namely a function which is a polynomial in the coordinates when $GL(V)$ is realised as $GL(d, \mathbb{R})$ with respect to a - any - basis of V) such that $G = \{x \in GL(V) \mid P(x) = 0\}$; (over a general field such an object is defined using a finite collection of polynomials, but over the reals a single polynomial serves the purpose!). Many groups like $SL(V)$, the subgroup consisting of elements which correspond to upper triangular matrices with respect to a fixed basis, the orthogonal subgroups with respect to inner products and more generally with respect to quadratic forms, are algebraic subgroups. A subgroup is said to be *almost algebraic* if it is a subgroup of finite index in an algebraic subgroup. We note that the centraliser of any subset of $GL(V)$ and the normaliser of an algebraic subgroup are algebraic subgroups of $GL(V)$. We note also that images of almost algebraic groups under homomorphisms which are restrictions of algebraic homomorphisms of the ambient algebraic groups, are almost algebraic. (see [2] and [3] for general references in this respect; see also [1] and [30], Chapter 3).

Theorem 4.3 *Let V be a finite-dimensional vector space, and let G be an almost algebraic subgroup of $GL(V)$. Let μ be a probability measure on G. Then $F(\mu)/Z(\mu)$ is relatively compact; that is, for any sequence $\{\lambda_n\}$ of factors of μ there exists a sequence $\{z_n\}$ in $Z(\mu)$ such that $\{\lambda_n z_n\}$ is relatively compact.*

Proof: To begin with we note that as $Z(\mu)$ is almost algebraic, its normaliser is an almost algebraic subgroup. Also the latter contains the normaliser of $G(\mu)$ and hence by Proposition 4.1 the supports of all factors of μ are contained in it. Therefore, replacing G by the normaliser we may without loss of generality assume that $Z(\mu)$ is normal in G. Now let $\{\lambda_n\}$ be a sequence of factors of μ. Let $\{g_n\}$ be a sequence in $N(\mu)$ such that the assertions as in Proposition 4.2 are satisfied. We view μ as a measure on $E = \mathrm{Hom}(V, V)$. For each n let $\alpha_n : E \to E$ be the map defined by $\alpha_n(x) = g_n x g_n^{-1}$ for all $x \in E$. Then $\alpha_n \in GL(E)$ for all n and by Proposition 4.2

$\{\alpha_n(\mu)\}$ and $\{\alpha_n^{-1}(\mu)\}$ are relatively compact. Then by Corollary 1.5 there exists a subsequence of $\{\alpha_n\}$, which also we shall write as $\{\alpha_n\}$ (by passing to a subsequence and modifying the notation), such that the support of μ is contained in the subspace, say E_1, consisting of v in E such that both $\{\alpha_n(v)\}$ and $\{\alpha_n^{-1}(v)\}$ are convergent. Since each α_n is an algebra automorphism of E it follows that E_1 is a subalgebra of E, containing the identity. Let W be the smallest subalgebra containing the support of μ and the identity. Then for any $g \in \operatorname{supp}\mu$ and any natural number k, $g(\operatorname{supp}\mu)^k$ is contained in W. Hence $gW \subset W$, and by dimension consideration $gW = W$ for all $g \in \operatorname{supp}\mu$. Since W contains the identity this implies that it contains g^{-1} for all $g \in \operatorname{supp}\mu$, and hence $G(\mu)$ is contained in W.

Now, W is contained in E_1, and hence $\{\alpha_n(v)\}$ and $\{\alpha_n^{-1}(v)\}$ are convergent for all $v \in W$. Furthermore, since each g_n normalises $G(\mu)$, W is invariant under each α_n. Now for every n let β_n be the restriction of α_n to W. By Proposition 1.6 the preceding condition implies that $\{\beta_n\}$ and $\{\beta_n^{-1}\}$ are both bounded sequences in $\operatorname{Hom}(W,W)$. Therefore $\{\beta_n\}$ is a bounded sequence in $GL(W)$. Now let G_1 be the subgroup of G consisting of all elements g such that $gWg^{-1} = W$. Then G_1 is an almost algebraic subgroup of $GL(V)$. Let $\eta : G_1 \to GL(W)$ be the map associating to each $g \in G_1$ the map $w \mapsto gwg^{-1}$ for all $w \in W$. Then the image of η is an almost algebraic subgroup of $GL(W)$, and in particular it is a closed subgroup. We note that the kernel of η is $Z(\mu)$. In view of the preceding observation it now follows that $gZ(\mu) \mapsto \eta(g)$ is a homeomorphism of $G_1/Z(\mu)$ onto $\eta(G_1)$. We note that $\eta(g_n) = \beta_n$ for all n. Thus $\{\eta(g_n)\}$ is a bounded sequence in $GL(W)$. Since $\{g_n\}$ also satisfies the second statement in Proposition 4.2, choosing $F = Z(\mu)$, we get that there exists a sequence $\{z_n\}$ in $Z(\mu)$ such that $\{\lambda_n z_n\}$ is relatively compact. This proves the theorem. □

