MODULI OF PARAHORIC $\mathcal{G}$–TORSORS ON A COMPACT RIEMANN SURFACE

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Abstract. Let X be an irreducible smooth projective algebraic curve of genus $g \geq 2$ over the ground field $\mathbb{C}$ and let G be a semisimple simply connected algebraic group. The aim of this paper is to introduce the notion of a *semistable and stable parahoric* torsor under a certain Bruhat-Tits group scheme $\mathcal{G}$, construct the moduli space of semistable parahoric $\mathcal{G}$–torsors and identify the underlying topological space of this moduli space with certain spaces of homomorphisms of Fuchsian groups into a maximal compact subgroup of G. The results give a complete generalization of the earlier results of Mehta and Seshadri on parabolic vector bundles.

1. Introduction

Let X be a smooth projective curve defined over $\mathbb{C}$ of genus $g \geq 2$. Let $\mathbb{H}$ be the *upper half space* and π a subgroup of the discontinuous group of automorphisms of $\mathbb{H}$ such that $X = \mathbb{H}/\pi$. We note that the action of π is *not* assumed to be free. It is well known that the set of fixed points of π are finite and the isotropy subgroups are cyclic of finite order. Let $z_i \in \mathbb{H}$ be the set of fixed points of the action of π and let the isotropy subgroups be denoted by

$$\pi_{z_i} = \langle C_i \rangle$$

with C_i as generators. Let $q : \mathbb{H} \to X$ be the quotient projection and let $\mathcal{R} \subset X$ be the points of X over which the map q is ramified and let n_i be the *ramification index* at $x_i \in \mathcal{R}$ which are the images of $z_i \in \mathbb{H}$. Thus, $m = |\mathcal{R}|$ and each C_i is an element of order n_i.

Let G be a connected reductive algebraic group over $\mathbb{C}$ and let $K_G \subset G$ be a maximal compact subgroup of G.

1.0.1. Definition. *The* type of a homomorphism $\rho : \pi \to G$ *is defined to be the set of conjugacy classes in* G *of the images* $\rho(C_i)$ *and is denoted by* $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$.

1.1. *Notation.* Let $R^{\boldsymbol{\tau}}(\pi, K_G)$ denote the space of homomorphisms $\rho : \pi \to K_G$ of type $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$.

Following Mehta-Seshadri [17], to a homomorphism $\rho : \pi \to K_G$ one can associate a (π, G)–bundle E on $\mathbb{H}$ i.e, E is a principal G–bundle over $\mathbb{H}$ and the action of π on $\mathbb{H}$ lifts to an action on E.

If G is the full-linear group, the (π, G)–bundles on $\mathbb{H}$ have an equivalent description as π–vector bundles on $\mathbb{H}$. We recall ([28], [17]) that if V is a π–vector bundle

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on $\mathbb{H}$, the vector bundle $W = q_*^\pi(V)$(invariant direct image by q) on X acquires a *parabolic structure* which consists of the data assigning a *flag to the fibre of* W *at every ramification point in* X *for the covering* q together with a tuple of *weights*;

The invariant direct image functor $V \mapsto q_*^\pi(V)$ gives a fully faithful embedding of the category of π-vector bundles on $\mathbb{H}$ into the category of parabolic vector bundles on X (morphisms being taken as isomorphisms). Moreover, we can realise every parabolic bundle as q_*^π for a suitable π and V.

This translates easily into an equivalent description of $(\pi, GL(n))$–bundles on $\mathbb{H}$ as principal $GL(n)$–bundles on X with *parabolic structures*. Now one can define the concepts of *stability* (resp. *semistability*) for π–vector bundles (or equivalently parabolic bundles on X) and construct the corresponding moduli space of equivalence classes of semistable objects (fixing some invariants) as a normal projective variety.

Further, for these moduli spaces the underlying topological spaces can be identified with equivalence classes of elements in $R^{\boldsymbol{\tau}}(\pi, U(n))$, i.e *unitary representations of* π (see Mehta-Seshadri [17], Seshadri [28]), which generalize the results in Narasimhan-Seshadri[19] and Seshadri [27].

The purpose of this paper is to further generalize the above results when the structure group G is no longer the full-linear group.

Let us suppose hereafter that the group G is *semisimple and simply connected* (over $\mathbb{C}$) unless otherwise stated. One can again give an equivalent description of (π, G)–bundles on $\mathbb{H}$ as certain intrinsically defined objects on X. However, the picture is more subtle than the case when G is the full-linear group ; for instance, *it is not possible, in general, to associate in a natural manner a principal* G*–bundle on* X *to a* (π, G)*–bundle on* $\mathbb{H}$. We call these new objects on X, which give an equivalent description of (π, G)–bundles on $\mathbb{H}$, as parahoric bundles or *parahoric torsors*. These parahoric torsors are describable as pairs $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a torsor (i.e principal homogeneous space) on X under a parahoric Bruhat-Tits group scheme $\mathcal{G}$, together with *weights* θ, which are elements of the set of *rational* one-parameter subgroups of G (see Definition 1.0.2 and Definition 6.2.3). We define notions of semistability and stability of such parahoric torsors and construct moduli spaces of these objects; they are projective varieties and their points are equivalence classes of semistable parahoric $\mathcal{G}$–torsors.

The parahoric torsors that we consider here have been defined earlier by Pappas and Rapoport, without however the notion of weights, where they made some conjectures on the moduli stack of such torsors (see [21] and [22]). Heinloth has since settled many of their conjectures (see [14]; we note that Heinloth works over arbitrary ground fields not just $\mathbb{C}$). We were led independently to the description of these parahoric torsors in trying to interpret (π, G)–bundles on $\mathbb{H}$ as objects on X (inspired by A. Weil's work [35], as was the case in [17] and [28]). In this context, one can say that the central observation in our paper is contained in Section 2, which links explicitly the ideas from the paper of Weil and Bruhat-Tits theory. We need to define a few technical terms before we can state the main results of our paper.

Let $\mathcal{G}$ be a smooth affine group scheme over X satisfying the following conditions (compare this with the definition in Heinloth [14]):

1.0.2. Definition.

(a) *Let $Ram(\mathcal{G}) \subset X$ be the finite set of points $x \in X$ and let A_x the complete local ring at $x \in X$. Then $\mathcal{G}_{A_x}$ is a Bruhat-Tits* parahoric group scheme *over Spec A_x as defined by* [7, Definition 5.2.6].

(b) *The generic fibre of $\mathcal{G}$ is isomorphic to G. Moreover,*

$$\mathcal{G}|_{X-\mathcal{R}am(\mathcal{G})} \simeq G \times (X - \mathcal{R}am(\mathcal{G})) \tag{1.0.2.1}$$

In what follows, we will consider parahoric Bruhat-Tits group schemes where the subset $Ram(\mathcal{G}) \subset X$ is the set $\mathcal{R}$ of ramification points of the map $q : \mathbb{H} \to X$.

For notations and conventions see 2.2. Let $A_{x_i} \simeq \widehat{\mathcal{O}}_{X,x_i}$ and K_{x_i} its quotient field, $x_i \in \mathcal{R}$. Let $\mathbb{E} \simeq Y(T) \otimes \mathbb{R}$ and Ω denote a collection of subsets $\{\Omega_i\} \subset \mathbb{E}^m$, where $|\mathcal{R}| = m$. As is well-known (see 2.2) to each subset $\Omega_i \subset \mathbb{E}$, there is a notion of a parahoric subgroup $\mathcal{P}_{\Omega_i}(K) \subset G(K_{x_i})$, $i = 1, \dots m$, and furthermore, associated to each parahoric subgroup $\mathcal{P}_{\Omega_i}(K)$, there is a smooth group scheme $\mathcal{G}_{\Omega_i}$ over *Spec* A_{x_i} known as the Bruhat-Tits group scheme. Let $\mathcal{G}_{\Omega,X}$ denote the parahoric Bruhat-Tits group scheme on a projective curve X associated to the local datum $\Omega = \{\Omega_i\}$ (see Notation 5.1). We will call such a group scheme a *parahoric Bruhat-Tits group scheme* over the curve X (see Remark 5.1.3 for its relationship with Heinloth's definition).

We observe that there is a picture of parahoric subgroups using the notion of *alcoves* which are products of simplices (see Remark 3.0.7). The points of the alcove $\mathcal{A}$ parametrize parahoric groups upto conjugacy by $G(K)$.

We define, in Section 5 of this paper, the concept of *semistable and stable $\mathcal{G}$–* torsors on X. Our main results can be formulated as follows (see Theorem 8.1.8):

1.0.3. Theorem.

(1) *To every set $\boldsymbol{\tau}$ of conjugacy classes, we can associate a collection $\Omega(\boldsymbol{\tau}) = \{\Omega_i\}$ such that, we have a bijective correspondence between the space $R^{\boldsymbol{\tau}}(\pi, K_G)/K_G$ of conjugacy classes of homomorphisms $\rho : \pi \to K_G$ of type $\boldsymbol{\tau}$ and the equivalence classes of* semistable *$\mathcal{G}_{\Omega(\boldsymbol{\tau})}$–torsors, where $\mathcal{G}_{\Omega(\boldsymbol{\tau})}$ is a parahoric Bruhat-Tits group scheme associated to the set $\boldsymbol{\tau}$.*

(2) *If $\mathcal{G}$ is a parahoric Bruhat-Tits group scheme, then there is a $\boldsymbol{\tau}$ such that equivalence classes of semistable and stable $\mathcal{G}$–torsors is identified with the equivalence classes of semistable and stable $\mathcal{G}_{\Omega(\boldsymbol{\tau})}$–torsors on X.*

(3) *Under this correspondence, the set of irreducible homomorphisms gets identified with the* stable *$\mathcal{G}_{\Omega(\boldsymbol{\tau})}$–torsors.*

1.0.4. *Remark.* If the points of $\Omega(\boldsymbol{\tau})$ correspond to points in the interior of the alcove $\mathcal{A}$, the group scheme $\mathcal{G}_{\Omega(\boldsymbol{\tau})}$ gets embedded in the constant group scheme $G \times X$ (see Remark 2.4.2).

1.0.5. Theorem.

(1) *The set $M_X(\mathcal{G}_{\Omega(\boldsymbol{\tau})})$ of equivalence classes of semistable $\mathcal{G}_{\Omega(\boldsymbol{\tau})}$–torsors on X gets a natural structure of an irreducible normal projective variety of dimension (see Theorem 7.3.2 and Corollary 8.1.9 for notation and details)*

$$dim_{\mathbb{C}}(G)(g-1) + \sum_{i=1}^{m} \frac{1}{2} e(\boldsymbol{\theta}) \tag{1.0.5.1}$$

(2) *The correspondence given in Theorem 1.0.3 above is a homeomorphism $R^{\boldsymbol{\tau}}(\pi, K_G)/K_G \simeq M_X(\mathcal{G}_{\Omega(\boldsymbol{\tau})})$ of the underlying topological spaces.*

1.0.6. *Remark.* We have assumed above that the group G is semisimple and simply connected. The construction of the moduli spaces when G is a general reductive group can be carried out as a consequence of the construction in the semisimple and simply connected case.

We make a few clarifying remarks on the paper.

(i) Let $G = SL(n)$ and let $q : \mathbb{H} \to X$ be the covering projection. Recall that to every homomorphism $\rho \in R^{\tau}(\pi, K_G)$, we can associate a (π, G)–bundle V_ρ on $\widetilde{X}$. The classical association of taking invariant direct image $q_*^{\pi}(V_\rho)$, immediately, by degree considerations, shows that the correspondence sends homomorphisms to bundles on X whose degree is *non-zero* (cf. [19], [28] and [17]). These were precisely the cases for which the *special parabolic structures* were introduced in Narasimhan-Seshadri [19]. In our setting, the moduli spaces of bundles with fixed determinants and non-zero degree gets realized as moduli spaces of torsors under maximal parahoric Bruhat-Tits group schemes and these are exactly n in number.

(ii) We observe that in the above theorem (for G simple), when the set $\boldsymbol{\tau}$ of conjugacy classes is identified with the points in the *interior* of the Weyl alcove, the parahoric Bruhat-Tits group scheme is a subgroup scheme of the constant group scheme $G \times X$ and the semistable and stable torsors are simply the parabolic G–bundles, i.e principal G–bundles with parabolic structures, which are analogues of the parabolic vector bundles. This case has been dealt with in Teleman-Woodward [32].

(iii) When $G = GL(n)$ the theory of Bruhat-Tits ensures that the only parahoric Bruhat-Tits group scheme are subgroup schemes of the Chevalley group schemes. This is the reason why in Mehta-Seshadri [17], the phenomenon of torsors under Bruhat-Tits group schemes does not figure in the correspondence.

(iv) (*Parabolic G–bundles*) Let $A_x = \widehat{\mathcal{O}}_{X,x}$. If for every $x \in \mathcal{R}_p$ the parahoric group $\mathcal{P}_{\Omega}(K_x)$ gets identified with the distinguished hyperspecial parahoric subgroup $G(A_x)$ the moduli space of parahoric torsors is the moduli space of principal G–bundles on X. If on the other hand, $\mathcal{P}_{\Omega}(K_x) \subseteq G(A_x)$ for every $x \in \mathcal{R}_p$, then under the evaluation map $ev : G(A_x) \to G(\mathbb{C})$, the subgroup $\mathcal{P}_{\Omega}(K_x)$ maps to a standard parabolic subgroup of G, so that in this case a quasi-parahoric torsor could indeed be called a *quasi-parabolic G–bundle* in the familiar sense of the term when G is the full-linear group, i.e the data consists of a principal G–bundle on X together with a parabolic subgroup of G (i.e a "flag") for every $x \in \mathcal{R}_p$.

(v) (*Parahoric torsors which are not principal G–bundles*) In general there is a third case, namely there are parahoric subgroups of $G(K_x)$ which cannot be conjugated to subgroups of $G(A_x)$ and indeed, barring $G(A_x)$, all the maximal parahoric subgroups of $G(K_x)$ fall under this third category (see [7]). It is this case which highlights the precise reason why we need to give a subtler description of (Γ, G)–bundles on Y as parahoric torsors on X which do not support a principal G–bundle on X. Evidence to this effect was shown using Tannakian considerations in Balaji-Biswas-Nagaraj [2], leading to the definition of a ramified bundle in [3]. The approaches in these papers give a general picture but does not give the precise correspondence obtained in this paper. More concrete examples were shown in [29] indicating what to expect in general. Furthermore, when the "weights" are in the far wall of the Weyl alcove, Teleman ([31, Section 9] and [32]) has already observed that the quasi-parabolic structures in this case should be given by generalized parabolic subgroups of the group $G((z))$ and that the "true" moduli spaces for these would correspond real analytically to certain space of representations into the maximal compact of G. These correspond to the parahoric torsors which are not parabolic G–bundles.

(vi) The striking cases which arise out of the present study are the non-hyperspecial maximal parahoric subgroups where a number of new phenomena show up. These correspond, on the side of the representations of the Fuchsian group (see 8.0.4.1), to those maps $\rho : \pi \to K_G$ such that centralizers of the images of the elements $\rho(C_i)$ are *proper semisimple* subgroups of G.

(vii) We remark that all the results which realize parahoric bundles on X as obtained from (Γ, G)–bundles from a Galois cover $Y \to X$ hold on $\mathbb{P}^1$ as well as elliptic curves. The only point which fails to hold in general is the existence of *stable parahoric bundles* on these curves. Once this is achieved by possibly choosing sufficiently large number of ramification points, the moduli construction also goes through.

(viii) After this paper was posted in the archives, we were informed by P. Boalch of his paper [5] where the parahoric structure is seen in the setting of regular singular connections. We believe that relating these with our paper should be a fruitful theme to pursue.

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2. Non-abelian functions and bounded groups

As the title suggests, the aim of this section is to tie up some ideas from the classical paper of A. Weil ([35]) and Bruhat-Tits theory ([7]). This section is central to this paper.

2.1. **Some preliminaries on root datum.** Let G be a semisimple, simply connected algebraic group defined over $\mathbb{C}$ and we fix a maximal torus T of G. Let $X(T) := Hom(T, \mathbb{G}_m)$ be the character group and $Y(T) := Hom(\mathbb{G}_m, T)$ the group of 1-parameter subgroups of T. Let $R = R(T, G) \subset X(T)$ be the root system associated to the adjoint representation of G and S be a system of simple roots.