Corollary 4.4 *Let G be an almost algebraic subgroup of $GL(V)$, where V is a finite-dimensional vector space. Let μ be a probability measure on G such that $Z(\mu)$ is compact. Then μ is factor compact.*

Proof: Under the hypothesis as above Theorem 4.3 implies every sequence of factors of μ is relatively compact. Any limit point of such a sequence can also be seen to be a factor of μ. Also, if $\{\lambda_n\}$ is a sequence in $F(\mu)$ such that $\lambda_n \to \lambda$, and $\{\nu_n\}$ is a sequence such that $\mu = \lambda_n\nu_n = \nu_n\lambda_n$ for all n, then $\{\nu_n\}$ has a subsequence converging to say a probability measure ν and then we get that $\mu = \lambda\nu = \nu\lambda$, and so $\lambda \in F(\mu)$. This shows that $F(\mu)$ is compact, namely μ is factor compact. □

The following examples show the necessity of the conditions as in Theorem 4.3 and Corollary 4.4.

Examples: Let H be the Heisenberg group consisting of all 3×3 upper triangular unipotent matrices, namely (x_{ij}) such that $x_{ij} = 0$ if $i > j$ and $x_{ij} = 1$ for $i = j$, the remaining variables being free. We equip H with the topology as a group of matrices. The center, say Z, of H can be seen to be the one-parameter subgroup consisting of matrices which satisfy furthermore the conditions $x_{12} = x_{23} = 0$, x_{13} being a free variable. Now let D be an infinite cyclic subgroup of Z. Then D is a closed normal subgroup of H. Let $G = H/D$, which is a connected Lie group. Let C be the subgroup Z/D. Then C is compact and it is the center of G. Let m be the normalised Haar measure on C, viewed as a measure on G. Let $\lambda \in P(G)$ be such that $\operatorname{supp} \lambda$ is not contained in any proper closed subgroup of G, and let $\mu = \lambda m$. Then $Z(\mu) = Z(\lambda) = C$, which is compact. On the other hand mg is a factor of μ for every $g \in G$, which shows that $F(\mu)$ is not compact.

In the above example the Lie group is not a linear group (not a subgroup of $GL(V)$ for any vector space V); (this follows from the fact that under any representation $\rho : G \to GL(V)$ the image $\rho([G,G])$ of the commutator subgroup must - by Lie's theorem - consist of unipotent elements, while on the other hand since $[G,G] = C$ it must be compact, which implies that it must be trivial). An example of a linear group (not almost algebraic) with a similar property can be given as follows. Let T be the two-dimensional torus, and let V be a 4-dimensional vector space equipped with a T-action which is faithful and has no nonzero fixed point. Let L be a one-parameter Lie subgroup of T which is dense in T; as a Lie subgroup L is isomorphic to $\mathbb{R}$. We now form a Lie group G, as the semidirect product of L and V, (under the action obtained by restricting the T-action to L). Under the condition as above there exists $t \in L$ such that the action of t on V has no nonzero fixed point. Now let λ be a probability measure on V invariant under the T-action on V, and let μ be the measure on G defined by $\mu = t\lambda$. Then it can be seen that $Z(\mu)$ is trivial. On the other hand for any $s \in L$, the point mass δ_s at s is a factor of μ, and hence $F(\mu)$ is noncompact.

Remark 4.5 Let G be a Lie group and $\mu \in P(G)$. Let $T(\mu) = \{g \in G \mid g\mu = \mu g\}$. As was noted earlier, for any $\lambda \in F(\mu)$ and $g \in T(\mu)$, $\lambda g \in F(\mu)$. Hence in analogy with Theorem 4.3 one may ask (the weaker question) whether $F(\mu)/T(\mu)$ is compact. This can be seen to hold in either of the above examples. The answer to the question is not known so far.

A larger class of Lie groups called 'weakly algebraic groups' was described in [11] with the property that $F(\mu)/Z(\mu)$ is compact, in the sense as in Theorem 4.3, for all $\mu \in P(G)$. A somewhat larger class of groups with this property was also described in [13].