Denote by $(\ , \ ) : Y(T) \times X(T) \to \mathbb{Z}$ the canonical bilinear form. The set S determines a system of positive roots $R^+ \subset R$ and a Borel subgroup $B \subset G$ with unipotent radical U. We now order the set $R^+ = \{r_i\}, i = 1, \dots, q$. We then have a family $\{u_r : \mathbb{G}_a \to G \mid r \in R\}$ of *root homomorphisms* of groups such that one gets an isomorphism of varieties:

$$\prod_{r \in R^+} u_r : \prod_{r \in R^+} \mathbb{G}_a \to U \tag{2.1.0.1}$$

For every root $r \in R$, we denote by $T_r = Ker(r)^0$, and $Z_r = Z_G(T_r)$, the centralizer of T_r in G. The derived group $[Z_r, Z_r]$ is of rank 1 and there exists a unique 1PS, $r^\vee : \mathbb{G}_m \to T \cap [Z_r, Z_r]$ such that $T = Im(r^\vee).T_r$ and $(r^\vee, r) = 2$. The element $r^\vee$ is the coroot (or 1–PS) associated to r. The $\{r^\vee \mid r \in R\}$ form a system of roots $R^\vee$.

For each $r \in R$ the *root homomorphism*

$$u_r : \mathbb{G}_a \to G \tag{2.1.0.2}$$

is such that

$$t.u_r(a).t^{-1} = u_r(r(t).a) \tag{2.1.0.3}$$

for any $\mathbb{C}$–algebra A and for any $t \in T(A)$, $a \in A$ and such that the tangent map du_r induces an isomorphism

$$du_r : Lie(\mathbb{G}_a) \to (LieG)_r$$

The functor $A \mapsto u_r(\mathbb{G}_a) = u_r(A)$ gives $U_r(A) \subset G(A)$. This determines a closed subgroup U_r of G and is called the *root group* corresponding to r.

Denote by $\{\alpha^* \mid \alpha \in S\}$ to be the basis dual to $\{\alpha \in S\}$, i.e $(\alpha^*, r) = \delta_{\alpha,r}$. Define

$$\mathbb{E} := Y(T) \otimes_{\mathbb{Z}} \mathbb{R} \tag{2.1.0.4}$$

$$\mathbb{E}' := X(T) \otimes_{\mathbb{Z}} \mathbb{R} \tag{2.1.0.5}$$

Most often, we in fact work with $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.2. **Parahoric subgroups.** Let K be the field $\mathbb{C}((z))$ or more generally a field equipped with a discrete valuation $v : K^\times \to \mathbb{Z}$ and which is *complete*. Let A be the ring of integers, with residue field $\mathbb{C}$.

For the notion of Bruhat-Tits buildings and their behaviour under field extensions see J.Tits [33, Page 43].

Once we fix a root datum for G, we see that we have a choice of an affine apartment; the choice of the maximal torus T then identifies $\mathbb{E}$ with an affine apartment $App(G, K)$ in the Bruhat-Tits building $\mathcal{B}(G, K)$.

A subset $M \subset G(K)$ is said to be *bounded* if for any regular function $f \in K[G]$, the values $v(f(m))$ are bounded below, when m runs over all elements of M. In particular, we may talk of *bounded subgroups*. A subgroup $M \subset G(K)$ is therefore bounded if the "order of poles" of elements of M is bounded. This can be made precise by taking a faithful representation $G \hookrightarrow GL(n)$ so that elements of M are represented by matrices with entries in K.

Let $\Omega \subset \mathbb{E}$ be a nonempty subset. Denote by $\mathcal{P}_\Omega(K) \subset G(K)$ the subgroup generated by $T(A)$ and the root groups $U_r(z^{m_r}A)$ for all the roots $r \in R$, where

$$m_r = m_r(\Omega) = -[inf_{\theta \in \Omega}(\theta, r)] \tag{2.2.0.6}$$

where $[h]$ stands for the biggest integer smaller than h.

The group $\mathcal{P}_\Omega(K)$ is a bounded subgroup, more precisely it is a *parahoric subgroup* of $G(K)$ in the sense of Bruhat-Tits and conversely, any parahoric subgroup is bounded in the above sense (cf. Bruhat-Tits [7]) .

The choice of a root datum identifies a parahoric subgroup $\mathcal{P}_\Omega(K) \subset G(K)$ as the stabilizer subgroup of $G(K)$ of a *facet* of the affine apartment $App(G, K)$. By Tits [33, Section 3.1, page 50], since we work with a semisimple and simply connected group G we could in turn take any point in *general position* i.e an interior point in the facet and consider the parahoric subgroup as the stabilizer of that point. Thus one can make an identification $\mathcal{P}_\Omega(K) \simeq \mathcal{P}_\theta(K)$ for an *interior point* θ in the facet determined by Ω.

By the main theorem of Bruhat-Tits ([7]), there exist smooth group schemes $\mathcal{G}_\Omega$ over $Spec(A)$ such that the group $\mathcal{G}_\Omega(A) = \mathcal{P}_\Omega(K)$ and moreover, since A is a complete discrete valuation ring, the group scheme is uniquely determined upto unique isomorphism by its A–valued points (see [7, Section 1.7]).

Let $\theta \in \mathbb{E}$. Thus,

$$m_r = m_r(\theta) = -[(\theta, r)] \tag{2.2.0.7}$$

In other words, we have:

$$\mathcal{P}_{\theta}(K) = \langle T(A),\ U_r(z^{m_r(\theta)}A),\ r \in R\rangle \tag{2.2.0.8}$$

i.e the subgroup generated by $T(A)$ and the root groups $\{U_r(z^{m_r(\theta)}A), r \in R\}$.

To summarize, since we work with a semisimple and simply connected group G, all *parahoric groups* are, upto conjugacy by elements of $G(K)$, precisely the collection of groups $\{\mathcal{P}_{\theta}(K)\}_{\theta \in \mathbb{E}}$ (see [33, Section 3.1, page 50]), and as such we will work with these groups. In particular, associated to the "origin" $0 \in \mathbb{E}$ we have the group $\mathcal{P}_{0}(K)$, which is nothing but the maximal bounded subgroup $G(A) \subset G(K)$.

Note that if $\theta \in Y(T)$ itself, then there exists $t \in T(K)$ such that

$$\mathcal{P}_{\theta}(K) = t.\mathcal{P}_{0}(K).t^{-1} \tag{2.2.0.9}$$

2.2.1. *Remark.* Again we note that if $m_r(\theta) < 1$ for all $r \in R$, then $\mathcal{P}_{\theta}(K) \subset G(A)$. These parahoric subgroups then correspond to the standard parabolic subgroups of G.

2.3. **Non-abelian functions and parahoric subgroups.** Let $X \simeq \mathbb{H}/\pi$ be as in the introduction.

2.3.1**.** A result due to A.Selberg ([25]) states that if $A \subset GL(n, \mathbb{C})$ is a finitely generated subgroup, then A has a *normal subgroup* A_0 of finite index with no torsion. It follows from this that the discrete group $\pi \subset Aut(\mathbb{H})$ has a normal subgroup π_0 of finite index such that π_0 operates *freely* on $\mathbb{H}$. Let $Y = \mathbb{H}/\pi_0$ and $\Gamma = \pi/\pi_0$. Then there is a canonical action of Γ on Y such that $X = Y/\Gamma$. Let $p : Y \to X$ be the covering map and note that $\Gamma = Gal(Y/X)$. It is easily seen that the study of (Γ, G)–bundles on Y is equivalent to the study of (π, G)–bundles on $\mathbb{H}$ and thus the study of (π, G)–bundles on $\mathbb{H}$ reduces to an algebraic problem since Y is a compact Riemann surface and hence a smooth projective curve.

Let $q_1 : \mathbb{H} \to Y$ be the simply connected covering projection of Y. We have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\ q_1\ } & Y \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array} \tag{2.3.1.1}$$

with $q = p \circ q_1$. Let y_i be the image of z_i in Y and let $\mathcal{R}^*_{p} = \{y_i \mid 1 \leq i \leq m\}$.

The map $q_1 : \mathbb{H} \to Y$ is a local isomorphism; in fact, if $z \in \mathbb{H}$ maps to $y \in Y$, then q_1 induces an isomorphism $\pi_z \xrightarrow{\sim} \Gamma_y$ of isotropy subgroups of π and Γ respectively, as well as an isomorphism of a sufficiently small (formal) neighbourhood of z onto that of y, respecting the actions of the isotropy groups. Now a (Γ, G)–bundle E on Y is *locally* a (Γ_y, G)–bundle at y.

Recall that this (Γ_y, G)–bundle is *defined by a representation* (see for example Grothendieck [12, Proposition 1, page 06] and in the formal setting see the more recent paper of Teleman-Woodward [32, Lemma 2.5]); i.e, if N_y is a sufficiently small Γ_y–stable formal neighbourhood of y, then this bundle is isomorphic to the

(Γ_y, G)–bundle $N_y \times G$, for the *twisted* Γ_y-action on $E \times G$ given by a representation $\rho_y : \Gamma_y \longrightarrow G$, defined as follows:

$$\gamma \cdot (u, g) = (\gamma u, \rho_y(\gamma) g), \; u \in N_y, \; \gamma \in \Gamma_y. \tag{2.3.1.2}$$

2.3.2. *Observation. It is easily seen that these* (Γ_y, G)*–bundles given by representations are isomorphic as* (Γ_y, G)*–bundles if and only if the defining representations are equivalent. We call the representations* ρ_y *the* local representations *associated to a* (Γ, G)*–bundle.*

Let $D_x = Spec(A)$, where A is the complete discrete valuation ring obtained by taking the completion of the local ring $\mathcal{O}_{X,x}$ and let $K = K_x$ be its quotient field. Similarly, for $y \in \mathcal{R}_p^*$, let $N_y = Spec(B)$, where B is the integral closure of A in $L = K(\omega)$, where ω is a primitive d^{th}–root of z, where $d = |\Gamma_y|$ and z is the uniformizer of A. Let $p : N_y \to D_x \simeq N_y / \Gamma_y$ be the totally ramified covering projection. Let E be the (Γ, G)–bundle on Y and $y \in \mathcal{R}_p^*$. Consider the restriction of E to N_y. Then as we have seen above in (2.3.1.2), as a (Γ_y, G) bundle we can identify $E|_{N_y}$ with the trivial bundle $N_y \times G$ together with the twisted Γ_y–action.

2.3.3. Definition. *Define* U_y *to be the group:*

$$\mathsf{U}_y = Aut_{(\Gamma_y, G)}(E|_{N_y}) \tag{2.3.3.1}$$

of (Γ_y, G) *automorphisms of* E *over* N_y*. We call* U_y the unit group *(or more precisely* the local unit group *at* $y \in Y$*) associated to* E*.*

We work with notations fixed above. Let $\rho : \Gamma_y \to G$ be a representation. Let $\ell = rank(G)$ and we represent the maximal torus $T \subset G$ in the diagonal form as follows:

$$T = \begin{bmatrix} t_1 & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & t_\ell \end{bmatrix} \tag{2.3.3.2}$$

where $\{t_1, \ldots, t_\ell\}$ is a basis of $X(T)$.

Since Γ_y is cyclic, we can suppose that the representation ρ of Γ_y in G factors through T (by a suitable conjugation).

The action of Γ_y on N_y canonically determines a character as follows. Since N_y is "1–dimensional", the action determines an action of Γ_y on the tangent space T_y to N_y at y. We denote this character by χ_o (which is of order d). Fix a generator γ in Γ_y. We can choose the coordinate function ω of N_y. Then the character χ_o is given by:

$$\chi_o(\gamma).\omega = \zeta.\omega \tag{2.3.3.3}$$

where ζ is a primitive d^{th}–root of unity.

2.3.4. Lemma. *Let* Γ_y *be a cyclic group of order* d *acting on* N_y *as above. Then we have a canonical identification*

$$Hom(\Gamma_y, T) \simeq \frac{Y(T)}{d.Y(T)} \tag{2.3.4.1}$$

Proof: This lemma is obvious. Observe that $X(\Gamma_y) \simeq \mathbb{Z}/d\mathbb{Z}$ by the canonical choice of character χ_o as in (2.3.3.3). Then, we see that

$$Hom(\Gamma_y, T) = Hom(X(T), X(\Gamma_y)) = Hom(X(T), \mathbb{Z}/d\mathbb{Z}) = \frac{Y(T)}{d.Y(T)}.$$

q.e.d

We now elaborate this identification for setting up the notations which play a key role in the next theorem.

Given a representation $\rho \in Hom(\Gamma_{_y}, T)$, the image $\rho(\gamma)$ takes the form

$$\rho(\gamma) = \begin{bmatrix} \chi_{_o}(\gamma)^{a_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \chi_{_o}(\gamma)^{a_\ell} \end{bmatrix} \tag{2.3.4.2}$$

i.e $\rho(\gamma)$ takes the form

$$\rho(\gamma) = \begin{bmatrix} \zeta^{a_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \zeta^{a_\ell} \end{bmatrix} \text{ with } a_i \in \mathbb{Z}. \tag{2.3.4.3}$$

We can suppose that $|a_i| < d$ for all i (or even $0 \leq a_i < d$) and take

$$\eta_i = a_i/d, \text{ so that } |\eta_i| < 1 \tag{2.3.4.4}$$

Note that the numbers $\{a_1, a_2, \ldots, a_\ell\}$ are determined uniquely modulo d.

In terms of the local coordinates ω and z, we may identify the function ω^{a_i} with z^{η_i} where $z = \omega^d$. Define the rational map $\Delta : N_{_y} \longrightarrow T$, or equivalently a morphism on the punctured disc $N_{_y} - (0)$ as follows:

$$\Delta = \Delta(\omega) = \begin{bmatrix} \omega^{a_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \omega^{a_\ell} \end{bmatrix} = \begin{bmatrix} z^{\eta_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & z^{\eta_\ell} \end{bmatrix} \tag{2.3.4.5}$$

Then we have

$$\Delta(\gamma u) = \rho(\gamma)\Delta(u), \quad u \in N_{_y} \tag{2.3.4.6}$$

where Δ can be taken as a function $\Delta : N_{_y} \longrightarrow G$ (through $T \hookrightarrow G$).

Consider the restriction of Δ to the *punctured disc* and view it as a 1PS i.e, $\Delta|_{_{Spec(L)}} : \mathbb{G}_{m,L} \longrightarrow G$. More precisely, the data of giving the function Δ together with its $\Gamma_{_y}$–equivariance automatically gives a rational 1–PS of G, i.e an element $\theta_{_\Delta} \in Y(T) \otimes \mathbb{Q}$ and the key point to note is that

$$d.\theta_{_\Delta} = \Delta \ i.e \ \theta_{_\Delta} \in \frac{Y(T)}{d.Y(T)} \tag{2.3.4.7}$$

The association $\rho \mapsto \theta_{_\Delta}$ gives explicitly the identification obtained in Lemma 2.3.4.

2.3.5. *Remark.* We note that the tuple of numbers $\{a_1, a_2, \ldots, a_\ell\}$ are determined uniquely modulo d through the above identification.

2.4. **The unit group.** The unit group U_y has been defined in Definition 2.3.3. The aim of this section is to prove the following:

2.4.1. THEOREM. *The unit group U_y is isomorphic to a parahoric subgroup $\mathcal{P}_{\theta_\Delta}(K)$ of $G(K)$ associated to the element $\theta_\Delta \in Y(T)\otimes\mathbb{Q}$. Conversely, if $\mathcal{P}_\theta(K)$ is any parahoric subgroup of $G(K)$ then there exists a positive integer d, a field extension $L = K(\omega)$ of degree d over K such that*

$$\mathcal{P}_\theta(K) \simeq \mathsf{U}_y \tag{2.4.1.1}$$

Proof: We first give a different description of the elements of U_y. By (2.3.1.2) a (Γ_y, G)–bundle on Y gets a Γ_y-equivariant trivialization; in other words, the Γ_y-action on $N_y \times G$ is given by a representation $\rho : \Gamma_y \longrightarrow G$

$$\gamma \cdot (u, g) = (\gamma u, \rho(\gamma) g),\ u \in N_y,\ \gamma \in \Gamma_y. \tag{2.4.1.2}$$

Let $\phi_0 \in \mathsf{U}_y$, i.e, the map

$$\phi_0 : N_y \times G \longrightarrow N_y \times G. \tag{2.4.1.3}$$

is equivariant for the Γ_y–action. This implies that (equivariance under G)

$$\phi_0(u, g) = (u, \phi(u) g)$$

where $\phi : N_y \longrightarrow G$ is a regular map satisfying the following Γ_y–equivariance:

$$\phi(\gamma \cdot u) \quad = \quad \rho(\gamma)\phi(u)\rho(\gamma)^{-1}, u \in N_y, \gamma \in \Gamma_y. \tag{2.4.1.4}$$

We may thus identify U_y with the following:

$$\mathsf{U}_y = \{\phi : N_y \to G \mid (2.4.1.4)\ holds\} = Mor^{\Gamma_y}(N_y, G) \tag{2.4.1.5}$$

Since $N_y = Spec(B)$, we can view $\mathsf{U}_y \subset G(B) \subset G(L)$.

Let Δ be as in (2.3.4.5). Consider the inner automorsphism defined by Δ:

$$i_\Delta : G(L) \to G(L) \tag{2.4.1.6}$$

given by $i_\Delta(\eta) = \Delta^{-1}.\eta.\Delta$. Define

$$\mathsf{U}'_y := i_\Delta(\mathsf{U}_y) \tag{2.4.1.7}$$

Let $\psi = i_\Delta(\phi) = \Delta^{-1}.\phi.\Delta$ with $\phi \in \mathsf{U}_y$. Then we observe that

$$\psi(\gamma u) = \psi(u)$$

so that $\psi \in G(L)^{\Gamma_y}$. That is, it *descends* to a rational function $\tilde{\psi} : D_x \longrightarrow G$, where $\tilde{\psi}(z) := \psi(\omega)$. In other words, we get

$$\mathsf{U}'_y \subset G(K) = G(L)^{\Gamma_y} \tag{2.4.1.8}$$

Then we *claim* the following:

$$\mathsf{U}'_y = \mathcal{P}_{\theta_\Delta}(K) \tag{2.4.1.9}$$

where $\theta_\Delta \in Y(T)\otimes\mathbb{Q}$ is as in (2.3.4.7). Recall the definition of the parahoric subgroup:

$$\mathcal{P}_{\theta_\Delta}(K) = \langle T(A),\ U_r(z^{m_r(\theta_\Delta)}A),\ r \in R\rangle \tag{2.4.1.10}$$

Let $\psi \in \mathsf{U}'_y$ and let $\psi = i_\Delta(\phi)$, with $\phi \in \mathsf{U}_y$. Thus,

$$\phi = \Delta\psi\Delta^{-1}.$$

Consider the map $\phi : N_y \to G$. Let $G^o \subset G$ denote the *big cell* determined by the roots R, (i.e the inverse image in G of a dense B–orbit in G/B).

Let us assume for the moment that $\phi(N_y) \in G^o$. In other words, ϕ can be described *uniquely* as a tuple $\big(\{\phi_r\}_{r\in R}\ ,\ \phi_t\big)$, with $\phi_r(u) \in U_r$ and $\phi_t(u) \in T$ for $u \in N_y$.

We first consider the tuples $\big(\phi_r(u)\big)_{r\in R}$ and the corresponding tuple for ψ, namely, $\big(\psi_r(u)\big)_{r\in R}$ first, where the $\phi_t\ :\ N_y \to T$ and

$$\{\phi_r, \psi_r : \mathbb{G}_{a,L} \to G \mid r \in R\}.$$

The uniqueness of the decomposition of elements in the big cell and the invariance property of ϕ translates into invariance for each of the ϕ_r and ϕ_t. In other words, we have the following:

$$\phi_r(\omega) = \Delta\psi_r(\omega)\Delta^{-1}. \tag{2.4.1.11}$$

i.e

$$\phi_r(\omega) = \psi_r(\omega)\omega^{r(\Delta)} \tag{2.4.1.12}$$

In terms of $\tilde{\psi}$, this gives:

$$\phi_r(\omega) = \tilde{\psi_r}(z)z^{\frac{r(\Delta)}{d}} \tag{2.4.1.13}$$

Now interpreting the condition that $\tilde{\psi}$ should satisfy so that the ϕ's are regular functions in the variable ω at $\omega = 0$, we see that the *order of pole* for $\psi_r(z)$ at $z = 0$, is bounded above by $[\frac{r(\Delta)}{d}]$ (the biggest integer smaller than $\frac{r(\Delta)}{d}$) . In other words $\forall r \in R$,

$$\tilde{\psi_r}(z) \in U_r(z^{-[r(\theta_\Delta)]}A) = U_r(z^{m_r(\theta_\Delta)}A) \tag{2.4.1.14}$$

and hence $\tilde{\psi} \in \mathcal{P}_{\theta_\Delta}(K)$.

Now, towards completing the proof of the claim (2.4.1.9), if $\phi_t(u) \in T$, then by (2.4.1.4), it follows that ϕ_t is Γ_y–invariant and hence, $\tilde{\psi_t} \in T(A)$.

We now take a closer look at the map $\phi : N_y \to G$. In general, the $\phi(N_y)$ need not be contained in the big cell G^o. So we consider the element $\phi(y) = g_o \in G$. Since the point $y \in N_y$ is Γ_y–fixed, it implies that $g_o \in G^{\Gamma_y}$. Thus, by (2.4.1.4), the point $g_o \in C_G(\rho(\gamma))$, the centralizer in G of $\rho(\gamma)$, which is a Levi subgroup L_θ of the standard parabolic subgroup of G determined by the coroot $\theta = \theta_\Delta$. The Levi subgroup can be described in terms of the $u_r : \mathbb{G}_a \to G$ given as in (2.1.0.2); $C_G(\rho(\gamma)) = L_\theta = \langle T,\ u_r(\mathbb{C}) \mid\ r \in R,\ and\ m_r(\theta) = (\theta, r) = 0\rangle$.

Furthermore, by the equation (2.3.4.5) which defines the function $\Delta : \mathbb{G}_m \to T$, it is immediate from (2.1.0.2) that $\Delta^{-1}.u_r.\Delta = u_r$ if $m_r(\theta) = (\theta, r) = 0$. The same obviously holds for the elements of the maximal torus. Hence the elements which commute with $\rho(\gamma)$ also commute with Δ. This implies immediately that $g_o = i_\Delta(g_o)$ and therefore g_o is an element of the parahoric subgroup $\mathcal{P}_{\theta_\Delta}(K)$.

Now define $\phi_1 : N_y \to G$ by $\phi_1(u) = g_o^{-1}\phi(u)$. Then, $\phi_1(y) = 1$ and hence lies in G^o. Hence by the openness of G^o and the fact that N_y is a formal neighbourhood of y, it follows that $\phi_1(N_y) \subset G^o$. Also, clearly ϕ_1 satisfies (2.4.1.4) and hence by the earlier argument together with the fact that $i_\Delta(g_o) \in \mathcal{P}_{\theta_\Delta}(K)$, we see that $i_\Delta(\phi) = \psi$ is an element in $\mathcal{P}_{\theta_\Delta}(K)$. This completes the proof of the claim (2.4.1.9) without any assumptions.

Conversely, we show that any parahoric subgroup of $G(K)$ can be identified, upto conjugation by a $g \in G(K)$, with a *unit group* U_y. Let $\theta \in \mathbb{E}$ and let $\mathcal{P}_\theta(K)$ be a parahoric subgroup. We would like to modify θ to a θ_Δ for a suitable $\Delta \in Y(T)$ so that, interpreted as unit groups we get $\mathcal{P}_\theta(K) \simeq \mathcal{P}_{\theta_\Delta}(K) \simeq \mathsf{U}_y$.

We observe firstly that the parahoric subgroup $\mathcal{P}_{\theta}(K)$ given by $\theta \in \mathbb{E}$ remains the same when another choice of θ is made in a neighbourhood. In other words, we may assume without loss of generality that $\theta \in Y(T) \otimes \mathbb{Q}$. Expressing it in terms of generators and clearing denominators, we see that there exists a positive integer d so that $d.\theta \in Y(T)$. Then the obvious choice is $\Delta = d.\theta$ which therefore forces $\Delta \in Y(T)$.

Now we view Δ as a "rational" map $\Delta : N_{y} \to T$ and hence Δ can be expressed as in (2.3.4.5), the a_i's being determined by the following considerations: for $r \in R$ be any root we define

$$r(\Delta) = d.(\theta, r)$$

By the discussion following Lemma 2.3.4, we have a $\theta_{\Delta} \in \frac{Y(T)}{d.Y(T)}$ and the identification of Lemma 2.3.4 gives the representation $\rho : \Gamma_y \to T \subset G$. The representation ρ gives the action on the root groups $U_r(B) \subset G(B)$ which are given by (see 2.1.0.3):

$$\rho(\gamma).U_r(B).\rho(\gamma)^{-1} = U_r(\zeta^{r(\Delta)}B) \tag{2.4.1.15}$$

Retracing the steps in the first half of the proof, it is easy to see that $\mathcal{P}_{\theta}(K) \simeq \mathsf{U}_y$ completing the proof of the theorem.

q.e.d

2.4.2. *Remark.* In the notations used above, if $m_r(\theta_{\Delta}) < 1$ for all $r \in R$, such elements $\theta_{\Delta} \in \mathbb{E}$ are precisely the points of the interior of the alcove $\mathcal{A}$ (see Remark 2.2.1 and Remark 3.0.7).

2.4.3. Definition. *Let $\theta \in Y(T) \otimes \mathbb{Q}$. Let $\Delta = d.\theta$ as above. To this data we associate a representation $\rho_{\theta} : \Gamma_{y} \to G$ which acts on the root groups as in* (2.4.1.15). *In particular, for $\alpha \in S$ a simple root, let $\theta_{\alpha} := \frac{\alpha^*}{c_\alpha}$ (see the definition*(3.0.3.2) *below) and ρ_{θ_α} denote the corresponding representation.*

2.4.4. *Remark.* It is remarked in [29, Case III, Page 8] that it was not clear whether the unit group in the situation considered there is a parahoric subgroup at all. In fact, this is indeed the case as can be seen from Theorem 2.4.1. Moreover, it is not too hard to check by some elementary computations that the unit group considered in [29, Case III, Page 8] does contain the standard Iwahori subgroup but only after a conjugation by a suitable element of $G(K)$.

2.4.5. *Example.* Let us now take $G = GL(m)$. We invite the reader to compare this discussion with the one in Weil ([35, page 56]). Then we can write $\phi = ||\phi_{ij}(\omega)||,\ \tilde{\psi} = ||\tilde{\psi_{ij}}(z)||,\ 1 \leq i,j \leq m$ (as matrices). Then the equation (2.4.1.13) takes the form

$$\phi_{ij}(\omega) = \tilde{\psi_{ij}}(z) z^{\alpha_i - \alpha_j}. \tag{2.4.5.1}$$

We can suppose that $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m < 1$. Since $|\alpha_i - \alpha_j| < 1$, we deduce easily that $\tilde{\psi_{ij}}$ are regular i.e. $\mathsf{U}_y \subset G(A)$. (To see this suppose that $\tilde{\psi_{ij}}$ is *not* regular. Then considered as a function in ω ($z = \omega^d$), ψ_{ij} has a pole of order $\geq d$, whereas $z^{\alpha_i - \alpha_j}$ could have only a pole of order d (as a function in ω). But $\phi_{ij}(\omega)$ is regular, which leads to a contradiction).

2.4.6. *Remark.* We refer the reader to the papers by Gille ([11, Lemma I.1.3.2]), Larsen ([16, Lemma 2.4]) and Serre ([26, Proposition 8, p. 546]) where similar phenomena are considered in the general setting of Bruhat-Tits theory.

2.5. **The Bruhat-Tits group scheme as invariant direct image.** The aim of this subsection is to prove Theorem 2.5.2 below, in other words to get a parametrized version of Theorem 2.4.1. Towards this, we first give a different description of the Bruhat-Tits group scheme associated to the parahoric subgroup $\mathcal{P}_{\theta_\Delta}(K)$.

Let $p : Y \to X$ be a a finite flat morphism of smooth quasi-projective curves. Let $\mathcal{G}_Y$ be a smooth group scheme over Y. Following [6], we can define the *direct image functor* p_* as the Weil restriction of scalars, i.e, we have a group scheme $p_*(\mathcal{G}_Y) := \mathrm{Res}_{Y/X}(\mathcal{G}_Y)$ with the following property; for any X–scheme T, we have a canonical bijection:

$$Hom_X(T, p_*(\mathcal{G}_Y)) \simeq Hom_Y(T \times_X Y, \mathcal{G}_Y) \tag{2.5.0.1}$$

which is functorial in T and $\mathcal{G}_Y$.

It is a fact that the group scheme $\mathcal{G}_Y$ is quasi-projective, being over a smooth quasi-projective curve. Hence, $\mathrm{Res}_{Y/X}(\mathcal{G}_Y)$ is in fact representable by a group scheme (see [6, Theorem 4 and Proposition 6]).

Now we consider the situation with a Galois group action. Let $p : Y \to X$ be a a Galois cover (possibly ramified) of smooth quasi-projective curves and let $\Gamma = Gal(Y/X)$. Suppose also that Γ acts on the group scheme $\mathcal{G}_Y$. Then there is a natural (right) action of Γ on $\mathrm{Res}_{Y/X}(\mathcal{G}_Y)$ given by:

$$z.g = \rho_{\mathcal{G}}(g).z.\rho_T(g)^{-1} \tag{2.5.0.2}$$

where T is a scheme over X, and $z \in \mathrm{Res}_{Y/X}(\mathcal{G}_Y)(T) = Hom_Y(T \times_X Y, \mathcal{G}_Y)$, $g \in \Gamma$ and $\rho_{\mathcal{G}}(g)$ the action on $\mathcal{G}_Y$ induced by g and $\rho_T(g) =$ the action on $T \times_X Y$ induced by its action on Y (see [10, Construction 2, page 292]).

We can now take the *fixed points* subscheme under the action of Γ. The generality on fixed point subschemes given in [10, Section 3] can be applied to our situation since we are in characteristic 0 and we have a canonically defined *smooth* closed X–subgroup scheme $\mathrm{Res}_{Y/X}(\mathcal{G}_Y)^\Gamma \subset \mathrm{Res}_{Y/X}(\mathcal{G}_Y)$.

2.5.1. Definition. *(Invariant direct image) Let $p : Y \to X$ be a a Galois cover (possibly ramified) of smooth quasi-projective curves and let $\Gamma = Gal(Y/X)$. Let $\mathcal{G}_Y$ be a smooth group scheme over Y. We define the* invariant direct image *of $\mathcal{G}_Y$ as:*

$$p_*^\Gamma(\mathcal{G}_Y) := Res_{Y/X}(\mathcal{G}_Y)^\Gamma \tag{2.5.1.1}$$

i.e, for any X–scheme S, we have $p_*^\Gamma(\mathcal{G}_Y)(S) = \mathcal{G}_Y(S \times_X Y)^\Gamma$.

We now return to our setting, i.e $p : N_y \to D_x$. Recall that $\Gamma_y = Gal(N_y/D_x)$. Let $M \subset G(L)$ be the *maximal bounded subgroup* defined by $M := \Delta^{-1}G(B)\Delta$, with Δ as in (2.3.4.5). Let $\mathcal{G}_\Delta$ be the Bruhat-Tits group scheme on $Spec(B)$ uniquely defined by M, i.e

$$\mathcal{G}_\Delta(B) = M = \Delta^{-1}G(B)\Delta \tag{2.5.1.2}$$

Observe that the B–group scheme $\mathcal{G}_\Delta$ is isomorphic to the trivial group scheme $G \times SpecB$.

The content of Theorem 2.4.1 is that

$$\mathcal{G}_\Delta(B)^{\Gamma_y} = \mathcal{P}_{\theta_\Delta}(K) \tag{2.5.1.3}$$

2.5.2. Theorem. *Let $\mathcal{G}_{\theta_\Delta}$ be the Bruhat-Tits group scheme defined by the $\mathcal{P}_{\theta_\Delta}(K)$. Let $D_x = Spec\ A$ and $N_y = Spec\ B$. Let $\mathcal{G}_\Delta$ be the group scheme on N_y defined by the bounded subgroup M. Then*

$$\mathcal{G}_{\theta_\Delta} \simeq p_*^{\Gamma_y}(\mathcal{G}_\Delta) \tag{2.5.2.1}$$

In particular, if T is a scheme over $\mathbb{C}$, then we have the identification:

$$Mor^{\Gamma_y}(N_y \times T, G) \simeq Mor(D_x \times T, \mathcal{G}_{\theta_\Delta}) \tag{2.5.2.2}$$

Proof: By Bruhat-Tits ([7, Section 1.7]), the smooth group scheme $\mathcal{G}_{\theta_\Delta}$ on D_x is uniquely determined by its A–valued points which is the parahoric group $\mathcal{P}_{\theta_\Delta}(K)$.

By the functorial property of the functor $p_*^{\Gamma_y}$, we see (by (2.3.4.2)) that

$$p_*^{\Gamma_y}(\mathcal{G}_\Delta)(A) = \mathrm{Res}_{B/A}(\mathcal{G}_\Delta)^{\Gamma_y} = \mathcal{G}_\Delta(B)^{\Gamma_y} = \mathcal{P}_{\theta_\Delta}(K) \tag{2.5.2.3}$$

Thus, by the uniqueness of the Bruhat-Tits group scheme, we have an isomorphism of $Spec(A)$–group schemes: $\mathcal{G}_{\theta_\Delta} \simeq p_*^{\Gamma_y}(\mathcal{G}_\Delta)$.