A probability measure μ on a locally compact group is said to be *root compact* if the subset of $P(G)$ consisting of all λ^k where k is a natural number and λ is a l-th root of μ for some $k \leq l$, is relatively compact; each such λ^k is a factor of μ and hence it follows that every factor compact measure is root compact. This fact is used crucially in [12] in proving the embedding theorem (see the section of M. McCrudden in these notes).

5 Levy's probability measures

Let G be a locally compact group. A probability measure μ on G is called *Levy's measure* if it is the limit of a sequence of measures of the form $\{g_i\alpha_i(\mu_1 * \mu_2 * \cdots \mu_i)\}$, where $\{g_i\}$ is a sequence in G, $\{\alpha_i\}$ is a sequence of automorphisms of G, and $\{\mu_i\}$ is a sequence in $P(G)$ such that $\{\alpha_i(\mu_j) \mid 1 \leq j \leq i\}$ is an infinitesimal triangular system, that is, $\{\alpha_i(\mu_j)\}$ converges to the point mass at the identity as $i \to \infty$, uniformly in j. Levy's measures on Euclidean spaces were described by K. Urbanik [28], and Riddhi Shah [27] studied them in the case of nilpotent Lie groups. In a paper of Riddhi Shah and the present author [15] it was shown that the class of nilpotent groups is the natural setting for the measures, at least among connected Lie groups, in the sense that if a connected Lie group admits a Levy's measure such that $G(\mu) = G$ and $\operatorname{supp}\mu$ contains the identity element, then G is a nilpotent Lie group. Here I discuss the role of asymptotics of measures under automorphisms in proving such a result. The proof crucially depends on the following theorem.

Theorem 5.1 *Let $G = GL(n, \mathbb{R})$, the group of all $n \times n$ matrices with real entries. Let U be the subgroup consisting of all upper triangular unipotent matrices in G. Let $\{g_i\}$ be a sequence in G, and for each i let α_i be the inner automorphism of G corresponding to g_i, (defined by $\alpha_i(x) = g_i x g_i^{-1}$ for all $x \in G$). Let e be the identity element in G and*

$$H = \{x \in G \mid \alpha_i(x) \to e \text{ as } i \to \infty\}.$$

Also let δ_e be the point mass at e and

$$\mathcal{C} = \{\mu \in P(G) \mid \alpha_i(\mu) \to \delta_e \text{ as } i \to \infty\}.$$

Then the following conditions are satisfied:

i) H is a closed subgroup of G, and there exists $\kappa \in G$ such that $H \subseteq \kappa^{-1}U\kappa$;

ii) for any $\mu \in \mathcal{C}$, $\operatorname{supp}\mu$ is contained in H; and

iii) if $\{\mu_i\}$ is a sequence in $\mathcal{C}$ and $\alpha_i(\mu_i) \to \mu \in P(G)$, then there exists $\sigma \in G$ such that $\operatorname{supp}\mu$ is contained in $\sigma U \sigma^{-1}$.

For the proof we need the following lemma, which is a variation of a Lemma of Azencott. Given a locally compact second countable space X, a compact subset K of X, a sequence $\{x_i\}$ in X is said to converge to K, and we write $x_i \to K$, if for every neighbourhood Ω of K there exists i_0 such that $x_i \in \Omega$ for all $i \geq i_0$.

Lemma 5.2 *Let G be a locally compact second countable group, $\mu \in P(G)$ and $\{\alpha_i\}$ be a sequence of automorphisms of G such that $\alpha_i(\mu) \to \nu \in P(G)$. If the support of ν is contained in a compact subset K of G, then there exists a subsequence $\{\alpha_{k_i}\}$ of $\{\alpha_i\}$ such that $\alpha_{k_i}(x) \to K$ for μ-almost all x in G.*

Proof: Let $\{\Omega_i\}$ be a decreasing sequence of open neighbourhoods of K such that every neighbourhood of K contains Ω_i for some i; in particular it follows that $\cap \Omega_i = K$. Since $\alpha_i(\mu) \to \nu$ and $\nu(K) = 1$ it follows that for every i there exists k_i such that $\mu(\alpha_{k_i}^{-1}(\Omega_i)) > 1 - 2^{-i}$. Let $S = \cup_j \cap_{i=j}^{\infty} \alpha_{k_i}^{-1}(\Omega_i)$. Then clearly $\mu(S) = 1$ and $\alpha_{k_i}(x) \to K$ for any $x \in S$. This proves the lemma. $\square$