The identification (2.5.2.2) now follows from the functorial properties of restriction of scalars and fixed point schemes since,

$$Mor^{\Gamma_y}(N_y \times T, G) = p_*^{\Gamma_y}(\mathcal{G}_\Delta)(D_x \times T) = \mathcal{G}_{\theta_\Delta}(D_x \times T) \tag{2.5.2.4}$$

q.e.d

3. Remarks on Bruhat-Tits theory when G is simple

Let the notations be as in the beginning of §2. For this section alone we will assume G is moreover *simple*. In this situation, the parahoric groups have a nicer description which assist in more explicit computations in the later sections.

3.0.3. *Remark.* As we have seen earlier, when G is semisimple and simply connected, it is known that every parahoric subgroup of $G(K)$, upto conjugation by an element of $G(K)$, can be identified with a $\mathcal{P}_\theta(K)$ for a suitable $\theta \in \mathbb{E}$ (see for example [33, Section 3.1, page 50]). Again by [33, page 51], the conjugacy classes of maximal parahoric subgroups of $G(K)$ are the stabilizers of the vertices of the building and they are precisely $l+1$ in number, where $l = rank(G)$.

Let α_{max} denote the highest root, noting that G is simple. Then we can express it as:

$$\alpha_{max} = \sum_{\alpha \in S} c_\alpha \cdot \alpha \tag{3.0.3.1}$$

with $c_\alpha \in \mathbb{Z}^+$.

One can have a nicer choice of the points whose stabilizers give the maximal parahorics (see the last paragraph in [34, Page 662]), now that G is *simple.* For every $\alpha \in S$, we define

$$\theta_\alpha = \frac{\alpha^*}{c_\alpha} \in \mathbb{E}, \tag{3.0.3.2}$$

then in fact, $\{\mathcal{P}_{\theta_\alpha}(K) \mid \alpha \in S\}$ and the group $\mathcal{P}_0(K)$ represent the conjugacy classes under $G(K)$ of all maximal parahoric subgroups *of $G(K)$.* In other words, these are indexed precisely by the vertices of the *extended Dynkin diagram.*

We now recall the description of the set of conjugacy classes in a compact semisimple and simply connected group in terms of the affine Weyl group W_{aff}.

Fix a maximal torus $T \subset G$. Let $Y(T)$ denote the group generated by the coroots $\check{\alpha}_i$ (see Section 2 for notations). Then one has the identifications:

$$\{\text{Elements of finite order}\}/conjugation \simeq \{\text{Elements of finite order in T}\}/W$$

and this is the same as

$$(Y(T) \otimes \mathbb{Q}/\mathbb{Z})/W \simeq (Y(T) \otimes \mathbb{Q})/W_{\text{aff}}.$$

Further, since G is simple, $(Y(T) \otimes \mathbb{Q})/W_{\text{aff}}$ gets identified with the simplex (*the (rational) Weyl alcove*)

$$\mathcal{A} := \{x \in Y(T) \otimes \mathbb{Q} \mid (x, \alpha_{_{max}}) \leq 1, (x, \alpha_i) \geq 0, \forall \text{ } positive\ roots\ \alpha_i\}$$

3.0.4. *Observation.* It is well-known that the set of conjugacy classes of element in K_G is identified with points of the Weyl alcove. Under this identification, elements of finite order get mapped to the points of $\mathcal{A}$ defined above.

3.0.5. *Remark.* Recall that vertices of the alcove $\mathcal{A}$ correspond to the vertices of the extended Dynkin diagram. Furthermore, to each point of $\mathcal{A}$ one can associate a *parahoric subgroup* of $G(K)$ and hence a canonically defined parahoric Bruhat-Tits group scheme. Thus, for each tuple $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}_{i=1}^m$ of conjugacy classes of elements of finite order in K_G we have a subset $\Theta(\boldsymbol{\tau}) = \{\theta_i\}_{i=1}^m \subset \mathcal{A}^m$, where $m = \#\{\ of\ conjugacy\ classes\}$ and hence an associated parahoric Bruhat-Tits group scheme $\mathcal{G}_{\Theta(\boldsymbol{\tau})}$.

3.0.6. *Remark.* By the description of the (rational) Weyl alcove $\mathcal{A}$ (see Definition 3.0.4) and the fact that the parahoric subgroups are determined by *general points* of $\mathbb{E}$, G being simple and simply connected, it follows that upto conjugacy by $G(K)$, every parahoric subgroup of $G(K)$ can be identified with a $\mathcal{P}_{\theta}(K)$ for a suitable $\theta \in \mathcal{A}$. Moreover, by Remark 2.2.1, if $m_r(\theta) < 1$ for all $r \in R$, then $\mathcal{P}_{\theta}(K) \subset G(A)$.

3.0.7. *Remark.* We remark that when G is semisimple, we still have the notion of an *alcove* $\mathcal{A}$, but it will no longer be a simplex as in the case when G is simple since there is no unique $\alpha_{_{max}}$ but $\mathcal{A}$ will now be a product of the Weyl alcoves associated to the simple factors of G.

3.1. **Hyperspecial Parahorics.** In Bruhat-Tits theory, we encounter the so-called *hyperspecial* maximal parahorics which have the following characterizing property: each parahoric group $\mathcal{P}_{\Omega}(K)$ is identified with $\mathcal{G}_{\Omega}(A)$, the A–valued points of a certain canonically defined smooth group scheme $\mathcal{G}_{\Omega}$ defined over A. It is a fact that the parahoric subgroup $\mathcal{P}_{\theta_\alpha}(K)$ is hyperspecial if and only if $c_\alpha = 1$ in the description of the long root $\alpha_{_{max}}$. This can be checked by an inspection of the tables and some easy computations. In particular, type-wise we have the following description upto conjugation by $G(K)$:

(1) In type A_n, all the $n+1$ maximal parahoric subgroups are hyperspecial parahorics.
(2) In types B_n, C_n we have exactly 2 hyperspecial maximal parahoric subgroups.
(3) Type D_n, has exactly 4 hyperspecial maximal parahoric subgroups.
(4) Type E_6 has exactly 3 parahoric subgroups.
(5) Type E_7 has exactly 2 parahoric subgroups.

(6) In types G_2, F_4, E_8, we have only one hyperspecial maximal parahoric subgroup each.

3.2. **Standard parahorics.** (See Remark 2.2.1, Remark 2.4.2 and Remark 3.0.6) Following the loop group terminology, the *standard parahoric subgroups* of $G(K)$ are parahoric subgroups of the distinguished hyperspecial parahoric subgroup $G(A)$. These are realized as inverse images under the evaluation map $ev : G(A) \to G(k)$ of standard parabolic subgroups $P_I \subset G$, where $I \subset S$ is any subset of the simple roots. In particular, the *Iwahori subgroup* $\mathfrak{I}$ is a standard parahoric and indeed, $\mathfrak{I} = ev^{-1}(B)$, $B \subset G$ being the standard Borel subgroup containing the fixed maximal torus T.

Since the *standard parahoric subgroups* of $G(A)$ are also indexed by the subsets of the set of simple roots, to avoid any confusion, we will henceforth denote the *standard parahoric subgroups* of $G(A)$ by $\mathcal{P}^{st}_{I}(K)$ for every subset $I \subset S$. For instance let $\alpha \in S$. Then $P_{\alpha} \subset G$ is a maximal parabolic subgroup while $ev^{-1}(P_{\alpha}) = \mathcal{P}^{st}_{\alpha}(K)$ is a standard parahoric, and we have the obvious inclusions:

$$\mathfrak{I} \subset \mathcal{P}^{st}_{\alpha}(K) \subset \mathcal{P}_{\theta_\alpha}(K) \cap \mathcal{P}_{0}(K) \tag{3.2.0.1}$$

These standard parahorics will play a role when we re-look at Hecke correspondences.

4. The adèlic picture of (Γ, G)–bundles

We work with the notations of §2. Let E be a (Γ, G)–bundle on Y.

4.0.1. Definition. *The* local type *of E at y is defined as the equivalence class of the local representation ρ_y and is denoted by $\boldsymbol{\tau}_y$.*

We denote by $\boldsymbol{\tau}(\mathcal{R}^*_{p})$ the set $\{\boldsymbol{\tau}_y \mid y \in \mathcal{R}^*_{p}\}$. Let us denote by

$$Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G) = \left\{ \begin{array}{l} \text{the set of isomorphism classes of} \\ (\Gamma, G) \text{ bundles with fixed local type } \boldsymbol{\tau}(\mathcal{R}^*_{p}) \end{array} \right\} \tag{4.0.1.1}$$

Since the action of Γ on $Y - p^{-1}(\mathcal{R}_p)$ is free, there is a principal G–bundle F on $X - \mathcal{R}_p$ such that then $E|_{Y-p^{-1}(\mathcal{R}_p)} \simeq p^*(F)$.

Since G is semisimple and simply connected by the theorem of Steinberg and Borel-Springer, F is *trivial*. Hence so is $E|_{Y-p^{-1}(\mathcal{R}_p)}$ as a (Γ, G)–bundle.

Again, by [32, Lemma 2.5], it follows that around each point $y_i \in p^{-1}(\mathcal{R}_p)$, we have formal neigbourhoods $N_{y_i} = Spec\ B_{y_i}$ with Γ_{y_i}–equivariant trivializations of $E|_{N_{y_i}}$ (see (2.3.1.2)). Recall that by Theorem 2.4.1 each unit group U'_y, $y \in \mathcal{R}^*_{p}$ is identified with a parahoric group $\mathcal{P}_{\theta_i}(K_{x_i})$, $x_i \in \mathcal{R}_{p}$.

4.0.2. Proposition. *We have the following set-theoretic identification of $Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ with the adèlic type set of double cosets.*

$$Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G) \simeq \Big[\prod_{x \in \mathcal{R}_p} \mathcal{P}_{\theta_i}(K_{x_i}) \backslash {}^{\prod_{x \in \mathcal{R}_p} G(K_x)} / G(k(X)) \Big] \tag{4.0.2.1}$$

K_x being the quotient field of the complete local rings A_x at $x \in \mathcal{R}_{p}$ and $k(X)$ being the quotient field of X.

Proof: Recall (2.3.1.2), that (Γ_y, G)–bundles are locally given by

$$\gamma \cdot (u,g) = (\gamma u, \rho(\gamma) g),\ u \in N_{y},\ \gamma \in \Gamma_{y}. \tag{4.0.2.2}$$

Let $E \in Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ be a (Γ, G) bundle on Y of local type $\boldsymbol{\tau}$ (see Definition 4.0.1). Let

$$X_1 = X - x,\ \textit{and}\ Y_1 = p^{-1}(X_1) \tag{4.0.2.3}$$

Now Γ acts freely on Y_1 so that the restriction of E to X_1 goes down to a principal G-bundle on X_1 which is trivial (in the algebraic sense) since G is semi-simple. Hence we have:

$E|_{Y_1} \simeq Y_1 \times G$ with the action of Γ given by $\gamma \cdot (u,g) = (\gamma u, g),\ \gamma \in \Gamma$ and $u \in Y_1$. (4.0.2.4)

Let

$$E|_{N_y} = E_1\ \textit{and}\ E|_{Y_1} = E_2 \tag{4.0.2.5}$$

We note that E_1 is given by (4.0.2.2) and $E_2|_{N_y}$ by (4.0.2.4).

The (Γ, G)–bundle E is given by a (Γ, G)–isomorphism

$$\Theta : E_2|_{N_y \cap Y_1} \longrightarrow E_1|_{N_y \cap Y_1}. \tag{4.0.2.6}$$

Observe that if Q is any (Γ, G)–bundle in $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$, then $Q|_{N_y} \simeq E_1$ and $Q|_{Y_1} \simeq E_2$ as (Γ, G)–bundles by (4.0.2.2) and (4.0.2.4) above. Thus Q is defined by an isomorphism (i.e transition function) as in (4.0.2.6) above. Let us denote it by η. Then E is (Γ, G)–isomorphic to Q if and only if we have the following:

$$\phi\ \Theta\ \mu = \eta \tag{4.0.2.7}$$

where ϕ is a (Γ_y, G)–automorphism of E_1 and μ a (Γ, G)–automorphism of E_2 .

Observe that by (4.0.2.4) the map μ is given by a morphism:

$$\begin{aligned} &\mu : Y_1 \times G \longrightarrow Y_1 \times G, \\ &(u,g) \to (u, \mu(u) g), \end{aligned} \tag{4.0.2.8}$$

where $\mu(\gamma \cdot u) = \mu(u), \gamma \in \Gamma$. In other words, the map μ goes down to a morphism $X_1 \longrightarrow G$ and we can view μ as an element in $G(X - x)$.

We now trace the various identifications by restricting the above picture to the punctured disc $N_{y}^{*} = N_{y} - (0)$; note that the (Γ, G)–isomorphism Θ is completely characterized by its restriction to N_{y}^{*}.

We observe by (4.0.2.4) that the restriction of E_2 to N_{y}^{*} is the (Γ_{y}, G)–bundle $N_{y}^{*} \times G$ over N_{y}^{*} with the action of Γ_{y} given by

$$\begin{gathered} \gamma : N_{y}^{*} \times G \longrightarrow N_{y}^{*} \times G,\ \gamma \in \Gamma_{y} \\ \gamma(u,g) = (\gamma u, g). \end{gathered} \tag{4.0.2.9}$$

The restriction of E_1 to N_{y}^{*} is the (Γ_{y}, G)–bundle $N_{y}^{*} \times G$ on N_{y}^{*} with the action of Γ_{y} given by

$$\begin{aligned} &\gamma : N_{y}^{*} \times G \longrightarrow N_{y}^{*} \times G \\ &\gamma(u,g) = (\rho u, \rho(\gamma) g),\ \gamma \in \Gamma_{y}. \end{aligned} \tag{4.0.2.10}$$

The restriction of $\Theta|_{N_y^*}$ of Θ to N_{y}^{*} (denoted again by Θ) is then a (Γ_{y}, G)–isomorphism of the bundle in (4.0.2.8) with the one of (4.0.2.7). We see easily that Θ is defined by the map:

$$\begin{aligned} N_y^* \times G &\longrightarrow N_y^* \times G \\ (u,g) &\longrightarrow (u, \Theta(u)g) \end{aligned} \tag{4.0.2.11}$$

where $\Theta : N_y^* \to G$ is such that $\Theta(\gamma \cdot u) = \rho(\gamma)\Theta(u)$.

Recall that the map Δ as in (2.3.4.5) is a morphism $N_y^* \longrightarrow G$ and has similar properties. Thus we can write

$$\Theta = \Delta\Theta_o \text{ such that } \Theta_o(\gamma u) = \Theta_o(u) \tag{4.0.2.12}$$

i.e. Θ_o descends to a regular map $D_x^* \longrightarrow G$, $D_x^* = D_x - (0)$.

The equivalence relation (4.0.2.7) therefore takes the following form:

$$\phi\ (\Delta\Theta_o)\ \mu = \eta \tag{4.0.2.13}$$

Multiplying on either side by Δ^{-1} we get

$$(\Delta^{-1}\phi\Delta)\ \Theta_o\ \mu = \Delta^{-1}\eta = \eta_o. \tag{4.0.2.14}$$

Thus, by the proof of Theorem 2.4.1, ϕ identifies with an element $i_\Delta(\phi) = \psi$ of the unit group U'_y and we can write (4.0.2.14) as

$$\psi\ \Theta_o\ \mu = \Delta^{-1}\eta = \eta_o. \tag{4.0.2.15}$$

Thus, $\Theta_o \in G(K_x)$ and $\psi \in \mathsf{U}'_y$ and by (4.0.2.8), μ becomes a regular map $X_1 \longrightarrow G$ i.e. $\mu \in G(X - x)$. Thus from (4.0.2.15) together with the identification $\mathsf{U}'_y = \mathcal{P}_\theta(K_x)$, we deduce the following set-theoretic identification of $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ with a set of double cosets:

$$Bun_Y^{\boldsymbol{\tau}}(\Gamma, G) \simeq \Big[\mathcal{P}_\theta(K_x)\backslash^{G(K_x)}/G(X-x)\Big] \tag{4.0.2.16}$$

q.e.d

4.0.3. Definition. *Define the set of double cosets associated to the parahoric groups* $\{\mathcal{P}_{\Omega_i}(K_{x_i})\}$*:*

$$\mathcal{M}_X(\{\mathcal{P}_{\Omega_i}(K_{x_i})\}) = \Big[\prod_{x_i \in \mathcal{R}_p} \mathcal{P}_\Omega(K_{x_i})\backslash^{\prod_{x_i \in \mathcal{R}_p} G(K_{x_i})}/G(k(X))\Big] \tag{4.0.3.1}$$

4.0.4. *Remark.* If $\mathcal{P}_\Omega(K) = \mathcal{P}_0(K)$ is the distinguished hyperspecial parahoric $G(A) \subset G(K)$, then the set of double cosets $\mathcal{M}_X(\mathcal{P}_0(K))$ is simply the $\mathbb{C}$–points of the moduli functor $Bun_X(G)$ of isomorphism classes of principal G–bundles on X (see (5.2.0.2)).

The next proposition shows that $\mathcal{M}_X(\mathcal{P}_\Omega(K))$ depends only on the conjugacy class of the parahoric subgroup.

4.0.5. Proposition. *Let* $g \in G(K)$ *and consider parahoric subgroups* $\mathcal{P}_\Omega(K)_g = g.\mathcal{P}_\Omega(K).g^{-1}$ *and* $\mathcal{P}_\Omega(K)$ *of* $G(K)$*. Then there is a natural bijection*

$$\phi_g : \mathcal{M}_X(\mathcal{P}_\Omega(K)_g) \to \mathcal{M}_X(\mathcal{P}_\Omega(K)) \tag{4.0.5.1}$$

Proof: This follows easily from the following observation. Given a $g \in G(K)$, define the map

$$\phi_g : \mathcal{M}_X(\mathcal{P}_\Omega(K)_g) \to \mathcal{M}_X(\mathcal{P}_\Omega(K))$$

by $\phi_g(\theta) = g.\theta$. That this defines a bijection of set of double cosetss is easy to check.

q.e.d

5. Torsors under Bruhat-Tits group schemes

5.1. **Bruhat-Tits group schemes and patching.** By the main theorem of Bruhat-Tits ([7]), there exist smooth group schemes $\mathcal{G}_{\Omega}$ over $Spec(A)$ such that the group $\mathcal{G}_{\Omega}(A) = \mathcal{P}_{\Omega}(K)$.

We now recall a result due to Raghunathan and Ramanathan as formulated in [8, Lemma 3.18].

5.1.1. Lemma. *Let X be a smooth projective curve and $k(X)$ be its function field. Let $x \in X$ and let A_x be the completion of $\mathcal{O}_{X,x}$ and K_x the completion of $k(X)$. Assume that we are given a triple (G_1, G_2, f) consisting of:*

(a) *An affine group scheme G_1 over $U = X - x$ of finite type.*
(b) *An affine and finitely presented group scheme G_2 over A_x.*
(c) *A K_x–group scheme isomorphism $f : G_1 \times_U K_x \simeq G_2 \times_U K_x$.*

Then there exists a group scheme $\mathcal{G}$, affine and of finite type over X such that $\mathcal{G} \times_X U \simeq G_1$ and $\mathcal{G} \times_X A_x \simeq G_2$ and both isomorphisms are compatible with f. Furthermore, if G_i are smooth then so is $\mathcal{G}$.

We now have the following key observation in the context of parahoric Bruhat-Tits group schemes on X (Definition 1.0.2).

5.1.2. *Observation.* From Lemma 5.1.1 it follows that given a finite $\mathcal{R}_p \subset X$ a subset $\Omega \subset \mathbb{E}^m$ together with *patching data f* as in Lemma above, we have a parahoric Bruhat-Tits group scheme $\mathcal{G}^f_{\Omega,X}$ with $Ram(\mathcal{G}^f_{\Omega,X}) = \mathcal{R}_p$ (see Remark 3.0.5).

5.1. *Notation. In what follows, in the notations of Lemma 5.1.1, we will assume that $G_1 = G \times (X - \mathcal{R}_p)$ and $G_j = \mathcal{G}_{\Omega_j}$ for $j = 1, \ldots, m$, the $\mathcal{G}_{\Omega_j}$ being the Bruhat-Tits group schemes on Spec A_j associated to $\Omega_i \subset \mathbb{E}$. The patching isomorphism f is the identity map. The parahoric Bruhat-Tits group scheme thus obtained will be denoted by $\mathcal{G}_{\Omega,X}$.*

5.1.3. *Remark.* In Heinloth [14] the parahoric Bruhat-Tits group scheme is defined with a restriction that the fibres $\mathcal{G}_x$ for points $x \in Ram(\mathcal{G})$ are *not semisimple.* We do not make this assumption since there are the maximal hyperspecial parahoric group schemes whose closed fibres are in general reductive (and even semisimple when G is simple) and the moduli spaces of torsors for such group schemes are very much a part of the theory presented here. Indeed, in the case of $G = SL(n)$, the maximal parahoric groups are all hyperspecial and the moduli spaces of torsors for the corresponding parahoric Bruhat-Tits group schemes (in our definition) are precisely the spaces of vector bundles of rank n and fixed determinants.

5.2. **$\mathcal{G}$–torsors.** Let $\mathcal{G}$ be a parahoric Bruhat-Tits group scheme on X. Recall that we have assumed that the generic fibre is connected and $\mathcal{G}$ is a smooth group scheme of finite type. Under these conditions, one has as a consequence of theorems of Steinberg and Borel-Springer (see [14, page 511]) the following description of $\mathcal{G}$–torsors.

Any $\mathcal{G}$–torsor E can be obtained by gluing the trivial torsor on some open subset $U \subset X$ and the trivial torsors on the formal completions at the points $\mathcal{R}_p = X - U$. Thus, if $Bun_X(\mathcal{G}_{\Omega,X})$ denotes the set of isomorphism classes of $\mathcal{G}_{\Omega,X}$–torsors on X, then we have the following bijection which gives an adèlic description of $\mathcal{G}_{\Omega,X}$–torsors on X:

$$\mathcal{M}_X(\mathcal{P}_{\Omega}(K)) \simeq Bun(\mathcal{G}_{\Omega,X}) \tag{5.2.0.1}$$

Let $\mathcal{P}_{\Omega_i}(K) \subset G(K)$ $i = 1, \dots m$, be parahoric subgroups. Let $\Omega = \{\Omega_i\}$. From the discussions in 2.2 we have identifications $w_{\theta_i} : \mathcal{P}_{\Omega_i}(K) \simeq \mathcal{P}_{\theta_i}(K)$, where $\theta_i \in \mathbb{E}$, for each i. By Theorem 2.4.1 we can identify these parahorics $\mathcal{P}_{\theta_i}(K)$ with unit groups.

By abuse of notation, let $Bun(\mathcal{G}_{\Omega,X})$ and $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ also denote the *functors* given by:

$$Bun(\mathcal{G}_{\Omega,X})(T) = \left\{ \begin{array}{l} \text{the set of isomorphism classes of} \\ \mathcal{G}_{\Omega,X}\text{–torsors on } X \times T \end{array} \right\} \tag{5.2.0.2}$$

$$Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)(T) = \left\{ \begin{array}{c} \text{isomorphism classes of } (\Gamma, G) \text{ bundles} \\ \text{with fixed local type } \boldsymbol{\tau} \text{ on } Y \times T \end{array} \right\} \tag{5.2.0.3}$$

5.2.1. Lemma. *Let E be a family of (Γ, G)–bundles of local type $\boldsymbol{\tau}$ on Y parametrized by T. Then there is an étale cover $\phi : T' \to T$ such that the bundle $(id_Y \times \phi)^*(E)|_{(Y - R_p^*) \times T'}$ is trivial as a (Γ, G)–torsor.*

Proof: This lemma is an analogue of the theorem of Drinfeld-Simpson for the equivariant case. We work over $\mathbb{C}$. The first step is to show that the bundle E has a Γ–equivariant reduction of structure group to a Borel subgroup $B \subset G$. For this we follow the proof in Heinloth [14, Proposition 25 and Corollary 26]. Then to complete the proof one uses the (Γ, B)–reduction and complete the argument following that of Drinfeld-Simpson.

The notion of Γ–cohomology for Γ–sheaves on Y has been constructed and dealt with in great detail in [12]. These can be realised as higher derived functors of the sub-functor of Γ–*fixed points*, namely, $(H^0)^\Gamma$ of the section functor H^0. We note immediately that since we work over fields of characteristic zero, the sub-functor $(H^0)^\Gamma \subset H^0$ is in fact a direct summand (by the operation of averaging). Hence, we see immediately that the higher derived functors of the functor$(H^0)^\Gamma$ are all sub objects of the derived functors of H^0.

Observe that the infinitesimal deformation space for the (Γ, B) and (Γ, G) bundles are respectively the Γ–cohomology modules $H^1_\Gamma(Y, E_B \times^B (Lie\ B))$ and $H^1_\Gamma(Y, E \times^G (Lie\ G))$, for the standard adjoint action of G on $Lie\ G$. The vanishing of $H^1_\Gamma(Y, E_B \times^B (Lie\ G/Lie\ B))$ now follows by noting that it is a submodule of the usual cohomology module $H^1(Y, E_B \times^B (Lie\ G/Lie\ B))$.

Now one follows the argument in Heinloth [14] to get hold of a modified reduction section s' with the following property: let E'_B be the B–bundle obtained from the new section s'. Then $H^1(Y, E'_B \times^B (Lie\ G/Lie\ B)) = 0$ (see [14, Proof of Proposition 25]).

The vanishing of the obstruction implies that the space of Γ–equivariant reductions of E to B is smooth at $E_{t,B}$ for each $t \in T$ and hence there is a smooth neighbourhood T'_t such that the reduction extends to this neighbourhood.

Having obtained the (Γ, B)–reduction for families, we restrict to $(Y - R_p^*)$, to get the (Γ, B)–torsor $(id_Y \times \phi)^*(E_B)|_{(Y - R_p) \times T'}$. By usual Galois descent, since the action of Γ is free on $Y - R_p$, we see immediately that there is a G–torsor F on $(X - R_p) \times T'$ together with an isomorphism: $(id_Y \times \phi)^*(E)|_{(Y - R_p^*) \times T'} \simeq p^*(F)$ and such that F has a B–reduction on $(X - R_p) \times T''$.

Once we have the B–reduction, to complete the proof we simply observe that the proof in Drinfeld-Simpson [9, page 826] applies to the G–torsor F to give a new étale cover $T'' \to T$ such that F gets trivialized as a principal G–torsor on $(X - R_p) \times T''$. This in turn gives the required Γ–equivariant trivialization of E on $(Y - R_p^*) \times T''$.

q.e.d

We now have the following key theorem:

5.2.2. Theorem. *Fix a Bruhat-Tits group scheme $\mathcal{G}_{\Omega,X}$ over X ramified over $\mathcal{R}_p \subset X$. Then there exists a cover $p : Y \to X$ ramified along $\mathcal{R}_p \subset X$, such that we have an isomorphism of functors $p^\Gamma : Bun_Y^{\boldsymbol{\tau}}(\Gamma, G) \simeq Bun(\mathcal{G}_{\Omega,X})$.*

Proof: Without loss of generality, we will assume that $\mathcal{R}_p = \{x\}$. Let θ be a point in the interior of the facet determined by Ω. This gives an identification $w_\theta : \mathcal{P}_\Omega(K) \simeq \mathcal{P}_\theta(K)$. Now Theorem 2.4.1 and Proposition 4.0.2 immediately gives the ramified cover $p : Y \to X$ and the set-theoretic identification of the $\mathbb{C}$–points of the respective functors. The transformation (4.0.2.12) in the proof of Proposition 4.0.2 identifies the transition function of the (Γ, G)–bundle with that of the $\mathcal{G}_{\theta,X}$–torsor and the transformation $\phi \mapsto i_\Delta(\phi) = \psi$ in the proof of Theorem 2.4.1 identifies the local automorphism group U_y with the parahoric group $\mathcal{P}_\theta(K)$.

We need to show that we have an isomorphism of the corresponding functors.

Let T be a scheme over $\mathbb{C}$. The question boils down to defining a *family of (Γ, G)–bundles of local type $\boldsymbol{\tau}$*. Let $E = E_T \to Y \times T$ be a family of (Γ, G)–bundles. Then by [32, Lemma 2.5], for any $t \in T$, there exists an étale neighbourhood T_t of t and a formal neighbourhood $\tilde{N}_y$ of $y \in Y$, such that the action of Γ on $E|_{\tilde{N}_y \times T_t}$ gets a uniform trivialization by a representation $\rho : \Gamma_y \to G$. Thus, there exists an étale covering $T' \to T$ such that the pull-back $E|_{\tilde{N}_y \times T'}$ has uniform local type $\boldsymbol{\tau}$. By Theorem 2.4.1 and the discussion following it, together with Theorem 2.5.2, we get a trivial $\mathcal{G}_A$–torsor $\mathcal{E}|_{D_x \times T'}$, where $D_x = Spec(A)$, a formal neighbourhood of $x \in X$.

We now consider the restriction of E to $(Y - p^{-1}(x)) \times T$. By Lemma 5.2.1 we get an étale cover $T' \to T$ and a Γ–equivariant trivialization of E on $(Y - p^{-1}(x)) \times T'$

This immediately gives a trivial $\mathcal{G}$–torsor on $(X - x) \times T'$. Gluing as well as étale descent, gives us a $\mathcal{G}_\Omega$–torsor on $X \times T'$. For the descent by the map $T' \to T$, we use an étale descent argument as shown below in the argument for the converse; this gives a $\mathcal{G}_\Omega$–torsor on $X \times T$.

Conversely, let E be a $\mathcal{G}_\Omega$–torsor on $X \times T$. Since G is assumed to be semisimple and simply connected and since the group scheme $\mathcal{G}_\Omega$ is trivial on $X - x$, by the theorem of Heinloth ([14, Theorem 1]) there is an étale covering $T' \to T$ such that the pull-back E' has a trivializing cover $U_1 = (X - x) \times T'$. By Theorem 2.5.2 and the discussion above, E is trivial on $U_2 = D_x \times T'$ with a transition function $\psi : D_x^* \times T' \to \mathcal{G}_\Omega$.

Let X_S denote the product $X \times S$. Since E' comes as a pull-back, it comes equipped with the obvious descent datum, i.e an isomorphism:

$$\alpha : p_1^*(E') \simeq p_2^*(E') \tag{5.2.2.1}$$

which satisfies the "cocycle" condition:

$$p_{31}^*(\alpha) = p_{32}^*(\alpha) \cdot p_{21}^*(\alpha) \tag{5.2.2.2}$$

where $p_1 : X''_{T'} = X_{T'} \times_{X_T} X_{T'} \to X_{T'}$ is the first projection etc. Giving the isomorphism α in terms of the trivializing cover U_i implies that there are maps

$\alpha_i : U_i \to \mathcal{G}_{\Omega}$ such that on the inverse image of $U_1 \cap U_2$ in $X''_{T'}$ we have

$$\alpha_1 \psi = \psi \alpha_2 \tag{5.2.2.3}$$

Let $V_1 = (Y - p^{-1}(x)) \times T'$ and $V_2 = N_y \times T'$. By Theorem 2.5.2, giving the map $\alpha_2 : U_2 \to \mathcal{G}_{\Omega}$ is equivalent to giving a map $\beta_2 : V_2 \to G$ which is Γ-equivariant and the map α_1 obviously gives a Γ-equivariant map $\beta_1 : V_1 \to G$. We also get the transition function $\phi : N_y^* \times T' \to G$ which is Γ_y–equivariant, coming from ψ.

Further, $\beta_1 \phi = \phi \beta_2$ on the inverse image of $V_1 \cap V_2$ in $Y''_{T'}$. The gluing data, gives a (Γ, G)–torsor F' on $Y \times T'$.

This further gives a Γ–equivariant isomorphism:

$$\beta : p_1^*(F') \simeq p_2^*(F') \tag{5.2.2.4}$$

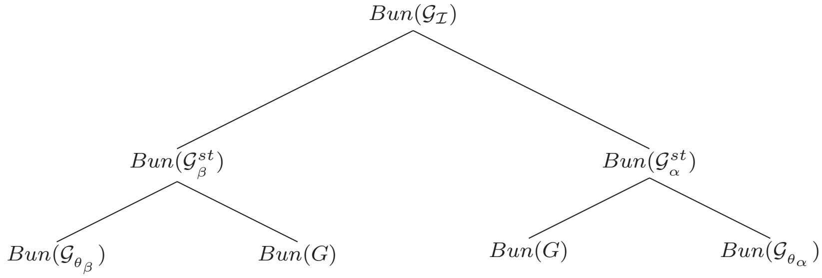
which satisfies the cocycle condition. Hence by étale descent of torsors, we get a (Γ, G)–torsor F on $Y \times T$.

q.e.d

5.2.3. *Remark.* We remark that the proof given above can be made more transparent as follows. Giving a (Γ, G)–bundle on $Y \times T$ is equivalent to giving (Γ, G)–bundles on $(Y - p^{-1}(x) \times T)$ and on $N_y \times T$ together with a Γ–equivariant patching isomorphism. This observation can be thought of as a parametrized analogue of Lemma 5.1.1, which of course needs to be formally proven. By Galois descent, we get a G–bundle on $(X - x) \times T$ and by Theorem 2.5.2, we get a $\mathcal{G}_{\Omega}$–torsor on $D_x \times T$; finally the patching datum also descends (being Γ–equivariant), giving a torsor on $X \times T$. The converse is similar.