Proof of Theorem 5.1: i) Let $M(n, I\!R)$ be the algebra of all $n \times n$ matrices and let $W = \{x \in M(n, I\!R) \mid g_i x g_i^{-1} \to 0\}$, where 0 denotes the zero matrix in $M(n, I\!R)$. Clearly, for $x \in GL(n, I\!R)$, $x \in H$ if and only if $x - e \in W$. This shows that H is closed. It is evidently a subgroup. Now let $g_i = k_i a_i \kappa_i$ be the polar decompositions of g_i, $i = 1, 2, \ldots$, where k_i and κ_i are orthogonal matrices and a_i are diagonal matrices with positive entries in the increasing order. Since $\{\kappa_i\}$ consists of orthogonal matrices it has a convergent subsequence, say $\kappa_{m_i} \to \kappa$. Now let $x \in H$. Then $g_i x g_i^{-1} \to 0$ and hence $a_i x_i a_i^{-1} \to e$, where $x_i = \kappa_i x \kappa_i^{-1}$ for all i. It is straightforward to verify (along the lines of Example 1.3) that the latter condition implies that the limit $\kappa x \kappa^{-1}$ of $x_{m_i} = \kappa_{m_i} x \kappa_{m_i}^{-1}$ is contained in U. Therefore $x \in \kappa^{-1} U \kappa$. This shows that H is contained in $\kappa^{-1} U \kappa$, thus proving i).

ii) Let $\mu \in \mathcal{C}$. Suppose that the support of μ is not contained in H. Let $g \in \operatorname{supp} \mu$ be an element not contained in H. Then there exists a neighbourhood Ω of e, and a subsequence $\{\alpha_{k_i}\}$ of $\{\alpha_i\}$ such that $\alpha_{k_i}(g) \notin \Omega$. Applying Lemma 5.2 to the subsequence $\{\alpha_{k_i}\}$ we deduce that there exists a subsequence $\{\alpha_{l_i}\}$ of $\{\alpha_{k_i}\}$ such that the support of μ is contained in the subgroup $\{x \in G \mid \alpha_{l_i}(x) \to e\}$, which by the assertion as in i) is a closed subgroup. But this is a contradiction since $g \in \operatorname{supp} \mu$, but $\alpha_{l_i}(g)$ does not converge to e, as it is outside Ω. This proves ii).

iii) Let $\{\mu_i\}$ be a sequence in $\mathcal{C}$ and suppose $\alpha_i(\mu_i) \to \mu \in P(G)$. By (i) for all i the support of μ_i is contained in $\kappa^{-1} U \kappa$, and hence the support of $\alpha_i(\mu_i)$ is contained in $(g_i \kappa^{-1}) U (g_i \kappa^{-1})^{-1}$. We write each $g_i \kappa^{-1}$ as $\sigma_i b_i$, where σ_i is an orthogonal matrix and b_i is an upper triangular matrix. Then $b_i U b_i^{-1} = U$ and hence $(g_i \kappa^{-1}) U (g_i \kappa^{-1})^{-1} = \sigma_i U \sigma_i^{-1}$ for all i. Since the orthogonal group is compact, passing to a subsequence we may assume that $\{\sigma_i\}$ is convergent, say $\sigma_i \to \sigma$. Then $\sigma_i^{-1}\alpha_i(\mu_i)\sigma_i \to \sigma^{-1}\mu\sigma$. Also, since for any i the support of $\alpha_i(\mu_i)$ is contained in $(g_i\kappa^{-1})U(g_i\kappa^{-1})^{-1} = \sigma_i U \sigma_i^{-1}$, it follows that $\sigma_i^{-1}\alpha_i(\mu_i)\sigma_i$ is supported on U. Therefore $\sigma^{-1}\mu\sigma$ is supported on U and so the support of μ is contained in $\sigma U \sigma^{-1}$. This proves the theorem. □

We shall now apply the theorem to deduce that the supports of Levy's measures on connected Lie groups are contained in cosets of nilpotent subgroups.