5.2.4. *Remark.* This theorem is the exact analogue of the fact that the *invariant direct image functor* p_*^{Γ} sets up an isomorphism between the stack of Γ–vector bundles and that of quasi-parabolic vector bundles; this is precisely the point of view in Seshadri [28] and Mehta-Seshadri [17] (see also Grothendieck[12] and §2.5 above).

5.3. **Hecke Correspondences.** In what follows, we consider parahoric subgroups $\mathcal{P}_{\Omega}(K)$ of $G(K)$ which contain a fixed Iwahori subgroup $\mathcal{I}$ (see 3.2 for notation). Using (3.2.0.1), we get $\mathcal{I} \subset \mathcal{P}_{\alpha}^{st}(K) \subset \mathcal{P}_{\theta_\alpha}(K) \cap \mathcal{P}_0(K)$. These maps of parahoric groups induce maps of the corresponding parahoric Bruhat-Tits group schemes, $\mathcal{G}_{\mathcal{I}} \to \mathcal{G}_{\alpha}^{st}$ and $\mathcal{G}_{\mathcal{I}} \to \mathcal{G}_{\theta_\alpha}$ and natural maps which are in fact morphisms at the level of stacks and get the following generalized Hecke correspondences. Recall that for the linear group case one has the classical Hecke correspondences due to Narasimhan and Ramanan [18]. The dimension formulae obtained later (see Corollary 8.1.9 below) reflect the picture accurately.



(5.3.0.1)

For example we have the Hecke correspondence induced by $\mathcal{G}_{\mathcal{I}} \to \mathcal{G}_{\Omega}$ and $\mathcal{G}_{\mathcal{I}} \to \mathcal{G}_{0} = G$ given by:

$$\begin{array}{ccc} & Bun_X(\mathcal{G}_{\mathcal{I}}) & \\ \swarrow & & \searrow^{G/B} \\ Bun_X(\mathcal{G}_{\Omega}) & & Bun_X(\mathcal{G}_0) \end{array} \tag{5.3.0.2}$$

5.3.1. *Remark.* It would be interesting to express these relations as morphisms between moduli spaces which have been constructed below as projective varieties.

6. Stability and semistability

The aim of this section is to introduce the notion of semistability and stability of torsors under parahoric group schemes introduced in the last section. We recall that once a root datum for G is fixed, there is a natural choice of an affine apartment and this identifies a parahoric subgroup $\mathcal{P}_{\Omega}(K) \subset G(K)$ as the stabilizer subgroup of $G(K)$ of a *facet* of the affine apartment $App(G, K)$ which can then be identified with $\mathbb{E}$ (see §2.2 for these notions). Again, as we have seen earlier, we could in turn take any point in *general position* i.e an interior point in the facet and consider the parahoric as the stabilizer of that point. Thus one can make an identification $\mathcal{P}_{\Omega}(K) \simeq \mathcal{P}_{\theta}(K)$ for an *interior point* θ in the facet determined by Ω.

6.1. **Quasi-parahoric torsors.** The notion of boundedness of subsets of $G(K)$ has been defined in §2.2 . Let $\mathcal{G}_{\Omega,X}$ be a Bruhat-Tits group scheme on the curve X associated to a subset $\Omega \in \mathbb{E}$ (see the notation 5.1). For simplicity we have assumed thet $|\mathcal{R}_p| = \{x\}$ but it is seen easily enough that the definitions go through when more generally we take $\Omega \subset \mathbb{E}^m$.

6.1.1. Definition. *Fix a parahoric subgroup $\mathcal{P}_{\Omega}(K) \subset G(K)$. A* quasi-parahoric *torsor is an element $\mathcal{E}$ of the set of double cosets $\mathcal{M}_X(\mathcal{P}_{\Omega}(K))$; equivalently (by* (5.2.0.1)*), giving $\mathcal{E}$ is giving an element in $Bun(\mathcal{G}_{\Omega,X})$.*

6.2. **Notion of weights and parahoric torsors.** Let $\mathcal{P}_{\Omega}(K)$ be a parahoric subgroup and let $\theta \in Y(T) \otimes \mathbb{Q}$ a point in the interior of the facet determined by Ω. This θ gives rise to an identification

$$w_{\theta} : \mathcal{P}_{\Omega}(K) \simeq \mathcal{P}_{\theta}(K) \tag{6.2.0.1}$$

of the parahoric subgroup $\mathcal{P}_{\Omega}(K)$ with a specific parahoric $\mathcal{P}_{\theta}(K)$ (and hence a local unit group U_y and not just the conjugacy class $\mathfrak{U}_x$) (see Theorem 2.4.1).

Equivalently, the choice of θ gives an isomorphism of group schemes:

$$w_{\theta} : \mathcal{G}_{\Omega} \simeq \mathcal{G}_{\theta} \tag{6.2.0.2}$$

and hence a consequent identification of the functors defined in(5.2.0.2):

$$f_{\theta} : Bun(\mathcal{G}_{\Omega,X}) \simeq Bun(\mathcal{G}_{\theta,X}) \tag{6.2.0.3}$$

6.2.1. Definition. *(Weights) Let $\mathcal{E} \in Bun(\mathcal{G}_{\Omega,X})$ be a quasi-parahoric torsor. A* weight *attached to $\mathcal{E}$ is an element $\theta \in Y(T) \otimes \mathbb{Q}$ in the interior of the facet determined by Ω.*

6.2.2. *Remark.* The choice of a weight gives the isomorphism w_{θ} which identifies $\mathcal{E}$ as an element in $Bun(\mathcal{G}_{\theta,X})$ via f_{θ}.

6.2.3. Definition. *(Parahoric torsor) A* parahoric torsor *is the pair* $(\mathcal{E}, \theta)$ *of a quasi-parahoric torsor* $\mathcal{E}$ *together with a rational weight* $\theta \in Y(T) \otimes \mathbb{Q}$*.*

6.2.4. *Remark.* Recall that by Theorem 5.2.2, fixing θ gives the following identification:

$$Bun_{X}(\mathcal{G}) \simeq Bun_{Y}^{\boldsymbol{\tau}}(\Gamma, G) \tag{6.2.4.1}$$

for a suitably defined covering $p : Y \to X$ with Galois group Γ. Thus, a family of parahoric torsors on X parametrized by T gets identified with a family of (Γ, G)–bundles on Y of local type $\boldsymbol{\tau}$.

6.2.5. *Remark.* The notion of weight defined above is the precise analogue of the classical weight for a parabolic vector bundle with *multiplicity* (cf. [17, Definition 1.5, page 211]).

6.3. **Parabolic line bundles.** Fix a finite subset $D \subset X$ with $|D| = m$.

6.3.1. Definition. *(see* [17, Definition 1.5, page 211]*) A* parabolic line bundle *on* (X, D) *is a pair* $(\mathcal{L}, \{\alpha_1, \dots, \alpha_m\})$*, where* $\mathcal{L}$ *is a line bundle on* X *together with a* m*–tuple of rational numbers* $(\alpha_1, \dots, \alpha_m)$ *with* $0 \le \alpha_i \le 1$*. The* parabolic degree *of a parabolic line bundle is defined as*

$$pardeg(\mathcal{L}) = deg(\mathcal{L}) + \sum_{i=1}^{m} \alpha_i$$

6.3.2. *Remark.* Let $p : Y \to X$ be a Galois cover ramified over $\mathcal{R}_p \subset X$ with ramification indices $n_{y_i}, i = 1, \dots, m$ at the points $y_i \in Y$ over $\mathcal{R}_p$ and let $Gal(Y/X) = \Gamma$.

Let L be a a Γ–line bundle on Y of local type $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$, where each $\boldsymbol{\tau}_i$ acts a character $\boldsymbol{\tau}_i(\zeta) = \zeta^{a_{y_i}}$ with $|a_{y_i}| < n_{y_i}, \forall i$. Then by [28] and [17], the *invariant direct image* $\mathcal{L} \simeq p_*^{\Gamma}(L)$ determines a parabolic line bundle on $(X, \mathcal{R}_p)$ with parabolic weights $(\frac{a_{y_1}}{n_{y_1}}, \dots, \frac{a_{y_m}}{n_{y_m}})$ and parabolic degree:

$$pardeg(p_*^{\Gamma}(L)) = deg(p_*^{\Gamma}(L)) + \sum_{i=1}^{m} \frac{a_{y_i}}{n_{y_i}}$$

6.3.3. *Remark.* In fact, all parabolic line bundles on (X, D) can be realized in this manner namely, as *invariant direct images*; this is done by constructing a cover ramified over D with suitable ramification indices.

6.4. **Parabolic subgroup scheme of Bruhat-Tits group schemes.** Let $\mathcal{G}_{\Omega,X}$ be a Bruhat-Tits group scheme on the curve X as in Notation 5.1. Following Heinloth [14, Definition 17], we have:

6.4.1. Definition. *A maximal parabolic subgroup* $P \subset \mathcal{G}_{\Omega,X}$ *of the group scheme* $\mathcal{G}_{\Omega,X}$ *is defined as the flat closure of a maximal parabolic subgroup of the generic fibre* $\mathcal{G}_K$ *of* $\mathcal{G}_{\Omega,X}$*.*

6.4.2. *Remark.* Since we work over $\mathbb{C}$, by Cartier's theorem flat group schemes are automatically smooth (cf. [14, Lemma 21]).

Let E be a $\mathcal{G}_{\Omega,X}$–torsor on X. Then we have (Heinloth [14, Lemma 23])

6.4.3. Lemma. *Let* $P_K \subset \mathcal{G}_K$ *be a maximal parabolic subgroup and let* E *be a* $\mathcal{G}_{\Omega,X}$*–torsor on* X*. Any choice of reduction section* $s_K \in E_K(\mathcal{G}_K/P_K) = E_K/P_K$ *defines a maximal parabolic subgroup* $P' \subset \mathcal{G}_{\Omega,X}$ *together with a reduction* s' *of* E *to* P'*.*

Proof: This follows immediately from [14] where the proof is given for Borel subgroup schemes. The only ingredients needed are Heinloth's uniformization theorem of ([14, Theorem 1]) and the projectivity of $E(G_K/P_K)$ over $X - Ram(\mathcal{G}_{\Omega,X})$.

6.4.4. *Remark.* Note however that $\mathcal{G}_{\Omega,X}/P$ need not be a projective scheme over A for all parahorics. It is so for instance if $\mathcal{G}_{\Omega,X}$ is a hyperspecial parahoric.

Let $\chi : P_K \to \mathbb{G}_{m,K}$ be a dominant character of the parabolic subgroup P_K. Then one knows that this defines an ample line bundle L_χ on $\mathcal{G}_K/P_K$. Of course, the quotient $\mathcal{G}_{\Omega,X}/P$ for a flat closure of P_K is not projective over X but $\mathcal{G}_K/P_K$ is projective over K. We see immediately that χ defines a line bundle L_χ on E_K/P_K as well and using a reduction section s_K, we therefore get a line bundle $s_K^*(L_\chi)$ on $X - x$.

6.4.5. Proposition. *Suppose that we are given the Bruhat-Tits group scheme $\mathcal{G} = \mathcal{G}_{\Omega,X}$ extending the generic group $\mathcal{G}_K$. Suppose further that we are given a weight $\theta \in Y(T) \otimes \mathbb{Q}$, a point in the facet determined by the parahoric subgroup $\mathcal{G}(A)$ and hence a $w_\theta : \mathcal{G}(A) \simeq \mathcal{P}_\theta(K)$. Let s_K be a generic reduction of structure group of E_K to P_K. Then the line bundle $s_K^*(L_\chi)$ on $X - x$ has a canonical extension L_χ^θ to X as a parabolic line bundle.*

Proof: By Theorem 2.4.1 once the identification w_θ is fixed along with the choice of $\theta \in Y(T) \otimes \mathbb{Q}$, we have a ramified cover $p : Y \to X$ with $\Gamma = Gal(Y/X)$ so that $\mathcal{G}_{\Omega,X}(A) = G(B)^\Gamma$. The data (E, w_θ), of a $\mathcal{G}$–torsor *together with weights* is therefore equivalent to giving a (Γ, G)–principal bundle F on Y.

The maximal parabolic subgroup $P_K \subset \mathcal{G}_K$ immediately gives a maximal parabolic $Q \subset G$ and the reduction s_K gives in turn a Γ–equivariant reduction of structure group t_L of F_L/Q_L, where L denotes the quotient field of B the local ring in Y over $x \in X$. By virtue of the projectivity of Y, the reduction section t_L extends to a Γ–equivariant reduction of structure group $t \in F/Q$. The dominant character χ gives a dominant character η of Q and the section t gives a Γ–line bundle $t^*(L_\eta)$.

Now observe that the GIT quotient of F/Q by the finite group Γ gives a natural *compactification* of $\mathcal{G}_{\Omega,X}/P'$. We observe that the line bundle $L_\chi^\theta := p_*^\Gamma(t^*(L_\eta))$ gives the required extension of $s_K^*(L_\chi)$. By the very definition of the invariant direct image (see Remark 6.3.2), we see that $L_\chi^\theta = p_*^\Gamma(t^*(L_\eta))$ gets the natural structure a *parabolic line bundle*.

q.e.d

6.4.6. Lemma. *Let $q_i : Y_i \to X$ be two coverings, ramified at $R \subset X$ with the same ramification indices. Let $\Gamma_i = Gal(Y_i/X)$ such that $\mathcal{G}_{\Omega,X}(A) = G(B_i)^{\Gamma_{i,y}}$, for $i = 1, 2$. Then parabolic line bundle L_χ^θ obtained in Proposition 6.4.5 is independent of the covering $q_i : Y_i \to X$.*

Proof: Since the ramification locus and ramification indices are the same for the coverings q_i, it follows that we have a common covering $p_i : Y \to Y_i$, such that Y dominates the coverings $q_i, i = 1, 2$ and the composite covering maps $q_i \circ p_i$ are also ramified over R. The existence of Y can be seen as follows: by (2.3.1), the ramification datum of the coverings $Y_i, i = 1, 2$, implies that there exist finite index normal subgroups $\pi_i \subset \pi$, such that π_i act freely on $\mathbb{H}$ and $\mathbb{H}/\pi_i \simeq Y_i$. It is immediate that $\pi_3 = \pi_1 \cap \pi_2$ is a normal subgroup of π of finite index and we can take $Y = \mathbb{H}/\pi_3$.

Thus to check the non-dependence of the parabolic line bundle L^{θ}_{χ} on the choice of the covering, it is enough to look at the map $p : Y \to Y_1 \to X$, with $p = q_1 \circ p_1$. Clearly, $p_1 : Y \to Y_1$ is étale. In other words, the local automorphism group of the covering $p : Y \to X$ is the same as $q_1 : Y_1 \to X$.

Let Γ_0 be the Galois group of the étale covering $p_1 : Y \to Y_1$. Then line bundle $t^*(L_{\eta})$ on Y_1 obtained in Proposition 6.4.5 from the character η when pulled-back to Y gets identified with the similar line bundle on Y defined the by the same character η. It is now easy to see that we have an identification of invariant direct images, $p^{\Gamma_1}_{*}(t^*(L_{\eta})) = p^{\Gamma}_{*}(t^*(L_{\eta}))$, where $\Gamma = Gal(Y/X)$.

q.e.d

6.4.7. *Remark.* The essential point in the independence statement made above is that the parabolic line bundle L^{θ}_{χ} depends only on the *local automorphism groups* and the ramification indices, i.e the datum which provides the parabolic weights and these are the same for the coverings q_i.

We have the following general definition of stability and semistability for (Γ, G)–bundles following A. Ramanathan [24, Lemma 2.1].

6.4.8. Definition. *(Semistability and stability) Let G be a reductive algebraic group. A (Γ, G)–bundle E on Y is called* Γ*-semi-stable (resp.* Γ*-stable) if for every maximal parabolic subgroup $P \subset G$ and every Γ–invariant reduction of structure group $\sigma : Y \to E(G/P)$, and for every dominant character $\chi : P \to \mathbb{G}_m$ we have* $\deg\ \sigma^*(L_{\chi}) \leq 0$. *(resp < 0).*

We then make the analogous definition:

6.4.9. Definition. *Let $\mathcal{G} = \mathcal{G}_{\Omega,X}$. A parahoric $\mathcal{G}$–torsor (E, θ) is called* stable (resp. semistable) *if for every maximal parabolic $\mathcal{P}_K \subset \mathcal{G}_K$, for every dominant character χ as above, for every reduction of structure group s_K, we have:*

$$pardeg(L^{\theta}_{\chi}) < 0 (resp. \leq 0)$$

6.4.10. Theorem. *The functorial identification*

$$Bun(\mathcal{G}_{\Omega,X}) \simeq Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$$

given by Theorem 5.2.2 identifies stable (resp. semistable) objects in the set $Bun(\mathcal{G}_{\Omega,X})$ with stable (resp. semistable) (Γ, G)–bundles of local type $\boldsymbol{\tau}$ on the ramified cover Y.

Proof: The proof of the theorem follows immediately from the above discussions together with Definition 6.4.8 .

q.e.d

6.4.11. *Remark.* (*Harder-Narasimhan reduction*) With the definition of semistability in place, it is routine now to define the Harder-Narasimhan reduction for a $\mathcal{G}$–torsor by using the identification of Theorem 6.4.10. The existence of a parahoric HN reduction is immediate from the exietence of a Γ–equivariant parabolic HN reduction for a (Γ, G)–bundle. The uniqueness follows as in the proof of the independence of the parabolic line bundle (see Lemma 6.4.6). Since the covering $p_1 : Y \to Y_1$ is étale and Galois, the uniqueness of the HN reduction for (Γ, G)–bundles shows the uniqueness of the HN reduction of a parahoric bundle as well.

7. Unitary representations of π

7.1. **Manifold of irreducible unitary representations of π.** Notations in this section are as in the introduction. Recall also the notion of local type of unitary representations $\rho : \pi \to K_G$ from Definition 1.0.1. We now recall the following result from Weil [36, Page 157].

7.1.1. Proposition. *Let ρ be a representation of π on a vector space V (over $\mathbb{R}$) such that $d = \dim V$ and ρ is unitary (or more generally leaving invariant a non-degenerate bilinear form on V). Then we have*

$$\dim_{\mathbb{R}} H^1(\pi,\rho) = 2d(g-1) + 2\dim_{\mathbb{R}} H^0(\pi,\rho) + \sum_{\nu=1}^{m} e_\nu$$

where e_ν is the rank of the endomorphism $(I - \rho(C_\nu))$ of V.

Let K_G be the maximal compact subgroup of G and $\kappa_G \simeq Lie(K_G)$ denote *the Lie algebra* of K_G, which is a real vector space of dimension d, where $d = dim(G)$.

As in the introduction, we assume that $X = \mathbb{H}/\pi$, with $x \in X$ corresponding to $z \in \mathbb{H}$. Let π_z be the stabilizer at z (cyclic of order n_x) and let γ be a generator of π_z and let $\rho : \pi \to K_G$ be a unitary representation of π.

7.2. **Explicit computation when G is simple:** Let $\alpha \in S$ and let ρ_{θ_α} be as in Definition 2.4.3. Let $\rho_{\theta_\alpha}(\gamma) \in K_G$ be the image of the generator γ of π_z. Note that the choice of the simple root α and identification of the representation ρ with ρ_{θ_α} amounts to *fixing the local type* of the representation $\rho : \pi \to K_G$, i.e the *conjugacy class* of $\rho(\gamma)$ in K_G.

We denote by Ad ρ_{θ_α}, the adjoint transformation on κ_G, namely if $M \in \kappa_G$, $M \mapsto \rho_{\theta_\alpha}(\gamma) M \rho_{\theta_\alpha}(\gamma)^{-1}$. Then we have:

7.2.1. Proposition. *Let $e(\theta_\alpha)$ denote the rank of $(Id - Ad\ \rho_{\theta_\alpha})$ on κ_G. Then*

$$e(\theta_\alpha) = dim_{\mathbb{R}}(K_G) - 2\mu(\alpha) - 2\nu(\alpha) - \ell = 2.(dim_{\mathbb{C}}(G/P_\alpha)) - \mu(\alpha)) \tag{7.2.1.1}$$

where P_α is the maximal parabolic subgroup of G associated to α and

$$\mu(\alpha) = \#\{r \in R^+ \mid r = c_\alpha.\alpha + \sum_{\beta \neq \alpha} x_\beta.\beta\} \tag{7.2.1.2}$$

$$\nu(\alpha) = \#\{r \in R^- \mid r \textit{ involves simple roots} \neq \alpha\} \tag{7.2.1.3}$$

and $\ell = \mid S \mid$.

Proof. Make K_G operate on itself by inner conjugation. Then, rank of (Id-Ad ρ_{θ_α}) acting on the Lie algebra κ_G equals *the dimension of the orbit through $\rho_{\theta_\alpha}(\gamma)$ for the action of K_G on itself by inner conjugation.*

We may assume for the purpose of this computation that $\rho_{\theta_\alpha}(\gamma)$ lies in the maximal torus. We firstly compute the number of roots $r \in R$ so that the corresponding root group $U_r(B)$ is centralized by $\rho_{\theta_\alpha}(\gamma)$. Recall from Definition 2.4.3 that the action of $\rho_{\theta_\alpha}(\gamma)$ on U_r is given as follows:

$$\rho_{\theta_\alpha}(\gamma).U_r(B).\rho_{\theta_\alpha}(\gamma)^{-1} = U_r(\zeta^{r(\Delta_\alpha)} B) \tag{7.2.1.4}$$

where as seen earlier, $r(\Delta_\alpha) = d.(\theta_\alpha, r)$. Since ζ is a primitive d^{th}–root of unity, we need to compute the $\#$ $\{r \in R \mid (\theta_\alpha, r) = \pm 1 \textit{ or } 0\}$. It is easy to see that

$$\{r \in R \mid (\theta_\alpha, r) = \pm 1 \textit{ or } 0\} = \bigcup_{i=1}^{4} A_i(\alpha) \tag{7.2.1.5}$$

where for $i = 1, 2$,

$$A_i(\alpha) = \{r \in R^{\pm} \mid r = \pm c_\alpha.\alpha + \sum_{\beta \neq \alpha} \pm x_\beta.\beta\} \quad (7.2.1.6)$$

$$A_3(\alpha) = \{r \in R^- \mid r \textit{ involves simple roots} \neq \alpha\} \quad (7.2.1.7)$$

and

$$A_4(\alpha) = \{r \in R^+ \mid r \textit{ involves simple roots} \neq \alpha\} \quad (7.2.1.8)$$

Since the maximal torus centralizes $\rho_{\theta_\alpha}(\gamma)$, we see that the dimension of the centralizer of $\rho_{\theta_\alpha}(\gamma)$ is

$$\#\{r \in R \mid (\theta_\alpha, r) = \pm 1 \textit{ or } 0\} + \mid S \mid \quad (7.2.1.9)$$

Observe that $\mid A_4 \mid = \mid A_3 \mid$ and $\mid A_1 \mid = \mid A_2 \mid$. To compute the rank of (Id - Ad ρ_{θ_α}), we simply subtract the above number (7.2.1.9) from the $dim_{\mathbb{R}}(K_G)$ to get the first expression for $e(\alpha)$. We see that

$$\nu(\alpha) = dim_{\mathbb{C}}(P_\alpha/B) \quad (7.2.1.10)$$

where P_α is the maximal parabolic subgroup of G defined by the simple root $\alpha \in S$. Thus,

$$dim_{\mathbb{R}}(K_G) - 2.\nu(\alpha) - \ell = dim_{\mathbb{C}}(G) - 2.\nu(\alpha) - \ell = 2.dim_{\mathbb{C}}(G/P_\alpha).$$

since $2.dim(B) - \ell = dim(G)$.

Hence, $e(\theta_\alpha) = 2.(dim_{\mathbb{C}}(G/P_\alpha)) - \mu(\alpha))$ and the proposition now follows.

q.e.d

7.2.2. Corollary. *Let $\alpha \in S$ be such that $\mathcal{P}_{\theta_\alpha}(K)^{hs}$ is a maximal parahoric subgroup in $G(K)$* which is hyperspecial. *Then $e(\theta_\alpha) = 0$ and conversely.*

Proof: By Bruhat-Tits theory, the hyperspecial parahorics are simply the maximal parahorics $\{\mathcal{P}_{\theta_\alpha}(K) \mid \forall \alpha \in S, \text{with } c_\alpha = 1\}$ upto conjugacy by $G(K)$. In these cases, the number $\mu(\alpha)$ will now be

$$\mu(\alpha) = \#\{r \in R^+ \mid r \text{ involves } \alpha\}$$

since the largest possible coefficient for such an α in any positive root is 1. Hence α is hyperspecial if and only if $\mu(\alpha) = dim(G/P_\alpha)$ and we are through by the Proposition 7.2.1.

q.e.d

7.3. **The moduli dimension.** Let G be semisimple and simply connected.

7.3.1. Corollary. *Let $\theta \in \mathbb{E}$ be an arbitrary element in the affine apartment $\mathbb{E}$ and let ρ_θ be the representation defined in Definition 2.4.3. Let $e(\theta)$ denote the rank of $(Id - Ad\ \rho_\theta)$ on κ_G. Then,*

$$e(\theta) = dim_{\mathbb{R}}(K_G) - \mid S \mid - \#\{r \in R \mid (\theta, r) = \pm 1 \textit{ or } 0\} \quad (7.3.1.1)$$

Proof: The proof is immediate from the above discussion. Note that when $\theta = \theta_\alpha$, the number $e(\theta)$ gets the explicit expression (7.2.1.1).

Let $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$ be a set of conjugacy classes and let $\boldsymbol{\theta} = \{\theta_i\} \in \mathbb{E}^m$ the corresponding set of points of the product of the affine apartments, with $m = |\mathcal{R}_p|$.

7.3.2. THEOREM. *The subset $R_o \subset R^{\boldsymbol{\tau}}(\pi, K_G)$ of* irreducible representations *is open and non-empty and is further smooth of real dimension equal to*

$$(2g-1)dim(K_G) + \sum_{i=1}^{m} e(\boldsymbol{\theta}). \tag{7.3.2.1}$$

Let K_G act on $R^{\boldsymbol{\tau}}(\pi, K_G)$ by inner conjugation. Let $\overline{K}_G = K_G/centre$. Then the equivalence classes of irreducible representations corresponds to the quotient space $R_o/\overline{K}_G$ and has the natural structure of a complex analytic orbifold *(i.e, with at most finite quotient singularities) of dimension*

$$dim_{\mathbb{C}}(R_o/\overline{K}_G) = dim_{\mathbb{C}}(G)(g-1) + \sum_{i=1}^{m} \frac{1}{2} e(\boldsymbol{\theta}) \tag{7.3.2.2}$$

Proof: The theorem follows in much the same fashion as in Seshadri [28, Page 180] and is an immediate consequence of Proposition 7.2.1.

q.e.d

8. THE MODULI SPACE OF PARAHORIC TORSORS

In this section we study the moduli space of semistable (Γ, G)–bundles on Y of local type $\boldsymbol{\tau}$ (see Definition 6.4.8) and prove the basic geometric properties of this space. We use these to conclude similar facts about the space of semistable and stable parahoric torsors by appealing to Theorem 6.4.10. We essentially follow the strategy of Balaji-Seshadri [4] and Balaji-Biswas-Nagaraj [2]. It is shown in [2, Theorem 5.8] that the moduli space $M_Y(\Gamma, G)$ of Γ–stable (Γ, G)–bundles is realized as a good quotient $Q_{(\Gamma,G)}//\mathcal{H}$ of a suitably defined scheme $Q_{(\Gamma,G)}$.

We fix a faithful representation $G \hookrightarrow GL(n)$ and consider the subscheme of a suitable "Quot–scheme" parametrizing Γ–vector bundles on the curve Y which are Γ–semistable of fixed local type $\boldsymbol{\tau}$ and we denote this scheme by $Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}$ (see [28] for details where this space is denoted $R^{\boldsymbol{\tau},ss}$). We may equivalently view the points in $Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}$ as Γ–semistable principal $(\Gamma, GL(n))$–bundles of local type $\boldsymbol{\tau}$.

We then define the scheme $Q^{\boldsymbol{\tau}}_{(\Gamma,G)}$ as the *space of Γ–equivariant reductions of structure group* of the bundles in $Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}$ which consists of those (Γ, G)–bundle which are of local type $\boldsymbol{\tau}$.

We now use the results in [28] which shows that there is an action of a certain reductive group $\mathcal{H}$ on $Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}$ and the good quotient $Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}//\mathcal{H}$ exists and gives a coarse moduli scheme for the functor of equivalence classes of Γ–stable principal $(\Gamma, GL(n))$–bundles on Y of local type $\boldsymbol{\tau}$.

The map $Q^{\boldsymbol{\tau}}_{(\Gamma,G)} \to Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}$ obtained by taking extension of structure groups via the inclusion $G \hookrightarrow GL(n)$, is shown to be *affine* and the action of $\mathcal{H}$ lifts to $Q^{\boldsymbol{\tau}}_{(\Gamma,G)}$ to give a good quotient $Q^{\boldsymbol{\tau}}_{(\Gamma,G)}//\mathcal{H}$ which we denote by $M^{\boldsymbol{\tau}}_Y(\Gamma, G)$ (see [4] and [2]). This scheme $M^{\boldsymbol{\tau}}_Y(\Gamma, G)$ is indeed the coarse moduli space for equivalence classes of (Γ, G)–bundles on Y which are Γ–semistable and of fixed local type $\boldsymbol{\tau}$.

In this paper we show that the points of the scheme $M^{\boldsymbol{\tau}}_Y(\Gamma, G)$ parametrizes *isomorphism classes of (Γ, G)–bundles of type $\boldsymbol{\tau}$ which are unitary and hence polystable*. Using this we show that $M^{\boldsymbol{\tau}}_Y(\Gamma, G)$ is normal and projective.

8.0.3. *Remark.* We note that the arguments of [2] are not sufficient for showing this since the local type of the bundles was not fixed in [2]. A key step in the arguments is the connectedness of the moduli space which fails if the local type is not fixed.

8.0.4**.** Let $g \geq 2$ be the genus of X. Recall that the Fuchsian group π can be identified with the group on the letters $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_m$, modulo the relations

$$[A_1, B_1] \cdots [A_g, B_g] \cdot C_1 \cdots C_m = I. \tag{8.0.4.1}$$

$$C_1^{n_1} = C_2^{n_2} = \cdots = C_m^{n_m} = I. \tag{8.0.4.2}$$

where π_{z_i} the isotropy subgroup of π at some $z_i \in \mathbb{H}$ is the cyclic subgroup of π of order n_i generated by C_i. Let y_i be the image of z_i in Y and let $\mathcal{R}_p^* = \{y_i \mid 1 \leq i \leq m\}$ be the set of ramification points of the cover $p : Y \to X$.

Let $\rho : \pi \to G$ be a homomorphism. Let $E(\rho)$ denote the (Γ, G)–bundle on Y defined by the *twisted action* given by (2.3.1.2). We observe that the *local type* $\boldsymbol{\tau}_i$ *of the bundle* $E(\rho)$ *at* y_i in the sense of Definition 4.0.1 is equivalently given by the *conjugacy class of* $\rho(C_i)$ in G. Thus if $\boldsymbol{\tau}(\mathcal{R}_p^*) = \{\boldsymbol{\tau}_i\}$, then we have

$$\rho \text{ is of type } \boldsymbol{\tau} = \{\boldsymbol{\tau}_i\} \iff E(\rho) \text{ is of local type } \boldsymbol{\tau}(\mathcal{R}_p^*) \tag{8.0.4.3}$$

8.0.5. Definition. *Fix a maximal compact subgroup K_G of G. If the representation ρ factors through K_G, one says that $E(\rho)$ is a* unitary (π, G)*–bundle.*

Recall the functor $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ defined in (5.2.0.3). It is well-known that the functor $Bun_Y(G)$ of principal G–bundles on Y is representable by an Artin stack, which we again denote by $Bun_Y(G)$. It is a fact (as for example shown in [28]) that the subfunctor $Bun_Y(\Gamma, G)$ of (Γ, G) bundles as well as $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ are both closed subfunctors of $Bun_Y(G)$ and hence representable by Artin substacks.

Let $\boldsymbol{\tau} = \boldsymbol{\tau}(\mathcal{R}_p^*)$. Then we get a canonical map

$$\psi_1 : R^{\boldsymbol{\tau}}(\pi, K_G) \to Bun_Y^{\boldsymbol{\tau}}(\Gamma, G) \tag{8.0.5.1}$$

which send $\rho \mapsto E(\rho)$.