Let G be a connected Lie group and μ be a Levy's measure on G. Let $\{g_i\}$ be a sequence in G, $\{\alpha_i\}$ be a sequence of automorphisms of G, and $\{\mu_i\}$ be a sequence in $P(G)$ such that $\{\alpha_i(\mu_j) \mid 1 \le j \le i\}$ is an infinitesimal triangular system and μ is the limit of $\{g_i\alpha_i(\mu_1 * \mu_2 * \cdots * \mu_i)\}$. For any $\lambda \in P(G)$ we denote by $\widetilde{\lambda}$ the measure defined by $\widetilde{\lambda}(E) = \lambda(\{g^{-1} \mid g \in E\})$ for all Borel subsets E of G. For each j let $\lambda_j = \mu_1 * \mu_2 * \cdots * \mu_j$ and let $\nu_j = \widetilde{\lambda}_j\lambda_j$. Then we have $\nu_j \to \delta_e$, the point mass at the identity, and $\alpha_j(\nu_j) \to \widetilde{\mu}\mu$. We shall show that under these conditions $\widetilde{\mu}\mu$ is supported on a nilpotent closed subgroup. The latter implies that μ is supported on a coset of the nilpotent subgroup, for if $g \in \operatorname{supp}\mu$ then for any $x \in \operatorname{supp}\mu$, $x^{-1}g \in \operatorname{supp}\widetilde{\mu}\mu$.

We note that the desired conclusion as above follows from Theorem 5.1 if G is a linear group, say $GL(V)$ where V is a finite-dimensional vector space, and the automorphisms in question are given by conjugation action of a sequence in $GL(V)$. The essential task in the proof of the following theorem, proving the above-stated result, is to reduce to this case.

Theorem 5.3 *Let G be a connected Lie group and $\{\alpha_i\}$ be a sequence of continuous automorphisms of G. Suppose that there exists a sequence $\{\nu_j\}$ in $P(G)$ such that $\nu_j \to \delta_e$, the point mass at the identity, and $\{\alpha_j(\nu_j)\}$ converges to ν in $P(G)$. Then $\operatorname{supp}\nu$ is contained in a closed nilpotent subgroup of G.*

Proof: We shall first show that in proving the theorem we may assume that the center of G is discrete. Let Z be the center of G and suppose it is not discrete. Let $\eta : G \to G/Z$ be the natural quotient map. Then the image of $\eta(\nu)$ on G/Z satisfies the condition in the hypothesis as above, as a measure on G/Z, and on the other hand if the support of $\eta(\nu)$ is proved to be contained in a nilpotent subgroup of G/Z it follows that the support of ν is contained in a nilpotent subgroup. Therefore it suffices to prove the theorem for G/Z in the place of G. The latter is a lower-dimensional Lie group, and hence by an obvious inductive argument we see that it suffices to prove the result in the case when the center is discrete.

When the center of G is discrete, the group of continuous automorphisms has only finitely many connected components; see [7]. Therefore infinitely many α_i's are contained in a single connected component, and by replacing ν by $\alpha(\nu)$ for a suitable automorphism α we may assume that $\{\alpha_i\}$ are all contained in the connected component of the identity automorphism.

Now let $\mathrm{Aff}\,(G)$ denote the group of affine automorphisms of G (namely homeomorphisms of the form $T_g \circ \alpha$ where T_g is the (left) translation by $g \in G$ and α is an automorphism of G). Let S be the connected component of the identity in $\mathrm{Aff}\,(G)$. By the reduction step as above the automorphisms α_i are all contained in S. We realise G as a subgroup of S consisting of left translations. Let V be the Lie algebra of S and let $\rho : S \to GL(V)$ be the adjoint representation of S. Now $\{\rho(\nu_i)\}$ is a sequence in $P(GL(V))$ such that $\rho(\nu_j) \to \delta_I$, the point mass at the identity I of $GL(V)$, and $\{\rho(\alpha_j(\nu_j))\}$ converges to $\rho(\nu)$. Moreover $\rho(\alpha_j(\nu_j)) = \rho(\alpha_j)\nu_j\rho(\alpha_j^{-1})$ for each j. Therefore by Theorem 5.1,(iii) the support of $\rho(\nu)$ is contained in a unipotent subgroup of $GL(V)$. Since the kernel of ρ is contained in the center of G this implies that ν is supported on a nilpotent subgroup of G, thus proving the theorem. $\square$

As indicated earlier Theorem 5.3 implies the following.

Corollary 5.4 *Let G be a connected Lie group and μ be a Levy's measure on G. Then there exists a closed nilpotent subgroup N of G such that the support of μ is contained in a coset of N. In particular if $\mathrm{supp}\,\mu$ is not contained in a proper closed subgroup of G and the identity element is contained in the support of μ then G is nilpotent.*

In [15] it is shown that in fact there exists a closed *connected* nilpotent subgroup N of G, containing the support of μ as above. The proof however, though similar in spirit and based on Theorem 5.1, involves also some more detailed features of Lie group theory. We shall not go into the details along those lines, since the purpose here has been to give a flavour of the application of the ideas on asymptotics of measures.

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