8.0.6. Proposition. *The moduli stack $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ of (Γ, G)–bundles on Y of fixed local type $\boldsymbol{\tau}$ is irreducible and smooth when the group G is semisimple and simply connected.*

Proof: This is immediate from Theorem 5.2.2, [14, Theorem 2] and [14, Proposition 1]. We now indicate a different proof of the connectedness from the Hecke correspondence shown above in (5.3.0.2). Observe that from Drinfeld-Simpson[9] (see also Ramanathan [23]), it follows that for the distinguished hyperspecial parahoric $\mathcal{P}_0(K) = G(A)$, the moduli stack $Bun_X(G)$ is irreducible because G is semisimple and simply connected. Further, the morphism $Bun(\mathcal{G}_{\mathcal{I},X}) \to Bun_X(G)$ is surjective and has fibre G/B, B being the Borel subgroup. Hence, $Bun(\mathcal{G}_{\mathcal{I},X})$ is connected. Now observe that the map $Bun(\mathcal{G}_{\mathcal{I},X}) \to Bun(\mathcal{G}_{\Omega,X})$ given by (5.3.0.2) is also surjective since it comes from the inclusion $\mathcal{I} \subset \mathcal{P}_\Omega(K)$. Hence $Bun(\mathcal{G}_{\Omega,X})$ is connected. The irreducibility follows from the formal smoothness of the functor of torsors (see [14, Proposition 1]; the obstruction to smoothness vanishes since we work on curves.

q.e.d

8.1. **Properness of the moduli of (Γ, G)-bundles.** Let $H = G/Z(G)$, the associated adjoint group. For such semisimple adjoint type groups we have the following obvious property. Let $\mathfrak{h} = Lie(H)$. Consider the adjoint representation $\rho : H \to GL(\mathfrak{h})$. It is clear that ρ is faithful irreducible representation.

Fix the representation $\rho : H \hookrightarrow GL(n)$ (where $n = dim\mathfrak{h}$) and a maximal compact K_H of H such that $K_H \hookrightarrow U(n)$. Consider the subset $Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s \subset Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)$ consisting of the stable $(\Gamma, GL(n))$-bundles.

8.1.1. Lemma. *Let $\phi : Bun_Y^{\boldsymbol{\tau}}(\Gamma, H) \longrightarrow Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)$ be the morphism induced by the representation ρ. Then the inverse image of the stable points $\phi^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s) = Bun_Y^{\boldsymbol{\tau}}(\Gamma, H)^s$, (when nonempty), consists of unitary (Γ, H)-bundles.*

Proof: We claim that a principal (Γ, H) bundle E is unitary if and only if the associated $(\Gamma, GL(\mathfrak{h}))$–bundle $E(\mathfrak{h})$ is so. If E is unitary obviously so is $E(\mathfrak{h})$.

We now show the converse. Let $A(\mathfrak{h})$ denote the stabilizer of the $GL(\mathfrak{h})$–action on the tensor space $\mathfrak{h}^* \otimes \mathfrak{h}^* \otimes \mathfrak{h}$ at the point $[\ ,\ ]$, i.e the Lie bracket. Since we have assumed that H is of adjoint type it implies that $A(\mathfrak{h}) = Aut(\mathfrak{h})$.

Now assume that $E(\mathfrak{h})$ comes from a unitary representation of π, then we take the Lie bracket morphism $E(\mathfrak{h}) \otimes E(\mathfrak{h}) \to E(\mathfrak{h})$. Either side comes from unitary representations of π and by *local constancy* ([17, Proposition 1.2]), i.e morphisms of bundles are induced by morphisms of π–modules. It now follows that $E(\mathfrak{h})$ gets a reduction of structure group to the group $A(\mathfrak{h}) = Aut(\mathfrak{h})$.

Since H is an adjoint group we have a short exact sequence:

$$1 \to H \to A(\mathfrak{h}) \to F \to 1$$

since H is the component of identity of $A(\mathfrak{h})$. Again we have a similar exact sequence of compact groups:

$$1 \to K_H \to K_{A(\mathfrak{h})} \to F \to 1$$

The bundle E is therefore such that $E(A(\mathfrak{h}))$ is a unitary bundle and comes from a representation $\bar{\chi} : \pi \to K_{A(\mathfrak{h})}$. Furthermore, the extended bundle $E(A(\mathfrak{h}))(F)$ is trivial since it comes with a section (giving E). By composing the representation $\bar{\chi}$ with the map $K_{A(\mathfrak{h})} \to F$, we see that the triviality of $E(A(\mathfrak{h}))(F)$ forces the composite to be the trivial homomorphism, implying that $\bar{\chi}$ factors via $\chi : \pi \to K_H$ to give the bundle E (cf. Atiyah-Bott [1, Lemma 10.12]).

Now using the main theorem of [28] we see that points of $Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s$, being stable bundles, are all unitary. Hence by the claim above the bundles in the inverse image $\phi^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ are also unitary.

q.e.d

8.1.2. Proposition. *Let ρ be the adjoint representation of H. Then the inverse image of $Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s$ by the induced morphism ϕ is nonempty.*

Proof: Let $\pi = Aut(\mathbb{H})$ .Then one knows that $\Gamma = Gal(Y/X)$ is the quotient of π by a normal subgroup π_o which acts freely on Y and by [17] a Γ-bundle is stable if and only if it arises from a unitary representation of π. The group π can be identified with the free group on the letters $A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_m$ modulo the relations (8.0.4.1) and (8.0.4.2).

So to prove that the inverse image $\phi^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ is nonempty, we need to exhibit a representation $\chi : \pi \to K_H$ such that the composition

$$\rho \circ \chi : \pi \to U(n) \text{ is } \textit{irreducible}. \qquad (8.1.2.1)$$

Choose elements $h_1, \cdots, h_m \in K_H$ so that they are elements of order n_i, where $i = 1, \cdots, m$ (these correspond to fixing the *local type* $\boldsymbol{\tau}$ of our bundles).

It is a well–known fact that every element of a compact connected real semisimple Lie group is a commutator. Further it is well-known (see for example [30, Lemma 3.1]) that there exists a dense subgroup $\langle \alpha, \beta \rangle$ of K_H generated by two general elements $\{\alpha, \beta\}$. Recall that the genus $g \geq 2$ and define the representation $\chi : \pi \to K_H$ as follows :

$$\chi(A_1) = \alpha, \, \chi(B_1) = \beta, \, \chi(A_2) = \beta, \, \chi(B_2) = \alpha, \tag{8.1.2.2}$$

$$\chi(A_i) = a_i, \, \chi(B_i) = b_i, \, for \, i = 3, \cdots, g, \, \chi(C_j) = h_j, \, and \, j = 1, \cdots, m \tag{8.1.2.3}$$

It is clear that χ gives a representation of the group π. Since ρ is irreducible, and the image of χ contains a dense subgroup, the composition $\rho \circ \chi$ gives an irreducible representation of π in the unitary group $U(n)$. Therefore, it gives a *stable* Γ-linearized vector bundle, which comes as the extension of structure group of a H-bundle. This completes the proof of the Proposition.

q.e.d

8.1.3. Corollary. *In the stack $Bun_Y^{\boldsymbol{\tau}}(\Gamma, H)$ for the H-bundles, there is a non-empty Zariski open substack consisting of unitary bundles of local type $\boldsymbol{\tau}$.*

Proof: This follows immediately from the Lemma 8.1.1 and Proposition 8.1.2.

q.e.d

We now return to G which is as before a semisimple, simply connected algebraic group.

8.1.4. Proposition. *In the stack $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ for the G-bundles, there is a non-empty Zariski open substack consisting of stable unitary bundles of local type $\boldsymbol{\tau}$.*

Proof: Let $\eta : Bun_Y^{\boldsymbol{\tau}}(\Gamma, G) \to Bun_Y^{\boldsymbol{\tau}}(\Gamma, H)$ be the morphism induced by the quotient map $G \to H$. We claim that the required open subset of $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ is $(\phi \circ \eta)^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$.

Let E be a (Γ, G)–bundle in $(\phi \circ \eta)^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$. It follows that $E(H) \in \phi^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$. By Lemma 8.1.1 the H–bundle $E(H)$ comes from a unitary representation $\rho : \pi \to K_H$.

Recall that, by the structure of π described above, there is a central extension

$$1 \to Z_{\tilde{\pi}} \to \tilde{\pi} \to \pi \to 1 \tag{8.1.4.1}$$

where $\tilde{\pi}$ is generated by $A_1, \ldots, A_g, B_1, \ldots, B_g, C_1, \ldots, C_m$ together with a central element J satisfying the extra relation

$$[A_1, B_1] \cdots [A_g, B_g] \cdot C_1 \cdots C_m = J. \tag{8.1.4.2}$$

It is easy (as in [19]), by adding an extra lasso around a dummy point (other than the parabolic points) to choose a lift of ρ to a representation $\tilde{\rho} : \tilde{\pi} \to K_G$ so that the associated (Γ, G)–bundle $E(\tilde{\rho})$ also maps to $E(H)$. Thus, both E and $E(\tilde{\rho})$ give $E(H)$ under the quotient map $G \to H$. Therefore, by twisting by a central character of $\tilde{\pi}$, we get a representation $\tilde{\pi} \to K_G$ which gives the (Γ, G)–bundle E (cf. [23, Page 148]).

We observe that this representation $\tilde{\pi} \to K_G$ in fact descends to a representation $\pi \to K_G$. This follows from the fact that the local type of E at the dummy point is trivial.

From this we can now conclude that all bundles in $(\phi \circ \eta)^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ are unitary (cf. [1, Lemma 10.12]). Furthermore, it is easy to see that a (Γ, G)–bundle is stable if and only if the associated (Γ, H)–bundle is so (cf. [23, Proposition 7.1]). It follows that all points of $(\phi \circ \eta)^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ are also stable (Γ, G)–bundles.

q.e.d

Let $f : Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^{ss} \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ be the canonical quotient map obtained by the categorical quotient property of the moduli space $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$. Composing f with ψ_1 of (8.0.5.1), we get a *continuous map* $\psi = f \circ \psi_1 : R^{\boldsymbol{\tau}}(\pi, K_G) \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$.

8.1.5. *Remark.* The open substack obtained in Proposition 8.1.4 gets identified with the open subspace of $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ of (Γ, G)–bundles with *full holonomy* and is hence *smooth* since such bundles have trivial automorphism groups.

8.1.6. Theorem. *The map* $\psi : R^{\boldsymbol{\tau}}(\pi, K_G) \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ *is surjective and hence* $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ *is compact. Thus the variety* $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ *gets a structure of a normal projective variety. Moreover, this in turn implies that the stack* $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^{ss}$ *is* proper.

Proof: Consider the canonical categorical quotient map $f : Bun_Y^{\boldsymbol{\tau}}(\Gamma, G) \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$. Let

$$Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^s := (\phi \circ \eta)^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$$

Since f is surjective (and hence dominant), by Chevalley's lemma, the image $f(Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^s)$ in $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ contains a Zariski open subset.

By the Proposition 8.1.4 above the subset $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^s$ is nonempty and consists entirely of unitary bundles. That is, the image $f(Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^s)$ is a subset of the image $\psi(R^{\boldsymbol{\tau}}(\pi, K_G))$ in $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$. Thus, it follows that $\psi(R^{\boldsymbol{\tau}}(\pi, K_G))$ contains a Zariski open subset of $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$. But then, since $R^{\boldsymbol{\tau}}(\pi, K_G)$ is compact the image $\psi(R^{\boldsymbol{\tau}}(\pi, K_G))$ is closed in $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ and contains a dense subset, and is therefore the whole of $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$, since these moduli spaces $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ are *irreducible* (by Proposition 8.0.6).

This proves that $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ is topologically compact and hence by GAGA a projective variety. The normality follows from the smoothness of the stack $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^{ss}$, again by Proposition 8.0.6.

That this implies the properness of the stack $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^{ss}$ follows for instance from [4, Lemma 3.1].

q.e.d

8.1.7. Corollary. *The map* $\psi : R^{\boldsymbol{\tau}}(\pi, K_G) \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ *defined above descends to a map*

$$\psi^* : R^{\boldsymbol{\tau}}(\pi, K_G)/\overline{K}_G \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$$

which gives a homemorphism of topological spaces. Further, the subset $R_o/\overline{K}_G$ *of equivalence classes of irreducible unitary representations maps bijectively onto the subset of stable* (Γ, G)*-bundles.*

Proof: Follows from the above discussions. The fact that irreducible representations give stable bundles follows exactly as in [23].

q.e.d

Let $\mathcal{G}_{\Omega,X}$ be a parahoric Bruhat-Tits group scheme on X associated to a subset $\Omega \subset \mathbb{E}^m$. Let

$$M(\mathcal{G}_{\Omega,X}) = \left\{ \begin{array}{l} \text{the set of equivalence classes of} \\ \text{semistable } \mathcal{G}_{\Omega,X}\text{–torsors on } X \end{array} \right\} \tag{8.1.7.1}$$

and let $M(\mathcal{G}_{\Omega,X})^s \subset M(\mathcal{G}_{\Omega,X})$ the subset of *stable* torsors.

By Theorem 6.4.10 the identification

$$Bun(\mathcal{G}_{\Omega,X}) \simeq Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$$

given by Theorem 5.2.2 identifies stable (resp. semistable, polystable) objects in $Bun(\mathcal{G}_{\Omega,X})$ with stable (resp. semistable, polystable) (Γ, G)–bundles of local type $\boldsymbol{\tau}$ on the ramified cover Y. This gives the following set-theoretic identification:

$$M_Y^{\boldsymbol{\tau}}(\Gamma, G) \simeq M(\mathcal{G}_{\Omega,X}) \tag{8.1.7.2}$$

We then have the following:

8.1.8. Theorem. *The set $M(\mathcal{G}_{\Omega,X})$ gets a natural structure of an irreducible normal projective variety with $M(\mathcal{G}_{\Omega,X})^s$ as an open subset. It gives a* coarse moduli space *for the open subfunctor $Bun(\mathcal{G}_{\Omega,X})^{ss}$ of $Bun(\mathcal{G}_{\Omega,X})$ (see* (5.2.0.3)*). Furthermore, we have a homeomorphism*

$$\phi^* : R^{\boldsymbol{\tau}}(\pi, K_G)/\overline{K}_G \to M(\mathcal{G}_{\Omega,X}) \tag{8.1.8.1}$$

where $\Omega = \Omega(\boldsymbol{\tau})$, which identifies the sub-orbifold $R_o/\overline{K}_G$ with $M(\mathcal{G}_{\Omega,X})^s$.

Proof: This follows immediately from Theorem 8.1.6.

q.e.d

8.1.9. Corollary. *Fix the local types $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$ (see Definition 1.0.1) and let $\boldsymbol{\theta} = \theta_i \in \mathbb{E}^m$ be the corresponding point in the product of the affine apartment. Then the dimension of the moduli space $M(\mathcal{G}_{\Omega,X})$ is given by*

$$dim_{\mathbb{C}}(G)(g-1) + \sum_{i=1}^{m} \frac{1}{2} e(\boldsymbol{\theta}) \tag{8.1.9.1}$$

Proof: This follows from Theorem 7.3.2 and the Corollary 8.1.8 .

8.1.10. *Remark.* When G is simple the expression for $e(\boldsymbol{\theta})$ in the dimension formula takes a more concrete shape (see Proposition 7.2.1).

8.1.11. *Remark.* A proof of properness of the functor of (Γ, G)-semistable bundles along the lines of [4] or [13], i.e as a *semistable reduction theorem* can also be given.

8.1.12**. Extension to the case when the structure group is reductive.** We now close by indicating briefly how to extend the construction of the moduli space of (Γ, H)–bundles to the case when the structure group H is a connected reductive algebraic group and identify it with the space of homomorphisms from π to K_H. Let $S = [H, H]$ be the derived group, i.e *the maximal connected semisimple subgroup* of H. Let Z_0 be the connected component of the centre of H (which is a torus) and one know that S and Z_0 together generate H. Let $G = Z_0 \times S$. Then in fact, $G \to H$ is a finite covering map. It is easy to see (following [23, page 145]) that (Γ, G)–bundles gives rise to (Γ, H)–bundles and the stability and semistability of the associated (Γ, H)–bundles follows immediately from that of the (Γ, G)–bundles.

The problem of handling the reductive group G reduces to the problem of handling the semisimple group H but which is *not simply connected*. Let G be the semisimple, simply connected algebraic group which is the covering group of H.

We are in the situation of Proposition 8.1.4. Recall the central extension (8.1.4.1). By adding a dummy point other than the parabolic point, the theory of (π, H)–bundles is recovered from that of $(\tilde{\pi}, G)$–bundles. Notice that a homomorphism $\pi \to K_H$ has as many liftings $\tilde{\pi} \to K_G$ as the order of the centre of G. It follows quite easily, following arguments as in Lemma 8.1.1, that the number of connected components of the moduli space in the non-simply connected case is given by the order of the centre of G. In fact, $Hom(\tilde{\pi}, K_G)$ is a union of spaces labelled by elements of the centre of G. Let $Z_0 = Ker(G \to H)$. Then, there is an action of $H^1(X, Z_0)$ on a specific labelled subset of $Hom(\tilde{\pi}, K_G)$. A component of the moduli space of representations into K_H can be obtained as a quotient of each of these by the action of $H^1(X, Z_0)$. Details of these ideas are again found in [23, page 148] and follow the ideas of Narasimhan and Seshadri [19], where the data over a dummy point is called a *special parabolic structure*.

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